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On uniformly pr-ideals in commutative rings

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Abstract: Let R be a commutative ring with nonzero identity and I a proper ideal of R. Then I is called a uniformly pr-ideal if there exists $N \in \mathbb{N}$ such that $ab \in I$ with $ann(a) = 0$ then $b^N \in I$. We say that the smallest $N \in \mathbb{N}$ is called order of I and denoted by $ord_R(I) = N$. In this paper, we give some examples and characterizations of this new class of ideals.

Key words: r-ideal, pr-ideal, uniformly pr-ideal

1. Introduction

In this article all rings will be assumed to be commutative with a unity and all modules are nonzero unital. Let R always be such a ring and M be such an R-module. The aim of this article is to introduce uniformly pr-ideals of commutative rings and to give relations with some classical ideals such as uniformly primary ideal, strongly primary ideal, r-ideal. In [12], Mohammadian introduces r-ideals of commutative rings which is the generalization of pure ideals. Recall that a proper ideal I of R is called an r-ideal if whenever $ab \in I$ and $ann(a) = 0$ then $b \in I$ ($b^n \in I$, for some $n \in \mathbb{N}$). In terms of r-ideals the author characterizes quasiregular rings, rings satisfying property A (See [12], Theorem 4.2) and ([12], Proposition 3.5).

In commutative algebra, prime ideal and its generalizations have an important role. There have been lots of studies on this issue (See [2, 15]). Recall that a proper ideal Q is called a primary ideal if $ab \in Q$ implies either $a \in Q$ or $b^n \in Q$. In [4], Cox also studies a special class of primary ideals fixing the power of an element $b \in R$ in the above definition. A proper ideal Q is called uniformly primary ideal if there exists $N \in \mathbb{N}$ and whenever $ab \in Q$ then either $a \in Q$ or $b^N \in Q$ in that case N is called order of Q and denoted by $ord(Q) = N$. Also Q is called a strongly primary ideal if Q is a primary ideal and $\sqrt{Q}^N \subseteq Q$ for some $N \in \mathbb{N}$ where $\sqrt{Q}$ is the radical of Q. In this case N is called the exponent of Q and denoted by $\exp(Q) = N$. Note that the classes of primary ideals contain uniformly primary ideals and also the classes of uniformly primary ideals contain strongly primary ideals. With these motivations in this paper uniformly pr-ideals and strongly pr-ideals are investigated. For the completeness of the article, we begin with some definitions and notations which will be followed through the study. The set of maximal ideals, prime ideals and minimal prime ideals are denoted by $MaxR$, $SpecR$, and $MinR$, respectively. Also the set of all zero divisors of R is denoted by $zd(R)$. A commutative ring R is called von Neumann regular if for each $a \in R$, there exists $x \in R$ such that

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$a = a^2x$. Recently there have been many studies on von Neumann regular rings. See for example, [8, 11, 16]. Recall that R is said to satisfy $(*)$ condition if for all family of ideals $\{I_i\}_{i\in\Delta}$, $\sqrt{\bigcap_{i\in\Delta} I_i} = \bigcap_{i\in\Delta}\sqrt{I_i}$. R satisfies $(*)$ property if and only if R is a π-regular ring, i.e, Krull dimension of R is zero. Note that a ring R is von Neumann Ring if and only if R is reduced π-regular ring. A proper ideal I of R is called a uniformly pr-ideal if there exists $N \in \mathbb{N}$ such that whenever $ab \in I$ with $ann(a) = 0$, then $b^N \in I$. An ideal I is called a strongly pr-ideal if I is a pr-ideal and $\sqrt{I}^N \subseteq I$ for some $N \in \mathbb{N}$. Among other results in this paper it is shown that the classes of pr-ideals contain uniformly pr-ideals and also uniformly pr-ideals contain strongly pr-ideals (See Corollary 2.6). Further it is proved that pr-ideals, uniformly pr-ideals, and strongly pr-ideals are equal in any Noetherian ring. Moreover, in Corollary 2.9 it is shown that any power of minimal prime ideals are strongly pr-ideals. When a primary ideal becomes a pr-ideal and a uniformly primary ideal becomes a uniformly pr-ideal are demonstrated in the study (See Propositions 2.13 and 2.14). Also the behaviour of uniformly pr-ideals in factor rings, in direct product of rings, and in idealization of a module are investigated (See Proposition 2.15, Theorem 2.16, and Proposition 2.20). Finally, uniformly pr-ideals in polynomial rings and formal power series rings are examined (See Theorem 2.24, Proposition 2.25, and Theorem 2.27).

2. Characterization of uniformly pr-ideals

Definition 2.1 *An ideal I of R is called a uniformly pr-ideal if there exists $N \in \mathbb{N}$ such that whenever $ab \in I$ with $ann(a) = 0$, then $b^N \in I$. The smallest $N \in \mathbb{N}$ is called order of I and denoted by $ord_R(I) = N$.*

It is easily obtained by the definition that every uniformly pr-ideal is a pr-ideal.

Example 2.2 *Let R be a finite ring and I be a proper ideal of R. Assume that $ab \in I$ with $ann(a) = 0$ for some $a, b \in R$. As R is a finite ring, then the set of all units in R and all regular elements in R are equal so that a has an inverse in R. Thus, we conclude that $a^{-1}(ab) = b \in I$. Hence, I is a uniformly pr-ideal with $ord_R(I) = 1$.*

Example 2.3 *Consider the ring of integers $\mathbb{Z}_n$ of modulo n, where $n > 1$ is an integer. Then by Example 2.2, every proper ideal of $\mathbb{Z}_n$ is a uniformly pr-ideal of order 1.*

Definition 2.4 *An ideal I is called a strongly pr-ideal if I is a pr-ideal and $\sqrt{I}^N \subseteq I$ for some $N \in \mathbb{N}$. The smallest $N \in \mathbb{N}$ which has the aforementioned property is called the exponent of I and denoted by $e_R(I)$.*

Proposition 2.5 *J is a pr-ideal if and only if $\sqrt{J}$ is an r-ideal.*

Proof It follows from [12, Proposition 2.6]. □

Corollary 2.6 *Every strongly pr-ideal is also a uniformly pr-ideal.*

Proof Let I be a strongly pr-ideal. Thus, I is a pr-ideal. Since I is a strongly pr-ideal, there exists $N \in \mathbb{N}$ such that $\sqrt{I}^N \subseteq I$. Let $ab \in I$ with $ann(a) = 0$ for some $a, b \in R$. Since I is a pr-ideal, then $b \in \sqrt{I}$. As $\sqrt{I}^N \subseteq I$, we have $b^N \in \sqrt{I}^N \subseteq I$. Hence, I is a uniformly pr-ideal. Furthermore, $ord_R(I) \leq N$ and so $ord_R(I) \leq e_R(I)$. □

By Proposition 2.5, we have the following explicit result.

Corollary 2.7 *Let I be an ideal with $\sqrt{I}$ that is finitely generated. Then I is a pr-ideal if and only if I is a uniformly pr-ideal if and only if I is a strongly pr-ideal if and only if $\sqrt{I}$ is an r-ideal.*

Corollary 2.8 *Suppose that R is a Noetherian ring. Let I be a proper ideal of R. Then the followings are equivalent.*

(i) I is a pr-ideal.

(ii) $\sqrt{I}$ is an r-ideal.

(iii) I is a uniformly pr-ideal

(iv) I is a strongly pr-ideal.

Corollary 2.9 *For every minimal prime ideal P of R, P^n is a strongly pr-ideal for every $n \in \mathbb{N}$.*

Proof Let $P \in Min(R)$, where $Min(R)$ denotes the set of minimal prime ideals. By [7, Corollary 1.2], $P \subseteq zd(R)$. Thus, by [12, Remark 2.3 (f)], P is an r-ideal and so by Proposition 2.5, P^n is a strongly pr-ideal of R. □

Corollary 2.10 *Suppose that I is a uniformly pr-ideal with $S \subseteq R$ is a nonempty subset and $S \subsetneq I$. If S contains a regular element, then $(I : S)$ is a uniformly pr-ideal.*

Proof Let $ab \in (I : S)$ with $ann(a) = 0$. Then $abs \in I$ for every $s \in S$. Assume that $s_0 \in S$ is a regular element. Then as_0 is a regular element. Since $abs_0 \in I$ and I is a uniformly pr-ideal, then $b^N \in I$ for some $N \in \mathbb{N}$. Thus, $b^N \in I \subseteq (I : S)$. □

Lemma 2.11 *(i) Assume that I_i's are uniformly pr-ideals for each $i \in \Delta$ with $ord(I_i) = N_i$. Let $\sup\{N_i : i \in \Delta\} < \infty$. Then $I = \bigcap_{i\in\Delta} I_i$ is a uniformly pr-ideal with $ord(I) \leq \sup\{N_i : i \in \Delta\}$.*

(ii) Assume that I_i's are strongly pr-ideals for each $i = 1, 2, ..., n$ with $\exp(I_i) = N_i$. Then $I = \bigcap_{i=1}^{n} I_i$ is a strongly pr-ideal with $\exp(I) \leq \max\{N_1, N_2, ..., N_n\}$.

Proof (i) Let $N = \sup\{N_i : i \in \Delta\}$. Then for every $i \in \Delta$, $N_i \leq N$. Let $ab \in I$ with $ann(a) = 0$ for some $a, b \in R$. This implies that $ab \in I_i$ for each $i \in \Delta$. Since I_i's are uniformly pr-ideals, we conclude that $b^{N_i} \in I_i$ and so $b^N \in I_i$. Then we have $b^N \in \bigcap_{i\in\Delta} I_i = I$. The rest can be easily seen.

(ii) By (i), it is clear that $I = \bigcap_{i=1}^{n} I_i$ is a uniformly pr-ideal. Let $\max\{N_1, N_2, ..., N_n\} = N$. For every $i = 1, 2, ..., n$, $N_i \leq N$. Then note that $\sqrt{I} = \sqrt{\bigcap_{i=1}^{n} I_i} = \bigcap_{i=1}^{n} \sqrt{I_i}$. Since I_i's are strongly pr-ideals, $\sqrt{I_i}^{N_i} \subseteq I_i$ and so $\sqrt{I_i}^{N} \subseteq I_i$. As $\sqrt{I} \subseteq \sqrt{I_i}$, we get $\sqrt{I}^N \subseteq \sqrt{I_i}^N \subseteq I_i$ and so $\sqrt{I}^N \subseteq \bigcap_{i=1}^{n} I_i = I$. Hence I is a strongly pr-ideal. □

Recall that R is said to satisfy the $(*)$ condition if for all family of ideals $\{I_i\}_{i\in\Delta}$,

$$\sqrt{\bigcap_{i\in\Delta} I_i} = \bigcap_{i\in\Delta} \sqrt{I_i}.$$

R satisfies the $(*)$ property if and only if R is a π-regular ring, i.e, Krull dimension of R is zero.

The following corollary generalizes (ii) in the previous proposition for π-regular rings.

Corollary 2.12 *Let R be a π-regular ring. Assume that I_i's are strongly pr-ideals for each $i \in \Delta$ with $\exp(I_i) = N_i$. Let $\sup\{N_i : i \in \Delta\} < \infty$. Then $I = \bigcap_{i\in\Delta} I_i$ is a strongly pr-ideal with $\exp(I) \le \sup\{N_i : i \in \Delta\}$.*

Proof By the previous proposition (i), $I = \bigcap_{i=1}^{n} I_i$ is a uniformly pr-ideal. From the $(*)$ property, radical commutes with intersection. The rest can be proved similar to the previous proposition (ii). □

Proposition 2.13 *Let Q be a primary ideal of R. Then Q is a pr-ideal if and only if $Q \subseteq zd(R)$.*

Proof Suppose that Q is a primary ideal with $Q \subseteq zd(R)$. Let $ab \in Q$ and $ann(a) = 0$ for some $a, b \in R$. It is easy to see that $a \notin Q$ and so $b^n \in Q$ for some $n \in \mathbb{N}$. Thus, Q is a pr-ideal. Conversely, assume that Q is pr-ideal. Then $\sqrt{Q}$ is an r-ideal and so by [12, Remark 2.3 (d)], we have $Q \subseteq \sqrt{Q} \subseteq zd(R)$. □

Recall that a proper ideal Q of R is said to be a uniformly primary ideal if there exists $N \in \mathbb{N}$ whenever $ab \in Q$ with $a \notin Q$ then $b^N \in Q$. A proper ideal Q is a strongly primary ideal if Q is a primary ideal and $\sqrt{Q}^N \subseteq Q$ for some $N \in \mathbb{N}$ [4].

Proposition 2.14 *(i) Let Q be a uniformly primary ideal. Then Q is a uniformly pr-ideal if and only if $Q \subseteq zd(R)$.*

(ii) Let Q be a strongly primary ideal. Then Q is a strongly pr-ideal if and only if $Q \subseteq zd(R)$.

Proof (i) Let Q be a uniformly primary ideal. Assume that Q is a uniformly pr-ideal. Then Q is a pr-ideal. Also note that uniformly primary ideals are primary ideals so $Q \subseteq zd(R)$ by the previous proposition. Conversely, assume that $Q \subseteq zd(R)$ and $ab \in Q$ with $ann(a) = 0$. Then $a \notin zd(R)$ and so $a \notin Q$. Assume that $ord(Q) = N$. Since Q is a uniformly primary ideal, we have $b^N \in Q$ and so Q is a uniformly pr-ideal.

(ii) It is similar to (i). □

Proposition 2.15 *Let I be an r-ideal and $I \subseteq J$ for some ideal J of R. If J/I is a uniformly pr-ideal then J is a uniformly pr-ideal.*

Proof Suppose that $ab \in J$ with $ann(a) = 0$ for some $a, b \in R$. If $ab \in I$, then $b \in I$ since I is an r-ideal and so $b \in J$. Now assume that $ab \notin I$. We will show that $ann(a + I) = 0_{R/I}$. Let $(x + I)(a + I) = 0_{R/I}$ for some $x \in R$. Then $xa \in I$. Since $ann(a) = 0$ and I is an r-ideal, we have $x \in I$ and so $x + I = 0_{R/I}$. Thus we have $ann(a + I) = 0_{R/I}$. As J/I is a uniformly pr-ideal and $(a + I)(b + I) = ab + I \in J/I$, there exists $N \in \mathbb{N}$ such that $(b + I)^N = b^N + I \in J/I$. This implies $b^N \in J$ and so J is a uniformly pr-ideal. □

Let R_1, R_2 be two commutative rings and $R = R_1 \times R_2$ be a direct product of these rings. It is well known that all ideals of R have the form $I = I_1 \times I_2$, where I_1, I_2 are ideals of R_1 and R_2, respectively.

Theorem 2.16 *Let R_1, R_2 be two commutative rings and $R = R_1 \times R_2$. Suppose that $I = I_1 \times I_2$, where I_1, I_2 are ideals of R_1 and R_2, respectively. Then the followings are equivalent:*

(i) I is a uniformly pr-ideal.

(ii) $I_1 = R_1$ and I_2 is a uniformly pr-ideal or $I_2 = R_2$ and I_1 is a uniformly pr-ideal or I_1, I_2 are uniformly pr-ideals.

Proof $(i) \Rightarrow (ii)$: Suppose that I is a uniformly pr-ideal and $I_2 = R_2$. Let $ab \in I_1$ with $ann(a) = 0$. Then note that $ann(a,1) = (0,0)$ and also $(a,1)(b,0) = (ab,0) \in I$. Since I is a uniformly pr-ideal, we conclude that $(b,0)^N = (b^N,0) \in I_1 \times I_2$ and so $b^N \in I_1$. Hence, I_1 is a uniformly pr-ideal. Similarly, one can easily show that I_2 is a uniformly pr-ideal when $I_1 = R_1$. Assume that I_1, I_2 are proper ideals, similarly I_1, I_2 are uniformly pr-ideals. Furthermore, $ord(I_1 \times I_2) \leq \max\{ord(I_1), ord(I_2)\}$.

$(ii) \Rightarrow (i)$: Let I_1, I_2 be uniformly pr-ideals with $ord(I_i) = N_i$ for every $i = 1,2$. Put $N = \max\{N_1, N_2\}$. Let $(a_1,a_2)(b_1,b_2) = (a_1b_1, a_2b_2) \in I_1 \times I_2$ with $ann(a_1,a_2) = (0,0)$. This implies that $ann(a_1) = ann(a_2) = 0$. Note that $a_ib_i \in I_i$ with $ann(a_i) = 0$. Since I_i is uniformly pr-ideals with $ord(I_i) = N_i$, we conclude that $b_i^{N_i} \in I_i$. Then $b_i^N \in I_i$ and so $(b_1,b_2)^N = (b_1^N, b_2^N) \in I_1 \times I_2$. Thus I is a uniformly pr-ideal. Also note that $ord(I_1 \times I_2) \leq N$. In other cases, one can similarly prove that I is a uniformly pr-ideal. □

Theorem 2.17 *Let $R_1, R_2, ..., R_n$ be commutative rings and $R = R_1 \times R_2 \times ... \times R_n$. Suppose that $I = I_1 \times I_2 \times ... \times I_n$, where I_i's are ideals of R_i, respectively. Then the followings are equivalent:*

(i) I is a uniformly pr-ideal.

(ii) There exists $k_1, k_2, ..., k_t \in \{1,2,...,n\}$ such that $I_k = R_k$ for each $k \in \{k_1, k_2, ..., k_t\}$ and I_k is a uniformly pr-ideal for each $k \in \{1,2,...,n\} - \{k_1, k_2, ..., k_t\}$

Proof We use the mathematical induction on n. If $n = 1$, the claim is true. If $n = 2$, the claim follows from the previous theorem. Assume that the claim is true for all $k < n$. Suppose that $I = I_1 \times I_2 \times ... \times I_n$. Now put $L = I_1 \times I_2 \times ... \times I_{n-1}$. Then by the previous theorem $I = L \times I_n$ is a uniformly pr-ideal if and only if $L = R_1 \times R_2 \times ... \times R_{n-1}$ and I_n is a uniformly pr-ideal or $I_n = R_n$ and L is a uniformly pr-ideal or L, I_n are uniformly pr-ideals. By induction hypothesis the claim is true. □

Theorem 2.18 *Let R_1, R_2 be two commutative rings and $R = R_1 \times R_2$. Suppose that $I = I_1 \times I_2$, where I_1, I_2 are ideals of R_1 and R_2, respectively. Then the followings are equivalent:*

(i) I is a strongly pr-ideal.

(ii) $I_1 = R_1$ and I_2 is a strongly pr-ideal or $I_2 = R_2$ and I_1 is a strongly pr-ideal or I_1, I_2 are strongly pr-ideals.

Proof Note that $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$. The rest is similar to Theorem 2.16. □

As a consequence of the previous theorem we have the following theorem.

Theorem 2.19 *Let $R_1, R_2, ..., R_n$ be commutative rings and $R = R_1 \times R_2 \times ... \times R_n$. Suppose that $I = I_1 \times I_2 \times ... \times I_n$, where I_i's are ideals of R_i, respectively. Then the followings are equivalent:*

(i) I is a strongly pr-ideal.

(ii) There exists $k_1, k_2, ..., k_t \in \{1, 2, ..., n\}$ *such that* $I_k = R_k$ *for each* $k \in \{k_1, k_2, ..., k_t\}$ *and* I_k *is a strongly* pr*-ideal for each* $k \in \{1, 2, ..., n\} - \{k_1, k_2, ..., k_t\}$.

Let M be an R-module. Then the idealization of M, $R(+)M = \{(r, m) : r \in R, m \in M\}$ is a commutative ring with componentwise addition and multiplication $(a, m)(b, n) = (ab, an + bm)$ for every $a, b \in R$ and $m, n \in M$ [1, 13]. If I is an ideal of R and N is a submodule of M then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$ [1]. In that case, $I(+)N$ is called a homogeneous ideal of $R(+)M$. Note that $zd(R(+)M) = \{(a, m) : a \in zd(R) \cup z(M)\}$ where $z(M) = \{a \in R : am = 0 \text{ for some } 0 \neq m \in M\}$ [1]. Now we characterize the uniformly pr-ideal of R in terms of the uniformly pr-ideal of $R(+)M$.

Proposition 2.20 *Let* R *be a ring and* M *be an* R*-module. Then*

(i) If I *is a uniformly* pr*-ideal, then* $I(+)M$ *is a uniformly* pr*-ideal of* $R(+)M$.

(ii) If $z(M) = zd(R)$ *and* $I(+)M$ *is a uniformly* pr*-ideal of* $R(+)M$, *then* I *is a uniformly* pr*-ideal.*

Proof (i) Let $(a, m)(b, n) = (ab, an+bm) \in I(+)M$ with $ann(a, m) = (0, 0)$. Then we have $ann(a) = 0$. Note that $ab \in I$. Since I is a uniformly pr-ideal, we conclude that $b^N \in I$. This implies that $(b, n)^N = (b^N, Nb^{N-1}n) \in I(+)M$. Hence, $I(+)M$ is a uniformly pr-ideal of $R(+)M$.

(ii) Let $ab \in I$ with $ann(a) = 0$. Then $(a, 0)(b, 0) = (ab, 0) \in I(+)M$. Since $Z(M) = zd(R)$, note that $ann(a, 0) = (0, 0)$. As $I(+)M$ is a uniformly pr-ideal, we conclude that $(b, 0)^N = (b^N, 0) \in I(+)M$. This implies that $b^N \in I$ and so I is a uniformly pr-ideal. □

Corollary 2.21 *Let* R *be a ring and* M *be an* R*-module such that* $Z(M) = zd(R)$. *Then* $I(+)M$ *is a uniformly* pr*-ideal of* $R(+)M$ *if and only if* I *is a uniformly* pr*-ideal of* R.

Proposition 2.22 *Let* R *be a ring and* M *be an* R*-module. Then*

(i) If I *is a strongly* pr*-ideal, then* $I(+)M$ *is a strongly* pr*-ideal of* $R(+)M$.

(ii) If $Z(M) = zd(R)$ *and* $I(+)M$ *is a strongly* pr*-ideal of* $R(+)M$, *then* I *is a strongly* pr*-ideal.*

Proof (i) Let I be a strongly pr-ideal. Then I is a uniformly pr-ideal. By previous proposition (i), $I(+)M$ is a uniformly pr-ideal and so $I(+)M$ is a pr-ideal. Since I is a strongly pr-ideal, we have $\sqrt{I}^N \subseteq I$ for some $N \in \mathbb{N}$. By [1, Theorem 3.2], $\sqrt{I(+)M} = \sqrt{I}(+)M$ and so

$$\begin{aligned}\sqrt{I(+)M}^N &= (\sqrt{I}(+)M)^N \subseteq \sqrt{I}^N(+)M \\ &\subseteq I(+)M.\end{aligned}$$

(ii) Assume that $I(+)M$ is a strongly pr-ideal. Then $I(+)M$ is a uniformly pr-ideal so by the previous proposition (ii) I is a uniformly pr-ideal. Then I is a pr-ideal. Since $I(+)M$ is a strongly pr-ideal, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}\sqrt{I(+)M}^N &= (\sqrt{I}(+)M)^N \subseteq \sqrt{I}^N(+)M \\ &\subseteq I(+)M.\end{aligned}$$

This implies that $\sqrt{I}^N \subseteq I$ and so I is a strongly pr-ideal. □

Corollary 2.23 *Let R be a ring and M be an R-module such that $Z(M) = zd(R)$. Then $I(+)M$ is a strongly pr-ideal of $R(+)M$ iff I is a strongly pr-ideal of R.*

A ring R is said to satisfy the annihilator condition (for short, a.c.) if every finitely generated ideal I consisting zero divisors has a nonzero annihilator. $R[X]$ denotes the polynomial ring over R in indeterminates x. If $f(X) = a_0 + a_1X + ... + a_nX^n$, where $a_n \neq 0$, then $\deg(f) = n$. Also the content of f is defined as $c(f) = (a_0, a_1, ..., a_n)$. In [6, Theorem 28.1], $c(f)^{n+1}c(g) = c(f)^n c(fg)$, where $\deg f = n$. Let I be an ideal of R. Then $I[X] = \{f(X) = a_0 + a_1X + ... + a_nX^n : a_i \in I \text{ for } i = 0, 1, ..., n\}$ is an ideal of $R[X]$. A ring R is called Armendariz ring if for any $f(x) = a_0 + a_1x + ... + a_nx^n$, $g(x) = b_0 + b_1x + ... + b_kx^k$ with $f(x)g(x) = 0$, then $a_ib_j = 0$ for all i, j. [14] Note that all reduced rings are an example of Armendariz ring.

Theorem 2.24 *Let R be an Armendariz ring. Then I is a uniformly pr-ideal if and only if $I[X]$ is a uniformly pr-ideal of $R[X]$.*

Proof Suppose that $I[X]$ is a uniformly pr-ideal of $R[X]$. Let $ab \in I$ with $ann(a) = 0$. Now put $f(X) = a, g(X) = b$. Then $f(X)g(X) = ab \in I[X]$ with $ann_{R[X]}(f(X)) = 0$. Since $I[X]$ is a uniformly pr-ideal of $R[X]$, there exists $N \in \mathbb{N}$ such that $g(X)^N = b^N \in I[X]$ and so $b^N \in I$. Thus, I is a uniformly pr-ideal of R. Conversely, let I be a uniformly pr-ideal of R with $ord_R(I) = N$. Assume that $f(X)g(X) \in I[X]$ with $ann_{R[X]}(f(X)) = 0$. Since R is an Armendariz ring and $ann_{R[X]}(f(X)) = ann(c(f))[X] = 0$, we have $c(f) \nsubseteq zd(R)$ and so there exists $r \in c(f)$ such that $ann(r) = 0$. Also note that $c(fg) \subseteq I$. Then by [6, Theorem 28.1], $c(f)^{n+1}c(g) = c(f)^n c(fg) \subseteq I$, where $\deg f = n$. Since $r^{n+1} \in c(f)^{n+1}$ and $ann(r^{n+1}) = 0$, we get $r^{n+1}c(g) \subseteq I$. Now assume that $c(g) = (b_0, b_1, ..., b_k)$ for some $b_i \in R$. Then for any $b_i \in c(g)$, we have $b_i^N \in I$. It is clear that $c(g)^{(k+1)N} \subseteq I$. and so $c(g^{(k+1)N}) \subseteq c(g)^{(k+1)N} \subseteq I$ and this yields that $g(X)^{(k+1)N} \in I[X]$. Hence $I[X]$ is a uniformly pr-ideal of $R[X]$. □

Proposition 2.25 *Let R be a ring satisfying property (A). Then I is a strongly pr-ideal if and only if $I[X]$ is a strongly pr-ideal of $R[X]$.*

Proof Let I be a proper ideal of R. Then $I[X]$ is a proper ideal and so $R[X]/I[X]$ is isomorphic to ring $(R/I)[X]$. This isomorphism gives us $\sqrt{0_{R[X]/I[X]}} = \sqrt{I[X]}/I[X] \cong \sqrt{0_{(R/I)[X]}} = (\sqrt{I}/I)[X]$ and so $\sqrt{I[X]}/I[X] = \sqrt{I}[X]/I[X]$ and this yields $\sqrt{I[X]} = \sqrt{I}[X]$. Suppose that $I[X]$ is a strongly pr-ideal of $R[X]$. Then $I[X]$ is a uniformly pr-ideal and so by the previous theorem I is a uniformly pr-ideal. Then I is a pr-ideal. Now assume that $e_{R[X]}(I[X]) = N$. Then

$$\sqrt{I[X]}^N = (\sqrt{I}[X])^N = \sqrt{I}^N[X] \subseteq I[X]$$

and this gives $\sqrt{I}^N \subseteq I$. Hence, I is a strongly pr-ideal. Conversely, assume that I is a strongly pr-ideal with $e_R(I) = N$. Then I is a uniformly pr-ideal and so by the previous theorem $I[X]$ is a uniformly pr-ideal. Then $I[X]$ is a pr-ideal. Also note that $\sqrt{I}^N \subseteq I$ and so

$$\sqrt{I[X]}^N = (\sqrt{I}[X])^N = \sqrt{I}^N[X] \subseteq I[X].$$

Hence, $I[X]$ is a strongly pr-ideal of $R[X]$. □

Let $R[[x]]$ denote the formal power series ring with coefficient in R. For any $f(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$, we denote the ideal generated by the set $\{a_n : n \in \mathbb{N}\}$ with $C(f)$. In [[5], Theorem 2.6], Dedekind Mertens Theorem for formal power series ring is given as the following.

Theorem 2.26 *Let R be a Noetherian ring and $0 \neq g \in R[[x]]$. Let k be the maximum of the numbers $\mu(c(g)_m)$, taken all over the maximal ideals m of R. (In particular, $\mu(c(g)_m) \geq k$) Then for all $f \in R[[x]]$, we have $c(f)^k c(g) = c(f)^{k-1} c(fg)$.*

Recall that the commutative ring is called a $ps-$Armendariz ring if for any $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ with $f(x)g(x) = 0$, then $a_k b_n = 0$ for all k, n. Note that all $ps-$Armendariz rings are an Armendariz ring [9].

Theorem 2.27 *Let R be a Noetherian ps-Armendariz ring. Then I is a uniformly pr-ideal if and only if $I[[X]]$ is a uniformly pr-ideal of $R[[X]]$.*

Proof The "only if" part is clear. Assume that I is a uniformly pr-ideal of R. Take $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k \in R[[X]]$ such that $f(x)g(x) \in I[[X]]$ with $ann(f(x)) = 0$. Since R is a ps-Armendariz ring, then $ann(f(x)) = (ann(c(f))[[x]]$ where $c(f) =< a_n : n \in \mathbb{N} >$. Since R is a Noetherian ring, $c(f) =< a_1^*, a_2^*, ..., a_n^* >$ and so $ann(f(x)) = ann(< a_1^*, a_2^*, ..., a_n^* >)[[X]] = 0$. As R satisfies the property (A), $< a_1^*, a_2^*, ..., a_n^* > \subsetneq zd(R)$ and so there exists $r \in c(f)$ such that $ann(r) = 0$. Also note that $c(fg) \subset I$. Then by the previous theorem $c(f)^{n+1} c(g) = c(f)^n c(fg) \subseteq I$ where k is the maximum of the numbers $\mu(c(g)_m)$. As $r^{n+1} \in c(f)^{n+1}$, $ann(r^{n+1}) = 0$, we get $r^{n+1} c(g) \subseteq I$. Since R is a Noetherian ring, $c(g) =< b_n : n \in \mathbb{N} >=< b_1^*, b_2^*, ..., b_k^* >$ .Since $r^{n+1} b_i^* \in I$ and I is a uniformly pr-ideal with $ord(I) = N$, we can conclude that $(b_i^*)^N \in I$. Now put $t = (k+1)N$. A similar argument of Theorem 2.24 shows that $c(g^t) \subseteq I$ and hence $g^t \in I$. Therefore, $I[[X]]$ is a uniformly pr-ideal of $R[[X]]$. □

Proposition 2.28 *Let R be a Noetherian ps-Armendariz ring. Then I is a strongly pr-ideal if and only if $I[[X]]$ is a strongly pr-ideal of $R[[X]]$.*

Proof Suppose that I is a strongly pr-ideal. Then I is a uniformly pr-ideal by Corollary 2.6. Then by the previous theorem $I[[X]]$ is a uniformly pr-ideal and hence $I[[X]]$ is a pr-ideal. By [[3], Corollary 2.2.3], $\sqrt{I[[X]]} = \sqrt{I}[[X]]$. Assume that $\exp(I) = N$ and it is easy to see that $\sqrt{I[[X]]}^N = (\sqrt{I}[[X]])^N = \sqrt{I}^N[[X]] \subseteq I[[X]]$ since $\sqrt{I}^N \subseteq I$. Conversely, assume that $I[[X]]$ is a strongly pr-ideal with $ord(I[[X]]) = N$. Then the similar argument shows that I is a strongly pr-ideal. □

Theorem 2.29 *Let R be a Noetherian ps-Armendariz ring. Then the following are equivalent.*

(i) I is a pr-ideal of R.

(ii) $\sqrt{I}$ is a r-ideal of R.

(iii) I is a uniformly pr-ideal of R.

(iv) I is a strongly pr-ideal of R.

(v) $I[X]$ is a uniformly pr-ideal of $R[X]$.

(vi) $I[X]$ is a strongly pr-ideal of $R[X]$.

(vii) $I[[X]]$ is a uniformly pr-ideal of $R[[X]]$.

($viii$) $I[[X]]$ is a strongly pr-ideal of $R[[X]]$.

(ix) $I[[X]]$ is a pr-ideal of $R[[X]]$.

(x) $I[X]$ is a pr-ideal of $R[X]$.

Proof $(i) \Longleftrightarrow (iv)$ are equal from Corollary 2.9.

$(i) \Longleftrightarrow (v)-(viii)$ follows from Theorem 2.24, Proposition 2.25, previous proposition, and previous theorem.

$(viii) \Rightarrow (ix) \Rightarrow (x) \Rightarrow (i)$ is clear. □

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