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On hypercyclic fully zero-simple semihypergroups

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Abstract: Let $\mathfrak{I}$ be the class of fully zero-simple semihypergroups generated by a hyperproduct. In this paper we study some properties of residual semihypergroup $(H_+, \star)$ of a semihypergroup $(H, \circ) \in \mathfrak{I}$. Moreover, we find sufficient conditions for $(H, \circ)$ and $(H_+, \star)$ to be cyclic.

Key words: Semihypergroups, simple semihypergroups, fully semihypergroups

1. Introduction

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Many authors have been working on this field and in [4] numerous applications are presented for algebraic hyperstructures, such as geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence, and probabilities. The semihypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. Some scholars have studied different aspects of semihypergroups [2, 5, 8, 9, 19, 20, 22–24] and interesting problems arise in the study of their so-called fundamental relations [1, 7, 16, 21, 25], which leads to analyzing the conditions for their transitivity, and minimal cardinality problems. In [16] the authors found all simple and zero-simple semihypergroups of size 3, such that the fundamental relation β is not transitive, apart from isomorphisms. This semihypergroups of size 3 were used in [8–12] to characterize the fully simple semihypergroups and the fully zero-simple semihypergroups having all hyperproducts of size ≤ 2. In particular, in [11] the authors proved that if $(H, \circ)$ is a hypercyclic simple semihypergroup, generated by a hyperproduct of elements in H, then the relation β is transitive. Consequently, we have that in every fully simple semihypergroup the size of every hyperproduct is ≤ 2. This is not true for the fully zero-simple semihypergroups, as many examples show in this paper.

The plan of this paper is as follows: after introducing some basic definitions and notations to be used throughout the paper, in Section 2, we prove that if $(H, \circ)$ is a hypercyclic fully zero-simple semihypergroup generated by hyperproduct P and $(H_+, \star)$ is the residual semihypergroup of $(H, \circ)$ then the relation β_{H_+} is transitive. Moreover, if $(H, \circ)$ is generated by hyperproduct P then $(P \cap P^2) - \{0\} = \emptyset$. In Section 3, we introduce the definition of rank for a hyperproduct P, which is the smallest positive integer k such that

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$P \cap P^{k+1} - \{0\} \neq \emptyset$. By means of this notion, we characterize the subsemihypergroup $\widehat{P}$ generated by a special hyperproduct P, called strong, and in Section 4 we analyze properties of the fully zero-simple semihypergroups generated by a strong hyperproduct. In particular, we prove that if $(H, \circ)$ is a fully zero-simple semihypergroup generated by a strong hyperproduct P of rank a prime number r then $(H, \circ)$ is cyclic. In this case, rank can be seen as a generalization of the concept of period in group theory. It is known that if G is a cyclic group of size a prime number r then every element different from identity is a generator of G. The same property is true for semihypergroups in Theorem 30, but the commutative property of cyclic groups does not generally hold; see the example in Remark 31.

1.1. Basic definitions and results

Let H be a nonempty set and $P^*(H)$ be the set of all nonempty subsets of H. A hyperoperation $\circ$ on H is a map from $H \times H$ to $P^*(H)$. For all $x, y \in H$, the subset $x \circ y$ is called the hyperproduct of x and y. If A, B are nonempty subsets of H then $A \circ B = \bigcup_{x \in A, y \in B} x \circ y$.

A *semihypergroup* is a nonempty set H endowed with an associative hyperproduct $\circ$; that is, $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in H$.

A nonempty subset K of a semihypergroup $(H, \circ)$ is called a subsemihypergroup of $(H, \circ)$ if it is closed with respect to multiplication; that is, $x \circ y \subseteq K$ for all $x, y \in K$. If $(H, \circ)$ is a semihypergroup, then the intersection $\bigcap_{i \in I} S_i$ of a family $\{S_i\}_{i \in I}$ of subsemihypergroups of $(H, \circ)$, if it is nonempty, is again a subsemihypergroup of $(H, \circ)$. For every nonempty subset $A \subseteq H$ there is at least one subsemihypergroup of $(H, \circ)$ containing A, e.g., H itself. Hence, the intersection of all subsemihypergroups of $(H, \circ)$ containing A is a subsemihypergroup. We denote it by $\widehat{A}$, and we note that it is defined by two properties:

1. $A \subseteq \widehat{A}$;
2. if S is a subsemihypergroup of H and $A \subseteq S$, then $\widehat{A} \subseteq S$.

Furthermore, $\widehat{A}$ is characterized as the algebraic closure of A under the hyperproduct in $(H, \circ)$; namely, we have $\widehat{A} = \bigcup_{n \geq 1} A^n$. Moreover, if H is finite, the set $\left\{ r \in \mathbb{N} - \{0\} \mid \bigcup_{k=1}^{r} A^k = \bigcup_{k=1}^{r+1} A^k \right\}$ has minimum $m \leq |H|$ and then it is known that

$$\widehat{A} = \bigcup_{k=1}^{m} A^k = \bigcup_{k=1}^{m+1} A^k = \dots = \bigcup_{k=1}^{|H|} A^k. \tag{1}$$

If $x \in H$, we suppose $\circ x^1 = \{x\}$ and $\circ x^n = \underbrace{x \circ \ldots \circ x}_{\text{n times}}$ for every integer $n > 1$. We refer to $\widehat{x} = \bigcup_{n \geq 1} \circ x^n$ as the *cyclic subsemihypergroup of* $(H, \circ)$ *generated by the element* x. It is the smallest subsemihypergroup containing x.

If K is a subsemihypergroup of $(H, \circ)$, it is said to be *hypercyclic* if there exists a hyperproduct P of elements in K such that $K = \widehat{P}$.

If $(H, \circ)$ is a semihypergroup, an element $0 \in H$ such that $x \circ 0 = \{0\}$ (resp., $0 \circ x = \{0\}$) for all $x \in H$ is called a *right zero scalar element* or *right absorbing element* (resp., *left zero scalar element* or *left absorbing* *element*) of $(H, \circ)$. If 0 is both a right and left zero scalar element, then 0 is called a *zero scalar element* or *absorbing element*.

A semihypergroup $(H, \circ)$ is called *simple* if $H \circ x \circ H = H$, for all $x \in H$.

A semihypergroup $(H, \circ)$ with an absorbing element 0 is called *zero-simple* if $H \circ x \circ H = H$, for all $x \in H - \{0\}$.

Given a semihypergroup $(H, \circ)$, the relation β^* of H is the transitive closure of the relation $\beta = \cup_{n \geq 1} \beta_n$, where β_1 is the diagonal relation in H and, for every integer $n > 1$, β_n is defined recursively as follows:

$$x \beta_n y \Longleftrightarrow \exists (z_1, \ldots, z_n) \in H^n \ : \ \{x, y\} \subseteq z_1 \circ z_2 \circ \ldots \circ z_n.$$

The relations β, β^* are called *fundamental relations* on H [25]. Their relevance in semihypergroup theory stems from the following facts [21]: the quotient set H/β^*, equipped with the operation $\beta^*(x) \otimes \beta^*(y) = \beta^*(z)$ for all $x, y \in H$ and $z \in x \circ y$, is a semigroup. Moreover, the relation β^* is the smallest strongly regular equivalence on H such that the quotient H/β^* is a semigroup.

The interested reader can find all relevant definitions, many properties, and applications of fundamental relations, even in more abstract contexts, in [3, 4, 6, 14, 15, 17, 18, 21, 25].

A semihypergroup $(H, \circ)$ is said to be *fully zero-simple* if it fulfills the following conditions:

1. All subsemihypergroups of $(H, \circ)$ ($(H, \circ)$ itself included) are zero-simple.

2. The relation β in $(H, \circ)$ and the relation β_K in all subsemihypergroups $K \subset H$ of size ≥ 3 are not transitive.

Since in all semihypergroups of size ≤ 2 the relation β is transitive, it follows that every fully zero-simple semihypergroup has size ≥ 3.

We denote by $\mathfrak{F}_0$ the class of fully zero-simple semihypergroups. We use 0 to denote the zero scalar element of each semihypergroup $(H, \circ) \in \mathfrak{F}_0$. Moreover, we use the notation H_+ to indicate the set of nonzero elements in H; that is, $H_+ = H - \{0\}$. Finally, for the reader's convenience, we collect in the following lemma some preliminary results from [9].

Lemma 1 *Let* $(H, \circ) \in \mathfrak{F}_0$. *Then we have:*

1. $H \circ H = H$;

2. *if* S *is a subsemihypergroup of* H *such that* $0 \notin S$, *then* $|S| = 1$, *and moreover, if* $|S| \geq 2$ *then the zero element of* S *is* 0;

3. *there exist* $x, y \in H_+$ *such that* $0 \in x \circ y$;

4. *for every sequence* $z_1, \ldots, z_n$ *of elements in* H_+ *we have* $\prod_{i=1}^{n} z_i \neq \{0\}$;

5. *the set* H_+ *equipped with hyperproduct* $a \star b = (a \circ b) \cap H_+$, *for all* $a, b \in H_+$, *is a semihypergroup.*

By points 2 and 4 of Lemma 1 we deduce the following result:

Corollary 2 *Let S be a subsemihypergroup of $H \in \mathfrak{F}_0$. Then we have:*

1. *if $0 \notin S$ then there exists $a \in H_+$ such that $S = \{a\}$ and $a \circ a = \{a\}$;*

2. *if $|S| = 2$ then there exists $a \in H_+$ such that $S = \{a, 0\}$ and $\{a\} \subseteq a \circ a \subseteq \{0, a\}$.*

From point 5 of Lemma 1, we know that the set of nonzero element H_+ of a fully 0-simple semihypergroup $(H, \circ)$ is a simple semihypergroup equipped with hyperoperation $a \star b = (a \circ b) \cap H_+$, for all $a, b \in H_+$. This semihypergroup is called a *residual semihypergroup* of $(H, \circ)$.

The following results were proved in [13]:

Theorem 3 *Let $(H, \circ) \in \mathfrak{F}_0$. For all $x \in H$, we have $(x, 0) \in \beta$. Moreover, H/β^* is trivial.*

Lemma 4 *Let A, B be two nonempty subsets of $(H, \circ) \in \mathfrak{F}_0$ different from the singleton $\{0\}$. We have:*

1. $(A - \{0\}) \star (B - \{0\}) = A \circ B - \{0\}$.

2. *If $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$ then $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$.*

3. *If $0 \in A$ and $(A - \{0\}, \star)$ is a subsemihypergroup of $(H_+, \star)$ then $(A, \circ)$ is a subsemihypergroup of $(H, \circ)$.*

4. *If $A_+ = A - \{0\}$ and $(\widehat{A}, \circ), (\widehat{A}_+, \star)$ are the subsemihypergroups of $(H, \circ)$ and $(H_+, \star)$ generated from A and A_+ respectively, then $\widehat{A}_+ = \widehat{A} - \{0\}$.*

Proposition 5 *Let $(H_+, \star)$ be the residual semihypergroup of $(H, \circ) \in \mathfrak{F}_0$ and $[0, 0]_H = \{(a, b) \in H \times H \mid a = 0$ or $b = 0\}$. Then we have $\beta_{H_+} = \beta - [0, 0]_H$.*

2. Hypercyclic semihypergroup in $\mathfrak{F}_0$

In [11] the authors introduced the definition of hypercyclic semihypergroups and studied a class of semihypergroups $(H, \circ)$ such that for all hyperproducts P of elements in H the subsemihypergroup $\widehat{P}$ is hypercyclic. In this section we study some properties of the hypercyclic semihypergroups in $\mathfrak{F}_0$. For the reader's convenience we denote by $\mathfrak{I}_0$ the subclass of hypercyclic semihypergroups in $\mathfrak{F}_0$.

Proposition 6 *If $(H, \circ) \in \mathfrak{I}_0$ is generated by hyperproduct P, then $(H_+, \star)$ is hypercyclic generated by $P_+ = P - \{0\}$ and β_{H_+} is transitive.*

Proof If $(H, \circ) \in \mathfrak{F}_0$ is generated from the set P, then $H = \widehat{P}$ and, for Lemma 4, the residual semihypergroup $(H_+, \star)$ is generated from $P_+ = P - \{0\}$. Therefore, $(H_+, \star)$ is a simple hypercyclic semihypergroup. By Theorem 3.1 in [11], the relation β_{H_+} is transitive. □

Corollary 7 *If $(H, \circ) \in \mathfrak{I}_0$ and a, b, c are three elements in H such that $(a, b) \in \beta$, $(b, c) \in \beta$, and $(a, c) \notin \beta$, then $b = 0$.*

Proof By Theorem 3, we have that $a \neq 0$ and $c \neq 0$; otherwise, $(a, c) \in \beta$. If, for absurdity, $b \neq 0$, then $a, b, c \in H_+$ and, for Proposition 5, $(a, b) \in \beta_{H_+}$ and $(b, c) \in \beta_{H_+}$. Now, for Proposition 6, we obtain that $(a, c) \in \beta_{H_+}$, which is impossible because $\beta_{H_+} \subseteq \beta$ and $(a, c) \notin \beta$. Therefore, $b = 0$. □

Theorem 8 *If* $(H,\circ)\in\mathfrak{I}_0$ *then* $\left|H_+/\beta^*_{H_+}\right|\geq 2$.

Proof For absurdity, let $|H_+/\beta^*_{H_+}|=1$. If $a,b\in H$, we can distinguish two cases: 1) $a=0$ or $b=0$; 2) $a\neq 0$ and $b\neq 0$. In the first case, by Theorem 3, we have that $(a,b)\in\beta$. In the second case, for the hypothesis $|H_+/\beta^*_{H_+}|=1$ and Proposition 6, we obtain that $(a,b)\in\beta_{H_+}\subseteq\beta$. Thus, we have that $(a,b)\in\beta$, for all $a,b\in H$. Therefore, we conclude that β is transitive, which is an absurdity. □

By Corollary 2, if $(H,\circ)\in\mathfrak{F}_0$ and $K\subset H$ is a subsemihypergroup of size $|K|<3$, then there exists an element $c\in K$ such that $c\in cc$. Now we will prove that if $|K|\geq 3$ and K is hypercyclic generated by hyperproduct P then $(P\cap P^2)-\{0\}=\emptyset$. We give the following result:

Lemma 9 *Let* $(H,\circ)\in\mathfrak{F}_0$. *If* P *is a hyperproduct of elements in* H *such that* $(P\cap P^2)-\{0\}\neq\emptyset$, *then* $(P^k\cap P^{k+1})-\{0\}\neq\emptyset$ *for every integer* $k\geq 1$.

Proof By hypothesis the thesis is true for $k=1$. Therefore, we suppose it is true for $k\geq 1$ and let $a\in(P^k\cap P^{k+1})-\{0\}\neq\emptyset$. Obviously we have $aP\subseteq P^kP=P^{k+1}$ and $aP\subseteq P^{k+1}P=P^{k+2}$; hence, $aP\subseteq P^{k+1}\cap P^{k+2}$. From Lemma 1(4), we obtain that $aP\neq\{0\}$ since $a\neq 0$ and $P\neq\{0\}$. Thus, we have that $(P^{k+1}\cap P^{k+2})-\{0\}\neq\emptyset$. □

Proposition 10 *If* $(H,\circ)\in\mathfrak{I}_0$ *is generated by hyperproduct* P *then we have* $(P\cap P^2)-\{0\}=\emptyset$.

Proof For absurdity we suppose that $(P\cap P^2)-\{0\}\neq\emptyset$. By Lemma 9 we have $(P^k\cap P^{k+1})-\{0\}\neq\emptyset$ for every integer $k\geq 1$. From Lemma 4 (1), if $P_+=P-\{0\}$ then we obtain

$$\star P_+^k\cap\star P_+^{k+1}=(P^k-\{0\})\cap(P^{k+1}-\{0\})=(P^k\cap P^{k+1})-\{0\}\neq\emptyset.$$

Moreover, by Proposition 6, the semihypergroup $(H_+,\star)$ is hypercyclic generated from P_+ and β_{H_+} is transitive. Now, if $x,y\in H_+$, then there exist two integers $m,n\geq 1$ such that $x\in\star P_+^m$ and $y\in\star P_+^n$. If $m=n$ then $(x,y)\in\beta_{H_+}$. If $m\neq n$ we can suppose that $m<n$ and $(\star P_+^{m+k}\cap\star P^{m+k+1})-\{0\}\neq\emptyset$, for every $k\in\{0,1,\ldots,n-m-1\}$. Therefore, there exist $n-m$ elements $z_0,z_1,\ldots,z_{n-m-1}\in H_+$ such that

$$\{x,z_0\}\subseteq\star P_+^m,\{z_0,z_1\}\subseteq\star P_+^{m+1},\ \ldots\ ,\{z_{n-m-1},y\}\subseteq\star P_+^n.$$

In consequence, $x\beta_{H_+}z_0\beta_{H_+}z_1\beta_{H_+}\ \ldots\ \beta_{H_+}z_{n-m-1}\beta_{H_+}y$ and $(x,y)\in\beta_{H_+}$ since β_{H_+} is transitive. Thus, for every $x,y\in H_+$ we have $(x,y)\in\beta_{H_+}$ and $|H_+/\beta^*_{H_+}|=1$. This fact is impossible by Theorem 8. □

As an immediate consequence of the preceding proposition, we can state the following result:

Corollary 11 *Let* $(H,\circ)\in\mathfrak{F}_0$ *and let* $K\subseteq H$ *be a hypercyclic subsemihypergroup of size* $|K|\geq 3$. *If* P *is a hyperproduct of elements in* $K-\{0\}$ *such that* $K=\widehat{P}$ *then* $(P\cap P^2)-\{0\}=\emptyset$.

3. Strong hyperproduct

Let $(H,\circ)\in\mathfrak{F}_0$ and let $K\subseteq H$ be a subsemihypergroup generated by P with $|K|\geq 3$. Since $(K,\circ)\in\mathfrak{F}_0$, by Lemma 1, we have $K=K\circ K=\bigcup_{n\geq 2}P^n$, and hence there exists an integer $s\geq 2$ such that $(P\cap P^s)-\{0\}\neq\emptyset$. This fact suggests the following definition:

Definition 12 Let $(H, \circ)$ be a semihypergroup and let P be a hyperproduct of elements in H. The smallest positive integer k such that $P \cap P^{k+1} - \{0\} \neq \emptyset$ is called the *rank* of P. If no such k exists, then we say P has rank 0.

Clearly, by Corollary 11, if K is a hypercyclic subsemihypergroup of $(H, \circ) \in \mathfrak{F}_0$, with size $|K| \geq 3$, and P is a hyperproduct of elements in K such that $K = \widehat{P}$, then the rank of P is ≥ 2 .

In this section we will use the notion of rank to determine a sufficient condition for a hypercyclic semihypergroup $(H, \circ) \in \mathfrak{I}_0$ to be cyclic.

Definition 13 Let $(H, \circ) \in \mathfrak{F}_0$. An element $c \in H$ is called *quasi-idempotent* if $c \neq 0$ and $\{c\} \subseteq c \circ c \subseteq \{0, c\}$.

Definition 14 Let $(H, \circ) \in \mathfrak{F}_0$. A hyperproduct P of elements in H is called *strong* if it fulfills the following conditions:

1. P does not contain any quasi-idempotent element of H.
2. The subsemihypergroup $\widehat{P}$ possesses a quasi-idempotent element.
3. If $c \in \widehat{P}$ is a quasi-idempotent element then $P^s - \{0\} = \{c\}$, for all integers s such that $c \in P^s$.

An immediate consequence of the previous definition and point 2, point 4, of Lemma 1 is the following result:

Proposition 15 *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a strong hyperproduct then $0 \in \widehat{P}$ and $|\widehat{P}| \geq 3$.*

Proposition 16 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct. The semihypergroup $\widehat{P}$ has one and only one quasi-idempotent element.*

Proof Since P is a strong hyperproduct, $\widehat{P}$ has a quasi-idempotent element c_1. If there exists another quasi-idempotent element $c_2 \in \widehat{P}$, then there exist two positive integers s_1 and s_2 such that $P^{s_1} - \{0\} = \{c_1\}$ and $P^{s_2} - \{0\} = \{c_2\}$. Obviously we have

$$\{c_1\} = c_1^{s_2} - \{0\} = (P^{s_1})^{s_2} - \{0\} = (P^{s_2})^{s_1} - \{0\} = c_2^{s_1} - \{0\} = \{c_2\}$$

and so $c_1 = c_2$. □

Corollary 17 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of elements in H. If c is the quasi-idempotent element in $\widehat{P}$ and $s \in \mathbb{N} - \{0\}$ then P^s is a strong hyperproduct if and only if $c \notin P^s$.*

The next table shows a fully zero-simple semihypergroup with two quasi-idempotent elements c_1, c_2 and two strong hyperproducts P and Q such that $c_1 \in \widehat{P}$ and $c_2 \in \widehat{Q}$.

Example 18 Let $H = \{0, 1, 2, 3, 4, 5, 6\}$ and let $\circ$ be the hyperproduct defined in the following table:

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	0, 3	0, 3	0, 1, 2	5, 6	0, 4, 6	0, 4, 5, 6
2	0	0, 3	0, 3	0, 1, 2	0, 4, 6	5, 6	0, 4, 5, 6
3	0	0, 1, 2	0, 1, 2	3	0, 4, 5, 6	0, 4, 5, 6	0, 4, 5, 6
4	0	2, 3	0, 1, 3	0, 1, 2, 3	0, 6	0, 6	0, 4, 5
5	0	0, 1, 3	2, 3	0, 1, 2, 3	0, 6	0, 6	0, 4, 5
6	0	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3	0, 4, 5	0, 4, 5	0, 6

We have $(H, \circ) \in \mathfrak{F}_0$. The elements 3 and 6 are quasi-idempotent; the hyperproducts $P = 1 \circ 3$, $Q = 4 \circ 6$ are strong of rank two; and we have $3 \in \widehat{P}$, $6 \in \widehat{Q}$. Moreover, we note that $(H_+, \star)$ is not a commutative simple semihypergroup and $H_+/\beta^*_{H_+}$ is isomorphic to the following semigroup:

	1	2
1	1	2
2	1	2

Proposition 19 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of rank r. Then we have $r \geq 2$.*

Proof Let c be the quasi-idempotent element in $\widehat{P}$ and $s \geq 2$ the minimum integer such that $P^s - \{0\} = \{c\}$. If for absurdity we suppose $P \cap P^2 \neq \emptyset$ then $P^{s-1} \cap P^s - \{0\} \neq \emptyset$ and so $c \in P^{s-1}$. By definition of a strong hyperproduct, we have $P^{s-1} - \{0\} = \{c\}$. That is a contradiction for the minimality of s. □

We are ready to prove the following result:

Proposition 20 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of rank r. Then r is the minimum positive integer such that $P^r - \{0\} = \{c\}$, where c is the quasi-idempotent element in $\widehat{P}$.*

Proof Since $c \in \widehat{P}$ there exists a minimum positive integer s such that $P^s - \{0\} = \{c\}$. Hence, $\{c\} = (P^s)^r - \{0\} = (P^r)^s - \{0\}$ and $c \in \widehat{P^r}$. Clearly, there exists a minimum positive integer t such that $P^{rt} - \{0\} = \{c\}$. Suppose, for absurdity, that $t \geq 2$. By point 4 of Lemma 1, we have $P^{(t-1)r-1} \neq \{0\}$. Moreover, since $(P \cap P^{r+1}) - \{0\} \neq \emptyset$, we obtain

$$\emptyset \neq ((P \cap P^{r+1}) - \{0\}) \circ P^{(t-1)r-1} - \{0\} \subseteq (P^{(t-1)r} \cap P^{tr}) - \{0\} \subseteq P^{(t-1)r} \cap \{c\}.$$

By Definition 14, it follows that $P^{(t-1)r} - \{0\} = \{c\}$, which is a contradiction for the minimality of t. Therefore, $t = 1$ and $P^r - \{0\} = \{c\}$. Now, let s be a positive integer such that $P^s - \{0\} = \{c\}$, and then $\emptyset \neq P \cap P^{r+1} - \{0\} = P \cap cP - \{0\} = P \cap P^{s+1} - \{0\}$ and so $s \geq r$. Therefore, r is the minimum positive integer such that $P^r - \{0\} = \{c\}$. □

Proposition 21 *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a strong hyperproduct of elements in H of rank r then there exists a positive integer $t \leq 2r$ such that $0 \in P^t$.*

Proof Let c be the quasi-idempotent element in $\widehat{P}$. From Definition 13 and Proposition 20, we have $c \in c \circ c \subseteq \{0, c\}$ and $c \in P^r \subseteq \{0, c\}$. Moreover, by Proposition 15, we know that $0 \in \widehat{P}$; hence, we can distinguish two cases: $0 \in P^{2r}$ or $0 \notin P^{2r}$. In the first case we have the thesis. In the second case, we obtain $c \circ c = \{c\} = P^r$. Now there exists an integer $m \geq 1$ such that $0 \in P^m$. If $m > 2r$, by Euclidean division, there exist two nonnegative integers q, n such that $m = qr + n$ with $q \neq 0$ and $0 \leq n < r$. We have $n \neq 0$; otherwise, $0 \in P^m = P^{qr} = (P^r)^q = c^q = \{c\}$. Hence, we deduce $0 \in P^m = P^{qr+n} = (P^r)^q \circ P^n = c^q \circ P^n = c \circ P^n = P^r \circ P^n = P^{n+r}$ and so $0 \in P^{n+r}$ with $n + r < 2r$. □

Corollary 22 *Let $(H, \circ) \in \mathfrak{F}_0$. If P is a strong hyperproduct of elements in H of rank r then $\widehat{P} = \bigcup_{k=1}^{2r} P^k$.*

Proof Let c be the quasi-idempotent element in $\widehat{P}$. For all $m > 2r$ there exist $q, n \in \mathbb{N}$ such that $m = qr + n$, $q \geq 2$, and $0 \leq n < r$. By Proposition 20, if $n = 0$ then $P^m = (P^r)^q \subseteq \{0, c\}^q \subseteq \{0, c\} \subseteq \{0\} \cup P^r$. Otherwise, if $n \neq 0$, then $P^m = (P^r)^q \circ P^n \subseteq \{0, c\}^q \circ P^n \subseteq (\{0\} \cup c^q) \circ P^n = \{0\} \cup (c^q \circ P^n) = \{0\} \cup (\{0, c\} \circ P^n) = \{0\} \cup c \circ P^n = \{0\} \cup P^r \circ P^n = \{0\} \cup P^{n+r}$. Hence, for Proposition 21, we obtain $P^m \subseteq \bigcup_{k=1}^{2r} P^k$ and $\widehat{P} = \bigcup_{k=1}^{2r} P^k$. □

Lemma 23 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of elements in H of rank r. If c is the quasi-idempotent element of $\widehat{P}$ then we have:*

1. $\widehat{P} = (\{c\} \cup c \circ P \cup c \circ P^2 \cup ... \cup c \circ P^{r-1})$;
2. $c \circ P^i - \{0\} = P^i \circ c - \{0\}$, *for every* $i \in \{1, 2, ..., r\}$;
3. $(c \circ P)^i - \{0\} = c \circ P^i - \{0\}$, *for every* $i \in \{1, 2, ..., r\}$;
4. $(c \circ P^i) \circ (c \circ P^j) - \{0\} = c \circ P^{i+j} - \{0\}$, *for every* $i, j \in \{1, 2, ..., r\}$;
5. $(P^i \cap P^j) - \{0\} = \emptyset$, *for every* $i, j \in \{1, 2, ..., r\}$ *and* $i \neq j$;
6. $c \notin c \circ P^i - \{0\}$, *for every* $i \in \{1, 2, ..., r-1\}$;
7. $(c \circ P^i \cap c \circ P^j) - \{0\} = \emptyset$, *for all* $i, j \in \{1, 2, ..., r-1\}$ *and* $i \neq j$;
8. $P^i \subseteq c \circ P^i$, *for every* $i \in \{1, 2, ..., r-1\}$.

Proof For Corollary 22, we can put $\widehat{P} = P \cup P^2 \cup ... \cup P^{2r}$.

1. Since $\widehat{P} = (\widehat{P})^r = P^r \cup P^{r+1} \cup ... \cup P^{2r^2}$, by Proposition 20, it results that:

$$P^r - \{0\} = \{c\}$$
$$P^{r+1} - \{0\} = c \circ P - \{0\}$$
$$\dots\dots\dots$$
$$P^{2r-1} - \{0\} = c \circ P^{r-1} - \{0\}$$
$$P^{2r} - \{0\} = c \circ P^r - \{0\} = \{c\} = P^r - \{0\}.$$

At this point, taking into account Proposition 21, the assertion follows immediately.

2. Since $\{c\} = P^r - \{0\}$, we have
$$c \circ P^i - \{0\} = P^r \circ P^i - \{0\} = P^i \circ P^r - \{0\} = P^i \circ c - \{0\}.$$

3. By item 2, we have $(c \circ P)^i - \{0\} = \underbrace{(c \circ P) \circ (c \circ P) \circ \ldots \circ (c \circ P)}_{\text{i times}} - \{0\} = \underbrace{c \circ c \circ \ldots \circ c}_{\text{i times}} \circ \underbrace{P \circ P \circ \ldots \circ P}_{\text{i times}} - \{0\} =$ $c \circ P^i - \{0\}$.

4. By item 2., $(c \circ P^i) \circ (c \circ P^j) - \{0\} = c \circ c \circ P^i \circ P^j - \{0\} = (c \circ P^{i+j}) - \{0\}$.

5. For absurdity, let $(P^i \cap P^j) - \{0\} \neq \emptyset$ for some $i, j \in \{1, 2, ..., r\}$ with $i \neq j$. Supposing $i < j$, we obtain $(P^{r-j+i} \cap P^{r-j+j}) - \{0\} \neq \emptyset$, and hence $(P^{r-j+i} \cap P^r) - \{0\} \neq \emptyset$. Since $P^r - \{0\} = \{c\}$ we have $c \in P^{r-j+i} - \{0\}$, which is a contradiction because r is the minimum integer such that $\{c\} \in P^r - \{0\}$.

6. Let $i \in \{1, 2, ..., r-1\}$. Since $c \circ P - \{0\} = P^{r+1} - \{0\}$, we have
$$(P \cap P^{r+1}) - \{0\} \neq \emptyset \Rightarrow (P \cap c \circ P) - \{0\} \neq \emptyset \Rightarrow (P^i \cap (c \circ P)^i) - \{0\} \neq \emptyset.$$
Moreover, by item 3, we obtain $P^i \cap c \circ P^i - \{0\} \neq \emptyset$. Now, if $c \in (c \circ P)^i - \{0\}$, then $c \circ P^i - \{0\} = P^r \circ P^i - \{0\} = P^{r+i} - \{0\}$, and hence $c \circ P^i - \{0\} = P^{r+i} - \{0\} = \{c\}$. Consequently, we have $c \in P^i - \{0\}$, which is impossible because $i < r$. Thus, $c \notin c \circ P^i - \{0\}$.

7. For absurdity, we suppose that there exists $i, j \in \{1, 2, ..., r-1\}$, with $i \neq j$, such that $(c \circ P^i \cap c \circ P^j) - \{0\} \neq \emptyset$. Letting $i < j$, by item 3, we obtain $(c \circ P^{r-j+i} \cap c \circ P^{r-j+j}) - \{0\} \neq \emptyset$. Having $c \circ P^r - \{0\} = \{c\}$, it follows that $c \in c \circ P^{r-j+i} - \{0\}$. Since $r - j + i < r$, by item 6, we have a contradiction.

8. Let $i \in \{1, 2, \ldots, r-1\}$ and $a \in P^i$. From item 1, there exists an integer s, with $1 \leq s \leq r-1$, such that $a \in c \circ P^s$. Clearly, it results that $c \circ a \subseteq c \circ P^i$ and $c \circ a \subseteq c \circ P^s$. Therefore, by point 4 of Lemma 1, we have $\emptyset \neq c \circ a \subseteq (c \circ P^i \cap c \circ P^s)$. Moreover, for item 7, $i = s$ and $a \in c \circ P^i$. □

Remark 24 From Lemma 23, if P is a strong hyperproduct of rank r, of elements in a fully zero-simple semihypergroup, then $\widehat{P}$ is partitioned by the family of subsets $\left\{\{c\}, c \circ P, c \circ P^2, ..., c \circ P^{r-1}\right\}$, where c is the quasi-idempotent element of $\widehat{P}$.

Lemma 25 *Let $(H, \circ) \in \mathfrak{F}_0$ and let P be a strong hyperproduct of rank r, with c a quasi-idempotent element in $\widehat{P}$. If Q is a hyperproduct such that $\emptyset \neq Q - \{0\} \subseteq P$, then:*

1. *Q is a strong hyperproduct with $c \in \widehat{Q}$, having the same rank r of P;*

2. *$c \circ Q^i - \{0\} = c \circ P^i - \{0\}$, for all $i \in \{1, 2, \ldots, r-1\}$;*

3. *$\widehat{Q} = \widehat{P}$.*

Proof

1. From Proposition 20, we have $Q^r - \{0\} \subseteq P^r - \{0\} = \{c\}$ and so $Q^r - \{0\} = \{c\}$ and $c \in \widehat{Q}$. Clearly $c \notin Q$ because $Q \subseteq P$ and $c \notin P$. Moreover, by point 3 of Definition 14, if $c \in Q^s - \{0\}$ then $c \in P^s - \{0\}$ and $Q^s - \{0\} = P^s - \{0\} = \{c\}$. Hence, Q is a strong hyperproduct of $(H, \circ)$ and c is a quasi-idempotent element in $\widehat{Q}$. From Proposition 20, if t is the rank of Q then $t \leq r$ because $Q^r - \{0\} = \{c\}$. Moreover, since $\{c\} = Q^t - \{0\} \subseteq P^t - \{0\}$, by point 3 of Definition 14 and Proposition 20, we have $P^t - \{0\} = \{c\}$ and $r \leq t$; therefore, $r = t$.

2. Let b be an element in $P - \{0\}$. We have $b \circ Q^{r-1} - \{0\} \subseteq b \circ P^{r-1} - \{0\} \subseteq P^r - \{0\} = \{c\}$ and so $b \circ Q^{r-1} - \{0\} = \{c\}$. Moreover, by item 1, we have $b \circ c - \{0\} = b \circ (Q^r - \{0\}) - \{0\} = b \circ (Q^{r-1} - \{0\}) \circ (Q - \{0\}) - \{0\} = c \circ (Q - \{0\}) - \{0\} = c \circ Q - \{0\}$ and so $b \circ c - \{0\} = c \circ Q - \{0\}$, for all $b \in P - \{0\}$. Thus, by point 2 of Lemma 23, we deduce that $c \circ P - \{0\} = P \circ c - \{0\} = (P - \{0\}) \circ c - \{0\} = \bigcup_{b \in P - \{0\}} (b \circ c) - \{0\} = \bigcup_{b \in P - \{0\}} (b \circ c - \{0\}) = c \circ Q - \{0\}$. Hence, $c \circ P - \{0\} = c \circ Q - \{0\}$. The proof of the item follows from point 4 of Lemma 23.

3. The result follows from previous item 2 and point 1 of Lemma 23. □

4. Semihypergroups in $\mathfrak{F}_0$ generated by a strong hyperproduct

In this section we consider hypercyclic fully simple semihypergroups generated by a strong hyperproduct. For the reader's convenience we give the following:

Definition 26 A semihypergroup $(H, \circ) \in \mathfrak{I}_0$ is called S*-hypercyclic* if there exists a strong hyperproduct P such that $H = \widehat{P}$.

Example 27 The next table shows an S-hypercyclic semihypergroup $(H, \circ) \in \mathfrak{I}_0$. For notational and descriptive simplicity, we denote $A = \{0, 1\}$, $B = \{0, 2, 3, 4\}$, $C = \{0, 5, 6\}$, and $D = \{0, 7, 8\}$.

$\circ$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	B	B	B	C	C	D	D
2	0	B	$0,5$	C	C	D	D	A	A
3	0	B	C	C	C	D	D	A	A
4	0	B	C	C	C	D	D	A	A
5	0	C	D	D	D	A	A	$2,3,4$	B
6	0	C	D	D	D	A	A	$2,3$	$2,4$
7	0	D	A	A	A	B	B	C	C
8	0	D	A	A	A	B	$2,4$	C	C

The element 1 is quasi-idempotent and, for example, $P = 6 \circ 7$ is a strong hyperproduct of rank four. Also, elements $2, 3, 4, 7, 8$ can be regarded as strong hyperproducts of rank four and $H = \widehat{P} = \widehat{a}$, for all $a \in \{2, 3, 4, 7, 8\}$. In this case $H_+/\beta^*_{H_+}$ is isomorphic to group $\mathbb{Z}_4$.

Proposition 28 *Let $(H, \circ) \in \mathfrak{I}_0$ be an S-hypercyclic semihypergroup generated by the strong hyperproduct P of rank r and let c be the quasi-idempotent element of H. We have:*

1. *If $a \in P - \{0\}$ then a is a strong hyperproduct of rank r and $H = \widehat{a}$.*

2. *If Q is a hyperproduct of elements in H_+ then $c \in \widehat{Q}$. Moreover, if $c \in Q$ then $Q - \{0\} = \{c\}$. Otherwise, if $c \notin Q$ then Q is strong and has rank $\leq r$.*

3. *If Q and T are strong hyperproducts of elements in H_+ then we have $c \circ Q^i \cap c \circ T^i - \{0\} = \emptyset$ or $c \circ Q^i - \{0\} = c \circ T^i - \{0\}$, for all $i \in \{1, 2, ..., r-1\}$.*

4. *The element c is the only identity of $(H, \circ)$.*

5. *The residual semihypergroup $(H_+, \star)$ of $(H, \circ)$ is a cyclic semihypergroup with identity.*

Proof

1. Immediate consequence of Lemma 25.

2. Let $a \in P - \{0\}$ and let $Q = \prod_{i=1}^{n} \alpha_i$ be a hyperproduct of elements in H_+. From point 1, $H = \widehat{a}$, and for every element α_i there exists an integer q_i such that $\alpha_i \in a^{q_i}$. Clearly, we have $Q = \prod_{i=1}^{n} \alpha_i \subseteq \prod_{i=1}^{n} a^{q_i} = a^u$, where $u = \sum_{i=1}^{n} q_i$. Hence, $Q^r - \{0\} \subseteq (a^u)^r - \{0\} = (a^r)^u - \{0\} = \{c\}$ and so $c \in \widehat{Q}$. Now, if $c \in Q$, then $c \in Q \subseteq a^u$ and we have $a^u - \{0\} = \{c\} = Q - \{0\}$. Moreover, if $c \notin Q$ and $c \in Q^s - \{0\}$ then $c \in Q^s - \{0\} \subseteq (a^u)^s - \{0\} = a^{us} - \{0\}$ and so $\{c\} = a^{us} - \{0\} = Q^s - \{0\}$. Hence, Q is a strong hyperproduct and the rank of Q is $\leq r$.

3. If $c \circ Q^i \cap c \circ T^i - \{0\} = \emptyset$ the thesis follows. Otherwise, if $c \circ Q^i \cap c \circ T^i - \{0\} \neq \emptyset$ then there exists $a \in H_+ - \{c\}$ such that $a \in c \circ Q^i$ and $a \in c \circ T^i$ and so, by point 2 of Lemma 25, $c \circ Q^i - \{0\} = c \circ a - \{0\} = c \circ T^i - \{0\}$.

4. By item 2, the element b is a strong hyperproduct for every $b \in H_+ - \{c\}$. From Lemma 23 (1), we have $\widehat{b} = \{c\} \cup c \circ b \cup ... \cup c \circ b^{s-1}$, where s is the rank of b. Hence, there exists $i \in \{1, 2, \ldots, s-1\}$ such that $b \in c \circ b^i$. Clearly, $c \circ b - \{0\} \subseteq c \circ b^i - \{0\}$ and, by item 7 of Lemma 23, we have $i = 1$ and $b \in c \circ b$. In the same way, by item 2 of Lemma 23, we obtain $b \in b \circ c$. Hence, $b \in c \circ b \cap b \circ c$ for all $b \in H_+$. Obviously, we also have $c \circ 0 = 0 \circ c = \{0\}$, and hence c is an identity of $(H, \circ)$. If $c' \in H - \{0, c\}$ is another identity and $Q = c \circ c'$, for item 2, we have $\{c, c'\} \subseteq Q - \{0\}$={c} and $c = c'$. Hence, element c is the only identity of $(H, \circ)$.

5. By point 1 and Lemma 4 (1), the semihypergroup $(H_+, \star)$ is cyclic. Moreover, from previous point 4, the element c is an identity of $(H_+, \star)$. □

Proposition 29 *Let $(H, \circ) \in \mathfrak{I}_0$ an S-hypercyclic semihypergroup generated by the strong hyperproduct P of rank r and suppose r is a prime number. Then $\widehat{c \circ P^h} = H$ for all $h \in \{1, 2, ..., r-1\}$.*

Proof By item 6 of Lemma 23 and the preceding proposition, $c \circ P^i$ is a strong hyperproduct for all $i \in \{1, 2, ..., r-1\}$. Now let $h \in \{1, 2, ..., r-1\}$. By item 1. of Lemma 23, we need to prove that $c \circ P^j \subseteq \widehat{c \circ P^h}$, for every $j \in \{1, 2, ..., r-1\}$. Since r is a prime number, the congruence $hx = j \,(\mathrm{mod\ r})$ has exactly one solution, $s \neq 0$ (mod r). Thus, $hs = j + kr$ and so

$$c \circ P^{j+kr} - \{0\} = c \circ P^{hs} - \{0\}.$$

By points 2, 3, and 4 of Lemma 23, we have $c \circ P^{hs} - \{0\} = c^s \circ P^{hs} - \{0\} = (c \circ P^h)^s - \{0\}$. Hence, $c \circ P^{j+kr} - \{0\} = (c \circ P^h)^s - \{0\}$. Clearly, if $k = 0$ we have $c \circ P^j \subseteq \widehat{c \circ P^h}$; otherwise, if $k \neq 0$ then $c \circ P^j - \{0\} \subseteq c \circ P^j \circ c - \{0\} = c \circ P^j \circ (P^r)^k - \{0\} = c \circ P^{hs} - \{0\} = (c \circ P^h)^s - \{0\}$ and also in this case $c \circ P^j \subseteq \widehat{c \circ P^h}$. □

Theorem 30 *Let $(H, \circ) \in \Im_0$ an $S-$ hypercyclic semihypergroup generated by a strong hyperproduct P of rank a prime number r and having c as quasi-idempotent element. Then:*

1. *Every element $a \in H_+ - \{c\}$ is a strong hyperproduct of rank r and $H = \widehat{a}$.*
2. *For every strong hyperproduct Q of H, $\widehat{Q} = H$ and Q has rank r.*
3. *$(H_+, \star)$ is a cyclic semihypergroup generated by every $a \in H_+ - \{c\}$.*
4. *$H_+/\beta^*_{H_+}$ is a cyclic semigroup.*

Proof

1. Let $a \in H_+ - \{c\}$. Then by item 1 of Lemma 23, there exists $h \in \{1, 2, ..., r-1\}$, such that $a \in c \circ P^h$. By Proposition 29, we have $\widehat{c \circ P^h} = H$. Moreover, $c \notin c \circ P^h$ and so, from Lemma 25, $H = \widehat{a}$.
2. Consequence of item 1.
3. Consequence of Lemma 4 (4) and item 1.
4. Immediate since $(H_+, \star)$ is a cyclic semihypergroup.

□

Remark 31 In Example 27, the S-hypercyclic semihypergroup $(H, \circ)$ is generated by a strong hyperproduct P of rank four, while elements 5 and 6 are strong hyperproducts of rank two and do not generate $(H, \circ)$. This fact shows that the hypothesis "rank of P is a prime number" in Theorem 30 cannot be deleted. The following product table shows an S-hypercyclic semihypergroup $(H, \circ)$ generated by a strong hyperproduct P of rank three.

$\circ$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	0, 2, 3, 4	0, 2, 3, 4	0, 2, 3, 4	0, 5, 6	0, 5, 6
2	0	0, 2, 3, 4	0, 5, 6	5, 6	0, 5, 6	0, 1	0, 1
3	0	0, 2, 3, 4	0, 5, 6	0, 5, 6	5, 6	0, 1	0, 1
4	0	0, 2, 3, 4	5, 6	0, 5, 6	0, 5, 6	0, 1	0, 1
5	0	0, 5, 6	0, 1	0, 1	0, 1	0, 2, 3	0, 4
6	0	0, 5, 6	0, 1	0, 1	0, 1	0, 2, 3, 4	0, 2, 3, 4

By the previous theorem, we have $H = \widehat{a}$, for every $a \in H_+ - \{1\}$, where 1 is the quasi-idempotent element of H. We note that if G is a group then every element a is a strong product and if a is a torsion element its rank is the period of the element a. Therefore, the rank can be seen as a generalization of the concept of period. Moreover, it is known that if G is a cyclic group of size a prime number r then every element different from identity is a generator of G. The same property is true for semihypergroups in Theorem 30, but the commutative property of cyclic groups does not generally hold. The hyperoperation in the previous example is not commutative.

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