

1-1-2019

## Some results related to the Berezin number inequalities

ULAŞ YAMANCI

MÜBARİZ TAPDIGOĞLU

Follow this and additional works at: <https://journals.tubitak.gov.tr/math>



Part of the [Mathematics Commons](#)

---

### Recommended Citation

YAMANCI, ULAŞ and TAPDIGOĞLU, MÜBARİZ (2019) "Some results related to the Berezin number inequalities," *Turkish Journal of Mathematics*: Vol. 43: No. 4, Article 10. <https://doi.org/10.3906/mat-1812-12>

Available at: <https://journals.tubitak.gov.tr/math/vol43/iss4/10>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals. For more information, please contact [academic.publications@tubitak.gov.tr](mailto:academic.publications@tubitak.gov.tr).

## Some results related to the Berezin number inequalities

Ulaş YAMANCI<sup>1,\*</sup>, Mubariz GARAYEV

<sup>1</sup>Department of Statistics, Faculty of Science, Süleyman Demirel University, Isparta, Turkey

<sup>2</sup>Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

Received: 05.12.2018

Accepted/Published Online: 27.05.2019

Final Version: 31.07.2019

**Abstract:** In this paper, we prove reverse inequalities for the so-called Berezin number of some operators. Also, by using the classical Jensen and Young inequalities, we obtain upper bounds for Berezin number of  $A^\alpha X B^\alpha$  and  $A^\alpha X B^{1-\alpha}$  for the case when  $0 \leq \alpha \leq 1$ .

**Key words:** Berezin number, positive operator, reproducing kernel Hilbert space, Berezin symbol, McCarthy inequality

### 1. Introduction

Denote by  $\mathcal{F}(\Omega)$  the set of all complex valued functions on some set  $\Omega$ . A reproducing kernel Hilbert space (RKHS for short) on the set  $\Omega$  is a Hilbert space  $\mathcal{H} \subset \mathcal{F}(\Omega)$  with a function  $k_\lambda : \Omega \times \Omega \rightarrow \mathcal{H}$ , which is called the reproducing kernel enjoying the reproducing property  $k_\lambda := k(\cdot, \lambda) \in \mathcal{H}$  for all  $\lambda \in \Omega$ , and

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$$

holds for all  $\lambda \in \Omega$  and all  $f \in \mathcal{H}$  (see [2, 28]).

Let  $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$  be the normalized reproducing kernel of the space  $\mathcal{H}$ . For any bounded linear operator  $A$  on  $\mathcal{H}$ , the Berezin symbol of  $A$  is the function  $\widetilde{A}$  defined by (see [5])

$$\widetilde{A}(\lambda) := \left\langle A \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \quad (\lambda \in \Omega).$$

The Berezin symbol is a very useful tool in studying operators on the RKHS, including Hardy, Bergman, and Fock spaces. For example, boundedness, invertibility, compactness, and positivity of some operators are characterized or related to their Berezin symbols (see [8, 10, 22, 26, 31]).

Following Coburn [9], note that since the Berezin map  $A \xrightarrow{B} \widetilde{A}$  is linear and in most familiar RKHSs it is one-to-one, it “encodes” operator-theoretic information into function theory in a striking but somewhat impenetrable way. In fact, since  $\widehat{k}_\lambda \rightarrow 0$  weakly as  $\lambda \rightarrow \partial\Omega$  (of course, if the space  $\mathcal{H}(\Omega)$  is standard in the sense of Nordgren and Rosenthal [27]), it is clear that  $B$  maps compact operators on these spaces into functions that vanish at the boundary  $\partial\Omega$ . Because of these properties, the mapping  $B$  has found useful applications in dealing with operators “of function-theoretic significance” such as Toeplitz and Hankel operators on the Hardy,

\*Correspondence: [ulasyamanci@sdu.edu.tr](mailto:ulasyamanci@sdu.edu.tr)

2010 *AMS Mathematics Subject Classification*: 47B35, 47A12

Bergman, and Fock spaces (for more information, see, for instance, Coburn [9], Berger and Coburn [6], and Engliš [15, 16]).

Recall that the Berezin set and the Berezin number for an operator  $A \in \mathcal{B}(\mathcal{H}(\Omega))$  were introduced by the second author in [22, 23] as follows:

$$Ber(A) := Range(\tilde{A}) = \left\{ \tilde{A}(\lambda) : \lambda \in \Omega \right\} \quad (\text{Berezin set}),$$

$$ber(A) := \sup \left\{ \left| \tilde{A}(\lambda) \right| : \lambda \in \Omega \right\} \quad (\text{Berezin number}).$$

Clearly,  $Ber(A) \subset W(A) := \{ \langle Ax, x \rangle : \|x\|_{\mathcal{H}} = 1 \}$  (numerical range) and  $ber(A) \leq w(A) := \sup \{ |\langle Ax, x \rangle| : \|x\|_{\mathcal{H}} = 1 \}$  (numerical radius). More information about numerical range and numerical radius can be found in [1, 4, 14, 19, 21, 24, 25]. Recently, some results about the Berezin number were obtained in [3, 18, 20, 29, 30].

In the present paper, by using some ideas from [12, 13], we will prove reverse inequalities for the so-called Berezin number of some operators acting in the reproducing kernel Hilbert space. Also, we obtain upper bounds for the Berezin number of  $A^\alpha X B^\alpha$  and  $A^\alpha X B^{1-\alpha}$  for the case when  $0 \leq \alpha \leq 1$ .

## 2. Relations between numerical radius and Berezin number

Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a RKHS of complex-valued functions on a set  $\Omega$ . A subset  $\mathcal{M}(\Omega)$  in  $\mathcal{H}(\Omega)$  is called the multiplier for the space  $\mathcal{H}(\Omega)$  if  $\mathcal{M}(\Omega) \mathcal{H}(\Omega) \subset \mathcal{H}(\Omega)$ , i.e.  $fg \in \mathcal{H}(\Omega)$  for any  $f \in \mathcal{M}(\Omega)$  and  $g \in \mathcal{H}(\Omega)$ .

The following two lemmas are well known (and very easy to verify).

**Lemma 1** *If  $f$  is a multiplier of  $\mathcal{H}(\Omega)$ , then  $\tilde{M}_f(\lambda) = f(\lambda)$  for all  $\lambda \in \Omega$ .*

**Proof** Indeed, if  $f$  is a multiplier, then we have

$$\begin{aligned} \tilde{M}_f(\lambda) &= \left\langle M_f \hat{k}_\lambda, \hat{k}_\lambda \right\rangle = \left\langle f \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \\ &= \frac{1}{\|\hat{k}_\lambda\|_{\mathcal{H}}^2} f(\lambda) \|\hat{k}_\lambda\|_{\mathcal{H}}^2 = f(\lambda) \end{aligned}$$

for all  $\lambda \in \Omega$ , as desired. □

**Lemma 2** *If  $f$  is a multiplier of  $\mathcal{H}(\Omega)$ , then  $f$  is bounded.*

**Proof** In fact, since  $f \in \mathcal{H}(\Omega)$  and  $fg \in \mathcal{H}(\Omega)$  for all  $g \in \mathcal{H}(\Omega)$ , by using Lemma 1, we have:

$$|f(\lambda)| = \left| \tilde{M}_f(\lambda) \right| = \left| \left\langle M_f \hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| \leq \|M_f\|.$$

Since  $M_f$  is a closed operator defined in hull space  $\mathcal{H}(\Omega)$ , by the closed graph theorem  $M_f$  is bounded (see, for instance, Aronzajn [2]). The last inequality shows that  $f$  is bounded. □

We set by  $\mathcal{H}_1$  the unit sphere of  $\mathcal{H}$ ,  $\mathcal{H}_1 = \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} = 1\}$ , and also we set  $\mathcal{J} := \{V \in \mathcal{B}(\mathcal{H}) : V \text{ is isometry}\}$ , where  $\mathcal{B}(\mathcal{H})$  is the Banach algebra of all bounded linear operators on  $\mathcal{H}$ .

In this short section, we prove the relations between the numerical radius and Berezin number of reproducing kernel Hilbert space operators, which improves some results in [17, 23].

**Theorem 1** Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a RKHS of complex-valued functions on  $\Omega$  with reproducing kernel  $k_\lambda$  such that it has a dense multiplier  $\mathcal{M}(\Omega)$  and  $k_0 = \mathbf{1}$ .

Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a bounded linear operator (i.e.  $A \in \mathcal{B}(\mathcal{H})$ ). Then

$$\sup_{V \in \mathcal{J}} \text{ber}(V^*AV) \leq w(A) \leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} \text{ber}(M_f^*AM_f).$$

**Proof** By assumption  $\mathcal{M}(\Omega)$  is dense in  $\mathcal{H}$ . Then it is standard to show that

$$\sup \{|\langle Af, f \rangle| : f \in \mathcal{H}_1\} = \sup \{|\langle Af, f \rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\}.$$

According to Lemma 2,  $\mathcal{M}(\Omega)$  consists of bounded functions of the space  $\mathcal{H}(\Omega)$ . Then we have:

$$\begin{aligned} w(A) &= \sup \{|\langle Af, f \rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\} \\ &= \sup \{|\langle AM_f\mathbf{1}, M_f\mathbf{1} \rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\} \\ &= \sup \{|\langle M_f^*AM_f\mathbf{1}, \mathbf{1} \rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\} \\ &= \|k_0\|_{\mathcal{H}}^2 \sup \left\{ \left| \left\langle M_f^*AM_f \frac{k_0}{\|k_0\|}, \frac{k_0}{\|k_0\|} \right\rangle \right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1 \right\} \\ &= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup \left\{ \left| \left\langle M_f^*AM_f \widehat{k}_0, \widehat{k}_0 \right\rangle \right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1 \right\} \\ &= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup \left\{ \left| \widetilde{M_f^*AM_f}(0) \right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1 \right\} \\ &\leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} \sup \left\{ \left| \widetilde{M_f^*AM_f}(\lambda) \right| : \lambda \in \Omega \right\} \\ &= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} \text{ber}(M_f^*AM_f), \end{aligned}$$

and hence

$$w(A) \leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} \text{ber}(M_f^*AM_f). \tag{2.1}$$

On the other hand, for any  $V \in \mathcal{J}$  and  $g \in \mathcal{H}$ , we have:

$$\begin{aligned} \text{ber}(V^*AV) &= \sup_{\lambda \in \Omega} \left| \widetilde{V^*AV}(\lambda) \right| = \sup_{\lambda \in \Omega} \left| \left\langle V^*AV \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \\ &= \sup_{\lambda \in \Omega} \left| \left\langle AV \widehat{k}_\lambda, V \widehat{k}_\lambda \right\rangle \right|. \end{aligned}$$

Since  $V$  is isometry,  $V \widehat{k}_\lambda \in \mathcal{H}_1$  for all  $\lambda \in \Omega$ . Then

$$\text{ber}(V^*AV) \leq \sup_{h \in \mathcal{H}_1} |\langle Ah, h \rangle| = w(A).$$

Thus,

$$\sup_{V \in \mathcal{J}} \text{ber}(V^*AV) \leq w(A). \tag{2.2}$$

It remains only to combine inequalities (2.1) and (2.2) to get the required result. The theorem is proven.  $\square$

**3. Reverse inequalities for the Berezin numbers of operators**

Our next results in this section are mainly motivated with Dragomir’s survey paper [13], where he proved relations only between the norm and numerical radius of operators. Here we investigate similar questions also for Berezin numbers of operators  $A$  and  $|A|^2 := A^*A$ ; here,  $|A| := (A^*A)^{1/2}$  is a so-called module of operator  $A$ .

**Theorem 2** *Let  $\mathcal{H} = \mathcal{H}(\Omega)$  be a RKHS on  $\Omega$  and  $A \in \mathcal{B}(\mathcal{H})$  be an operator. If  $\mu \in \mathbb{C} \setminus \{0\}$  and  $r > 0$  are such that*

$$\sqrt{\text{ber}(|A - \mu|^2)} \leq r, \tag{3.1}$$

then

$$(0 \leq) \sqrt{\text{ber}(|A|^2)} - \text{ber}(A) \leq \frac{1}{2} \frac{r}{|\mu|}. \tag{3.2}$$

**Proof** For any  $\lambda \in \Omega$ , we have from (3.1) that

$$\begin{aligned} \left\| A\widehat{k}_\lambda - \mu\widehat{k}_\lambda \right\| &= \left\| (A - \mu)\widehat{k}_\lambda \right\| = \left\langle (A - \mu)\widehat{k}_\lambda, (A - \mu)\widehat{k}_\lambda \right\rangle^{1/2} \\ &= \left\langle (A - \mu)^*(A - \mu)\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{1/2} \leq \text{ber}((A - \mu)^*(A - \mu))^{1/2} \\ &= \text{ber}(|A - \mu|^2)^{1/2} \leq r, \end{aligned}$$

and hence

$$\left\| A\widehat{k}_\lambda - \mu\widehat{k}_\lambda \right\|^2 \leq r^2,$$

or equivalently,

$$\left\| A\widehat{k}_\lambda \right\|^2 + |\mu|^2 - 2 \text{Re} \left[ \overline{\mu} \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right] \leq r^2.$$

Hence,

$$\left\| A\widehat{k}_\lambda \right\|^2 + |\mu|^2 \leq 2|\mu| \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + r^2. \tag{3.3}$$

Taking the supremum over  $\lambda \in \Omega$  in (3.3), we have the following inequality:

$$\text{ber}(|A|^2) + |\mu|^2 \leq 2|\mu| \text{ber}(A) + r^2. \tag{3.4}$$

By arithmetic-geometric mean inequality,

$$\text{ber}(|A|^2) + |\mu|^2 \geq 2|\mu| \sqrt{\text{ber}(|A|^2)}, \tag{3.5}$$

and hence by (3.4) and (3.5) we deduce the desired inequality (3.2), because it is elementary to see that actually

$\sqrt{\text{ber}(|A|^2)} - \text{ber}(A) \geq 0$ . Indeed,

$$\begin{aligned} |A(\lambda)| &\leq \|A\widehat{k}_\lambda\| = \langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{1/2} = \left(\widetilde{|A|^2}(\lambda)\right)^{\frac{1}{2}} \\ &\leq \text{ber}(|A|^2)^{1/2} \end{aligned}$$

for all  $\lambda \in \Omega$ , and hence  $\text{ber}(A) \leq \sqrt{\text{ber}(|A|^2)}$ . The theorem is proved.  $\square$

**Corollary 1** Let  $A \in \mathcal{B}(\mathcal{H})$  be an operator and  $\varphi, \psi \in \mathbb{C}$  with  $\psi \neq -\varphi$ ,  $\varphi$ . If

$$\text{Re}((A - \varphi)^*(\psi - A)) \geq 0, \tag{3.6}$$

then

$$\sqrt{\text{ber}(|A|^2)} - \text{ber}(A) \leq \frac{1}{4} \frac{|\psi - \varphi|}{|\psi + \varphi|}. \tag{3.7}$$

**Proof** Utilizing the fact that in any Hilbert space the following two statements are equivalent,

- (i)  $\text{Re} \langle y - x, x - z \rangle \geq 0$ ,  $x, z, y \in H$ ;
- (ii)  $\left\| x - \frac{z + y}{2} \right\| \leq \frac{1}{2} \|y - z\|$ ,

we conclude that (3.6) is equivalent to

$$\left\| A\widehat{k}_\lambda - \frac{\psi + \varphi}{2} \widehat{k}_\lambda \right\| \leq \frac{1}{2} \|\psi - \varphi\|$$

for any  $\lambda \in \Omega$ , which in its turn is equivalent with the following inequality:

$$\text{ber} \left( \left| A - \frac{\psi + \varphi}{2} \right|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2} |\psi - \varphi|.$$

Now, applying Theorem 2 for  $\mu = \frac{\psi + \varphi}{2}$  and  $r = \frac{1}{2} |\psi - \varphi|$ , we deduce the desired result (3.7).  $\square$

**Corollary 2** Assume that  $A, \mu, r$  are as in Theorem 2. If, in addition, there exists  $\rho \geq 0$  such that

$$\left| |\mu| - \text{ber}(A) \right| \geq \rho, \tag{3.8}$$

then

$$\text{ber}(|A|^2) - \text{ber}(A)^2 \leq r^2 - \rho^2. \tag{3.9}$$

**Proof** From (3.4) of Theorem 2, we have

$$\begin{aligned} \text{ber}(|A|^2) - \text{ber}(A)^2 &\leq r^2 - \text{ber}(A)^2 + 2\text{ber}(A)|\mu| - |\mu|^2 \\ &= r^2 - (|\mu| - \text{ber}(A))^2. \end{aligned}$$

On utilizing (3.4) and (3.8), we deduce inequality (3.9), as desired.  $\square$

**Remark 1** In particular, if  $\sqrt{\text{ber}(|A - \mu|^2)} \leq r$  and  $|\mu| = \text{ber}(A)$ ,  $\mu \in \mathbb{C}$ , then  $\text{ber}(|A|^2) - \text{ber}(A)^2 \leq r^2$ .

**Theorem 3** Let  $A \in \mathcal{B}(\mathcal{H})$  be a nonzero operator and  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $r > 0$  with  $|\mu| > r$ . If  $\sqrt{\text{ber}(|A - \mu|^2)} \leq r$ , then

$$\sqrt{1 - \frac{r^2}{|\mu|^2}} \leq \frac{\text{ber}(A)}{\sqrt{\text{ber}(|A|^2)}}. \tag{3.10}$$

**Proof** From (3.4), we have

$$\text{ber}(|A|^2) + |\mu|^2 - r^2 \leq 2|\mu| \text{ber}(A),$$

which implies, on dividing with  $\sqrt{|\mu|^2 - r^2} > 0$ , that

$$\frac{\text{ber}(|A|^2)}{\sqrt{|\mu|^2 - r^2}} + \sqrt{|\mu|^2 - r^2} \leq \frac{2|\mu| \text{ber}(A)}{\sqrt{|\mu|^2 - r^2}}. \tag{3.11}$$

By the arithmetic-geometric mean inequality,

$$2\sqrt{\text{ber}(|A|^2)} \leq \frac{\text{ber}(|A|^2)}{\sqrt{|\mu|^2 - r^2}} + \sqrt{|\mu|^2 - r^2},$$

and by (3.11) we get

$$\sqrt{\text{ber}(|A|^2)} \leq \frac{\text{ber}(A)|\mu|}{\sqrt{|\mu|^2 - r^2}}.$$

This is equivalent to (3.10), which proves the theorem. □

**Corollary 3** Let  $\varphi, \psi \in \mathbb{C}$  with  $\text{Re}(\psi\bar{\varphi}) > 0$ . If  $A \in \mathcal{B}(\mathcal{H})$  is an operator such that either (3.6) or

$$(A^* - \bar{\varphi})(\psi - A) \geq 0 \tag{3.12}$$

holds true, then

$$\frac{2\sqrt{\text{Re}(\psi\bar{\varphi})}}{|\psi + \varphi|} \leq \frac{\text{ber}(A)}{\sqrt{\text{ber}(|A|^2)}}$$

and

$$\text{ber}(|A|^2) - \text{ber}(A)^2 \leq \left| \frac{\psi - \varphi}{\psi + \varphi} \right| \text{ber}(|A|^2).$$

**Proof** If we put  $\mu = \frac{\psi + \varphi}{2}$  and  $r = \frac{1}{2} |\psi - \varphi|$ , then  $|\mu|^2 - r^2 = \left| \frac{\psi + \varphi}{2} \right|^2 - \left| \frac{\psi - \varphi}{2} \right|^2 = \operatorname{Re}(\psi\bar{\varphi}) > 0$ . It is easy by applying Theorem 2 to see that under condition (3.12) we have

$$\sqrt{\operatorname{ber}(|A|^2)} - \operatorname{ber}(A) \leq \frac{1}{4} \frac{|\psi - \varphi|}{|\psi + \varphi|}.$$

By considering all these and applying Theorem 3, we obtain the desired results. □

The next result maybe of interest as well.

**Theorem 4** Let  $A : \mathcal{H}(\Omega) \rightarrow \mathcal{H}(\Omega)$  be a nonzero bounded linear operator and  $\mu \in \mathbb{C} \setminus \{0\}$ ,  $r > 0$  with  $|\mu| > r$ . If

$$\sqrt{\operatorname{ber}(|A - \mu|^2)} \leq r,$$

then

$$\operatorname{ber}(|A|^2) - \operatorname{ber}(A)^2 \leq \frac{2r^2}{|\mu| + \sqrt{|\mu|^2 - r^2}} \operatorname{ber}(A). \tag{3.13}$$

**Proof** From the proof of Theorem 2, we have

$$\|A\widehat{k}_\lambda\|^2 + |\mu|^2 \leq 2 \operatorname{Re} [\bar{\mu} \langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle] + r^2 \tag{3.14}$$

for all  $\lambda \in \Omega$ .

Now, after dividing (3.14) by  $|\mu| |\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|$  (which, by (3.14), is positive), we obtain

$$\frac{\|A\widehat{k}_\lambda\|^2}{|\mu| |\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|} \leq \frac{2 \operatorname{Re} [\bar{\mu} \langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle]}{|\mu| |\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|} + \frac{r^2}{|\mu| |\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|} - \frac{|\mu|}{|\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle|} \tag{3.15}$$

for all  $\lambda \in \Omega$ . Hence,

$$\frac{|\widetilde{A}|^2(\lambda)}{|\mu| |\widetilde{A}(\lambda)|} \leq \frac{2 \operatorname{Re} [\bar{\mu} \widetilde{A}(\lambda)]}{|\mu| |\widetilde{A}(\lambda)|} + \frac{r^2}{|\mu| |\widetilde{A}(\lambda)|} - \frac{|\mu|}{|\widetilde{A}(\lambda)|} \tag{3.16}$$



for all  $\lambda \in \Omega$ . If we subtract in (3.16) the same quantity  $\frac{|\tilde{A}(\lambda)|}{|\mu|}$  from both sides, then we have

$$\begin{aligned} \frac{|\widetilde{A}|^2(\lambda)}{|\mu| |\tilde{A}(\lambda)|} - \frac{|\tilde{A}(\lambda)|}{|\mu|} &\leq \frac{2 \operatorname{Re} [\bar{\mu} \tilde{A}(\lambda)]}{|\mu| |\tilde{A}(\lambda)|} + \frac{r^2}{|\mu| |\tilde{A}(\lambda)|} - \frac{|\tilde{A}(\lambda)|}{|\mu|} - \frac{|\mu|}{|\tilde{A}(\lambda)|} \\ &= \frac{2 \operatorname{Re} [\bar{\mu} \tilde{A}(\lambda)]}{|\mu| |\tilde{A}(\lambda)|} - \frac{|\mu|^2 - r^2}{|\mu| |\tilde{A}(\lambda)|} - \frac{|\tilde{A}(\lambda)|}{|\mu|} \\ &= \frac{2 \operatorname{Re} [\bar{\mu} \tilde{A}(\lambda)]}{|\mu| |\tilde{A}(\lambda)|} - \left( \frac{\sqrt{|\mu|^2 - r^2}}{\sqrt{|\mu|} |\tilde{A}(\lambda)|} - \frac{\sqrt{|\tilde{A}(\lambda)|}}{\sqrt{|\mu|}} \right)^2 \frac{|\tilde{A}(\lambda)|}{|\mu|} - 2 \frac{\sqrt{|\mu|^2 - r^2}}{|\mu|}. \end{aligned} \tag{3.17}$$

Since  $\operatorname{Re} [\bar{\mu} \tilde{A}(\lambda)] \leq |\mu| |\tilde{A}(\lambda)|$  and

$$\left( \frac{\sqrt{|\mu|^2 - r^2}}{\sqrt{|\mu|} |\tilde{A}(\lambda)|} - \frac{\sqrt{|\tilde{A}(\lambda)|}}{\sqrt{|\mu|}} \right)^2 \geq 0,$$

by (3.17) we obtain

$$\frac{|\widetilde{A}|^2(\lambda)}{|\mu| |\tilde{A}(\lambda)|} - \frac{|\tilde{A}(\lambda)|}{|\mu|} \leq \frac{2 \left( |\mu| - \sqrt{|\mu|^2 - r^2} \right)}{|\mu|},$$

which implies the inequality

$$\begin{aligned} |\widetilde{A}|^2(\lambda) &\leq |\tilde{A}(\lambda)|^2 + 2 |\tilde{A}(\lambda)| \left( |\mu| - \sqrt{|\mu|^2 - r^2} \right) \\ &\leq \operatorname{ber}(A)^2 + 2 \operatorname{ber}(A) \left( |\mu| - \sqrt{|\mu|^2 - r^2} \right) \end{aligned}$$

for all  $\lambda \in \Omega$ , which implies that

$$\begin{aligned} \operatorname{ber}(|A|^2) - \operatorname{ber}(A)^2 &\leq 2 \operatorname{ber}(A) \frac{\left( |\mu| - \sqrt{|\mu|^2 - r^2} \right) \left( |\mu| + \sqrt{|\mu|^2 - r^2} \right)}{|\mu| + \sqrt{|\mu|^2 - r^2}} \\ &= \frac{2r^2 \operatorname{ber}(A)}{|\mu| + \sqrt{|\mu|^2 - r^2}}, \end{aligned}$$

as desired. The proof is complete. □

**Corollary 4** Let  $\varphi, \psi \in \mathbb{C}$  with  $\operatorname{Re}(\psi\bar{\varphi}) > 0$ . If  $A \in \mathcal{B}(\mathcal{H})$  is an operator such that either (3.6) or (3.12) holds true, then

$$\operatorname{ber}(|A|^2) - \operatorname{ber}(A)^2 \leq \left[|\psi + \varphi| - 2\sqrt{\operatorname{Re}(\psi\bar{\varphi})}\right] \operatorname{ber}(A).$$

**Remark 2** If  $M \geq m > 0$  are such that either  $\operatorname{Re}((A^* - m)(M - A)) \geq 0$  for all  $\lambda \in \Omega$ , or simply  $(A^* - m)(M - A)$  is self-adjoint and  $(A^* - m)(M - A) \geq 0$ , then it follows from the first claim of Corollary 3 that

$$\frac{\sqrt{\operatorname{ber}(|A|^2)}}{\operatorname{ber}(A)} \leq \frac{M + m}{2\sqrt{mM}},$$

which is equivalent to

$$\sqrt{\operatorname{ber}(|A|^2)} - \operatorname{ber}(A) \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} \operatorname{ber}(A),$$

while we have from (3.13) that

$$\operatorname{ber}(|A|^2) - \operatorname{ber}(A)^2 \leq (\sqrt{M} - \sqrt{m})^2 \operatorname{ber}(A).$$

Also, (3.7) becomes

$$\sqrt{\operatorname{ber}(|A|^2)} - \operatorname{ber}(A) \leq \frac{1}{4} \frac{(M - m)^2}{M + m}.$$

Our next result is based on the following refinement of Schwartz’s inequality obtained by Dragomir [11, Theorem 2] :

$$\|a\| \|b\| \geq |\langle a, b \rangle - \langle a, e \rangle \langle e, b \rangle| + |\langle a, e \rangle \langle e, b \rangle| \geq |\langle a, b \rangle|, \tag{3.18}$$

provided  $a, b, e \in H$  and  $\|e\| = 1$ . Since

$$|\langle a, b \rangle - \langle a, e \rangle \langle e, b \rangle| \geq |\langle a, e \rangle \langle e, b \rangle| - |\langle a, b \rangle|,$$

by the first inequality in (3.18) we deduce that

$$\frac{1}{2} (\|a\| \|b\| + |\langle a, b \rangle|) \geq |\langle a, e \rangle \langle e, b \rangle|. \tag{3.19}$$

Inequality (3.19) was proved by a different method earlier by Buzano [7]. Now we are ready to state our result.

**Theorem 5** Let  $A \in \mathcal{B}(\mathcal{H}(\Omega))$ . Then

$$\operatorname{ber}(A)^2 \leq \frac{1}{2} \left[ \operatorname{ber}(A^2) + \sqrt{\operatorname{ber}(|A|^2) \operatorname{ber}(|A^*|^2)} \right].$$

**Proof** Let us choose in (3.19)  $e = \widehat{k}_\lambda$ ,  $a = A\widehat{k}_\lambda$ , and  $b = A^*\widehat{k}_\lambda$  to get

$$\frac{1}{2} \left( \|A\widehat{k}_\lambda\| \|A^*\widehat{k}_\lambda\| + \left| \langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right| \right) \geq \left| \langle A\widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^2$$

for all  $\lambda \in \Omega$ . From this we have

$$\frac{1}{2} \left[ \sqrt{\text{ber}(|A|^2)} \sqrt{\text{ber}(|A^*|^2) + \text{ber}(A^2)} \right] \geq |\widetilde{A}(\lambda)|^2$$

or all  $\lambda \in \Omega$ , which obviously implies the desired result. □

#### 4. Other Berezin number inequalities for product of operators

The main goal of this section is to find upper bounds for the Berezin number of  $A^\alpha X B^\alpha$  and  $A^\alpha X B^{1-\alpha}$  for the case when  $0 \leq \alpha \leq 1$ .

The following lemma is a consequence of the classical Jensen and Young inequalities [11]. By using this lemma, we prove the next results.

**Lemma 3** For  $a, b \geq 0$ ,  $0 \leq \alpha \leq 1$ , and  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , we have:

(a)  $a^\alpha b^{1-\alpha} \leq \alpha a + (1 - \alpha) b \leq [\alpha a^r + (1 - \alpha) b^r]^{\frac{1}{r}}$  for  $r \geq 1$ ;

(b)  $ab \leq \frac{a^p}{p} + \frac{b^q}{q} \leq \left( \frac{a^{pr}}{p} + \frac{b^{qr}}{q} \right)^{\frac{1}{r}}$  for  $r \geq 1$ .

**Theorem 6** Let  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$  and  $pr, qr \geq 2$ . Let  $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$  be positive operators. Then

$$\text{ber}^r(A^\alpha X B^\alpha) \leq \|X\|^r \left( \frac{1}{p} \text{ber}(A^{pr}) + \frac{1}{q} \text{ber}(B^{qr}) \right)^\alpha$$

for all  $0 \leq \alpha \leq 1$ .

**Proof** By using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left| \langle A^\alpha X B^\alpha \widehat{k}_\lambda, \widehat{k}_\lambda \rangle \right|^r &= \left| \langle X B^\alpha \widehat{k}_\lambda, A^\alpha \widehat{k}_\lambda \rangle \right|^r \\ &\leq \|X B^\alpha \widehat{k}_\lambda\|^r \|A^\alpha \widehat{k}_\lambda\|^r \\ &\leq \|X\|^r \langle A^{2\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{r/2} \langle B^{2\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \rangle^{r/2}, \end{aligned}$$

and so

$$\left| \widetilde{A^\alpha X B^\alpha}(\lambda) \right|^r \leq \|X\|^r \left[ \widetilde{A^{2\alpha}}(\lambda) \right]^{r/2} \left[ \widetilde{B^{2\alpha}}(\lambda) \right]^{r/2}$$

for all  $\lambda \in \Omega$ . From the McCarthy inequality and Lemma 3, we obtain

$$\begin{aligned} & \|X\|^r \left[ \widetilde{A^{2\alpha}}(\lambda) \right]^{r/2} \left[ \widetilde{B^{2\alpha}}(\lambda) \right]^{r/2} \\ & \leq \|X\|^r \left( \frac{1}{p} \left( \widetilde{A^{2\alpha}}(\lambda) \right)^{pr/2} + \frac{1}{q} \left[ \widetilde{B^{2\alpha}}(\lambda) \right]^{qr/2} \right) \\ & \leq \|X\|^r \left( \frac{1}{p} \left( \widetilde{A^{pr}}(\lambda) \right)^\alpha + \frac{1}{q} \left( \widetilde{B^{qr}}(\lambda) \right)^\alpha \right) \end{aligned} \tag{4.1}$$

for all  $\lambda \in \Omega$ . From the concavity of  $t^\alpha$ , we have

$$\|X\|^r \left( \frac{1}{p} \left( \widetilde{A^{pr}}(\lambda) \right)^\alpha + \frac{1}{q} \left( \widetilde{B^{qr}}(\lambda) \right)^\alpha \right) \leq \|X\|^r \left( \frac{1}{p} \widetilde{A^{pr}}(\lambda) + \frac{1}{q} \widetilde{B^{qr}}(\lambda) \right)^\alpha \tag{4.2}$$

for all  $\lambda \in \Omega$ .

Combining (4.1) and (4.2), we get

$$\left| \widetilde{A^\alpha X B^\alpha}(\lambda) \right|^r \leq \|X\|^r \left( \frac{1}{p} \widetilde{A^{pr}}(\lambda) + \frac{1}{q} \widetilde{B^{qr}}(\lambda) \right)^\alpha$$

for all  $\lambda \in \Omega$ . Taking the supremum in the last inequality, we get

$$\text{ber}^r (A^\alpha X B^\alpha) \leq \|X\|^r \left( \frac{1}{p} \text{ber} (A^{pr}) + \frac{1}{q} \text{ber} (B^{qr}) \right)^\alpha$$

for all positive operators  $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ . This proves the theorem. □

**Theorem 7** *Let  $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$  be positive operators. Then*

$$\text{ber}^r (A^\alpha X B^{1-\alpha}) \leq \|X\|^r \text{ber} (\alpha A^r + (1 - \alpha) B^r)$$

for all  $r \geq 2$  and  $0 \leq \alpha \leq 1$ .

**Proof** By using the Cauchy–Schwarz inequality, as in the proof of Theorem 8, we have

$$\begin{aligned} \left| \left\langle A^\alpha X B^{1-\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r &= \left| \left\langle X B^{1-\alpha} \widehat{k}_\lambda, A^\alpha \widehat{k}_\lambda \right\rangle \right|^r \\ &\leq \|X\|^r \left\| B^{1-\alpha} \widehat{k}_\lambda \right\|^r \left\| A^\alpha \widehat{k}_\lambda \right\|^r \\ &= \|X\|^r \left\langle B^{2(1-\alpha)} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{r/2} \left\langle A^{2\alpha} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{r/2} \end{aligned}$$

and therefore

$$\left| \widetilde{A^\alpha X B^{1-\alpha}}(\lambda) \right|^r \leq \|X\|^r \left( \widetilde{B^{2(1-\alpha)}}(\lambda) \right)^{r/2} \left( \widetilde{A^{2\alpha}}(\lambda) \right)^{r/2}$$

for all  $\lambda \in \Omega$ . Then we get from the McCarthy inequality and Lemma 3 that

$$\begin{aligned} \left| \widetilde{A^\alpha X B^{1-\alpha}}(\lambda) \right|^r &\leq \|X\|^r \left( \widetilde{A^r}(\lambda) \right)^\alpha \left( \widetilde{B^r}(\lambda) \right)^{1-\alpha} \\ &\leq \|X\|^r \left( \alpha \widetilde{A^r} + (1 - \alpha) \widetilde{B^r}(\lambda) \right) \end{aligned}$$

for all  $\lambda \in \Omega$ . Taking the supremum in the last inequality, we obtain

$$\text{ber}^r (A^\alpha X B^{1-\alpha}) \leq \|X\|^r \text{ber} (\alpha A^r + (1 - \alpha) B^r)$$

for all positive operators  $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ . This proves the theorem.  $\square$

### Acknowledgment

The authors thank the referee for his or her useful remarks. The second author thanks the Deanship of Scientific Research, College of Science Research Center, King Saud University for supporting this work.

### References

- [1] Abu-Omar A, Kittaneh F. Notes on some spectral radius and numerical radius inequalities. *Studia Mathematica* 2015; 272 (2): 97-109.
- [2] Aronzajn N. Theory of reproducing kernels. *Transactions of the American Mathematical Society* 1950; 68: 337-404.
- [3] Bakherad M. Some Berezin number inequalities for operator matrices. *Czechoslovak Mathematical Journal* 2018; 68 (4): 997-1009. doi: 10.21136/CMJ.2018.0048-17
- [4] Bakherad M, Shebrawi K. Upper bounds for numerical radius inequalities involving off-diagonal operator matrices. *Annals of Functional Analysis* 2018; 9 (3): 297-309.
- [5] Berezin FA. Covariant and contravariant symbols for operators. *Mathematics of the USSR-Izvestiya* 1972; 6: 1117-1151.
- [6] Berger CA, Coburn LA. Toeplitz operators on the Segal-Bargmann space. *Transactions of the American Mathematical Society* 1987; 301: 813-829.
- [7] Buzano ML. Generalizzazione della disuguaglianza di Cauchy-Schwarz. *Rendiconti del Seminario Matematico Università e Politecnico di Torino* 1971; 31: 405-409 (in Italian).
- [8] Chalendar I, Fricain E, Gürdal M, Karaev MT. Compactness and Berezin symbols. *Acta Scientiarum Mathematicarum* 2012; 78: 315-329.
- [9] Coburn LA. A Lipschitz estimate for Berezin's operator calculus. *Proceedings of the American Mathematical Society* 2004; 133: 127-131.
- [10] Das N, Sahoo M. Berezin transform of the absolute value of an operator. *Annals of Functional Analysis* 2018; 9 (2): 151-165.
- [11] Dragomir SS. Some refinements of Schwarz inequality. *Polytechnical Institute of Timișoara, Romania* 1985: 13-16.
- [12] Dragomir SS. Reverse inequalities for the numerical radius of linear operators in Hilbert spaces. *Bulletin of the Australian Mathematical Society* 2006; 73 (2): 255-262.
- [13] Dragomir SS. A survey of some recent inequalities for the norm and numerical radius of operators in Hilbert spaces. *Banach Journal of Mathematical Analysis* 2007; 1 (2): 154-175.
- [14] El-Haddad M, Kittaneh F. Numerical radius inequalities for Hilbert space operators. II. *Studia Mathematica* 2007; 182: 133-140.
- [15] Enliş M. Functions invariant under the Berezin transform. *Journal of Functional Analysis* 1994; 121: 233-254.
- [16] Enliş M. Toeplitz operators and the Berezin transform on  $H^2$ . *Linear Algebra and its Applications* 1995; 223/224: 171-204.
- [17] Garayev MT, Guediri H, Sadraoui H. Applications of reproducing kernels and Berezin symbols. *New York Journal of Mathematics* 2016; 22: 583-604.

- [18] Garayev MT, Gürdal M, Okudan A. Hardy-Hilbert's inequality and a power inequality for Berezin numbers for operators. *Mathematical Inequalities & Applications* 2016; 19 (3): 883-891.
- [19] Gustafson KE, Rao DKM. *Numerical Range*. New York, USA: Springer Verlag, 1997.
- [20] Hajmohamadi M, Lashkaripour R, Bakherad M. Improvements of Berezin number inequalities. *Linear and Multilinear Algebra* (in press). doi: 10.1080/03081087.2018.1538310
- [21] Hajmohamadi M, Lashkaripour R, Bakherad M. Some generalizations of numerical radius on off-diagonal part of  $2 \times 2$  operator matrices. *Journal of Mathematical Inequalities* 2018; 12(2): 447-457.
- [22] Karaev MT. Berezin symbol and invertibility of operators on the functional Hilbert spaces. *Journal of Functional Analysis* 2006; 238: 181-192.
- [23] Karaev MT. Reproducing kernels and Berezin symbols techniques in various questions of operator theory. *Complex Analysis and Operator Theory* 2013; 7: 983-1018.
- [24] Kittaneh F. Numerical radius inequalities for Hilbert space operators. *Studia Mathematica* 2005; 168: 73-80.
- [25] Kittaneh F, Moslehian MS, Yamazaki T. Cartesian decomposition and numerical radius inequalities. *Linear Algebra and Its Applications* 2015; 471: 46-53.
- [26] Li Y, Wang M, Lan W. Compactness of a class of radial operators on weighted Bergman spaces. *Advances in Operator Theory* 2018; 3 (2): 400-410.
- [27] Nordgren E, Rosenthal P., Boundary values of Berezin symbols. In: Feintuch A, Gohberg I (editors). *Nonselfadjoint Operators and Related Topics*. Operator Theory: Advances and Applications. Basel, Switzerland: Birkhäuser, 1994.
- [28] Saitoh S, Sawano Y. *Theory of Reproducing Kernels and Applications*. Singapore: Springer, 2016.
- [29] Yamancı U, Gürdal M. On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space. *New York Journal of Mathematics* 2017; 23: 1531-1537.
- [30] Yamancı U, Gürdal M, Garayev MT. Berezin number inequality for convex function in reproducing kernel Hilbert space. *Filomat* 2017; 31 (18): 5711-5717.
- [31] Zhao X, Zheng D. Invertibility of Toeplitz operators via Berezin transforms. *Journal of Operator Theory* 2016; 75: 475-495.