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Derivations of generalized quaternion algebra

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Abstract: The purpose of this paper is to determine derivations of the algebra $H_{\alpha,\beta}$ of generalized quaternions over the reals and hence to obtain the algebra $Der(H_{\alpha,\beta})$ of derivations of $H_{\alpha,\beta}$. Once we know derivations we might decompose $Der(H_{\alpha,\beta})$ in terms of its inner and/or central derivations whenever they exist. Apart from $Der(H_{\alpha,\beta})$ we would also be able to obtain generalized derivations, which have been studied by analysts in the context of algebras of some normed spaces, and of prime and semiprime rings.

Key words: Derivation, quaternion

1. Introduction

Derivations of an algebra give interesting insights for studying its algebraic structure. In particular, derivations of Lie algebras have been used in a control theoretical setting since they are intimately related with linear vector fields. By a linear vector field on a Lie group G we mean that its flow forms a 1-parameter subgroup of the group of G-automorphisms. Such a vector field (together with right invariant vector fields, called control vectors) is considered to define the dynamics of linear control systems on Lie groups. Moreover, it is well known that when G is a simply connected and nilpotent Lie group with the Lie algebra $\mathfrak{g}$, any derivation of $\mathfrak{g}$ induces a linear vector field on G. We refer the reader to [1, 2, 4, 5] for further information regarding the use of derivations in control theory contexts and also [3], in which we provided a simple computational algorithm for finding explicit derivations of Lie algebras.

In this paper, we consider exclusively the derivations of generalized quaternion algebra over $\mathbb{R}$ as a class of Lie algebra. We first state the conditions that a linear map should obey to become a derivation for the generalized quaternion algebra under consideration and then we obtain a typical derivation in its matrix form. Once we have on hand the explicit derivations, we also determine the algebra of derivations in terms of inner and/or central derivations whenever they exist. We recall that this was already done for Lie algebras in the literature and we find it convenient to present in this article an appropriate variant for quaternion algebra. We end our work with a brief section on generalized derivations since they are natural generalizations of classical derivations and there has been ongoing interest about this subject. However, we constrain ourselves only to the task of determining generalized derivations of quaternion algebra in the matrix form.

This paper is organized as follows: Section 2 only gives minimal information on quaternion algebra. Section 3 considers their derivations, which is the main subject of our work. We then find a typical derivation of

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quaternion algebra in its matrix form. In Section 4, we attempt to extend the derivations obtained to generalized derivations since such derivations are often of interest. We explicitly get a generalized derivation by means of its matrix representation.

2. Quaternion algebra

By an algebra A over a field F we mean a vector space over F provided with a bilinear map $\cdot : A \times A \to A$. We will be only considering the case $F = \mathbb{R}$ and A being quaternion algebra over $\mathbb{R}$. We start by giving the definition of generalized quaternions from [7] as follows:

Definition 2.1 *A generalized quaternion x is of the form $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3$ where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the quaternionic units e_0, e_1, e_2, and e_3 obey the following equations:*

$$\begin{aligned} e_1^2 &= -\alpha, \quad e_2^2 = -\beta, \quad e_3^2 = -\alpha\beta, \\ e_1 \cdot e_2 &= e_3 = -e_2 \cdot e_1, \\ e_2 \cdot e_3 &= \beta e_1 = -e_3 \cdot e_2, \\ e_3 \cdot e_1 &= \alpha e_2 = -e_1 \cdot e_3, \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$.

For simplicity, we sometimes use the notation $x = x_0e_0 + \mathbf{x}$, where $\mathbf{x} = x_1e_1 + x_2e_2 + x_3e_3$ is the vector part of x. We denote by $H_{\alpha,\beta}$ the set of generalized quaternions over the reals with the basis $\mathfrak{B}(H_{\alpha,\beta}) = \{e_0, e_1, e_2, e_3\}$ corresponding to the familiar $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. Note that e_0 acts as identity, which means $e_0 \cdot e_i = e_i \cdot e_0 = e_i$ for any i and hence the center of $H_{\alpha,\beta}$ is $Z(H_{\alpha,\beta}) = \mathbb{R} \cdot e_0 = \mathbb{R}$.

Given $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ and the real numbers $\alpha, \beta > 0$ there exists an inner product $\mathbf{g}(u, v)$ defined by $\mathbf{g}(u, v) = \alpha u_1v_1 + \beta u_2v_2 + \alpha\beta u_3v_3$. Denote by $\mathbb{R}^3_{\alpha,\beta}$ the linear space on $\mathbb{R}^3$ provided with the inner product $\mathbf{g}(u, v)$ to distinguish it from $\mathbb{R}^3$ with the standard inner product $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$. One might also define a slightly different vector product in $\mathbb{R}^3_{\alpha,\beta}$ as follows:

$$u \wedge v = \begin{vmatrix} \beta e_1 & \alpha e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Note that $e_1 \wedge e_2 = e_3, e_2 \wedge e_3 = \beta e_1$, and $e_3 \wedge e_1 = \alpha e_2$. The multiplicative product " $\cdot$ " for $H_{\alpha,\beta}$ is defined by

$$x \cdot y = (x_0y_0 - g(\mathbf{x}, \mathbf{y}))e_0 + x_0\mathbf{y} + y_0\mathbf{x} + \mathbf{x} \wedge \mathbf{y}.$$

Addition (and hence subtraction) is defined component-wise: $x + y = (x_0 + y_0)e_0 + \mathbf{x} + \mathbf{y}$. In particular, any scalar $c \in \mathbb{R}$ can be thought of as a quaternion $c = c + \mathbf{0}$ and hence multiplication of $x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3 \in H_{\alpha,\beta}$ by c is given by

$$c \cdot x = cx_0e_0 + c\mathbf{x}.$$

It follows that the multiplication rule is anticommutative but associative and distributive over addition.

3. Algebra of derivations for $H_{\alpha,\beta}$

Definition 3.1 *(Derivation) A derivation of an algebra A is a linear map $D : A \to A$ such that*

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y) \tag{3.1}$$

for all $x, y \in A$.

It is clear that the set of all derivations of an algebra A forms a vector space, which we denote by $Der(A)$. Recall that $\mathfrak{gl}(A)$ is a Lie algebra with Lie bracket given by $[f,g] = f \circ g - g \circ f$ for all $f, g \in \mathfrak{gl}(A)$. Note that D_1D_2 may fail to be a derivation of A, in general. However, the commutator $[D_1, D_2]$ is always a derivation since $Der(A) \subset \mathfrak{gl}(A)$ and

$$[D_1, D_2](x \cdot y) = [D_1, D_2](x) \cdot y + x \cdot [D_1, D_2](y)$$

for every $D_1, D_2 \in Der(A)$ and $x, y \in A$. We also recall that any associative algebra A can be made into a Lie algebra, say $L(A)$, by taking the commutator as the Lie bracket $[x, y] = x \cdot y - y \cdot x$ for all $x, y \in A$. It follows that if $D \in Der(A)$ then D is also a derivation of the corresponding Lie algebra, which means

$$D([x, y]) = [D(x), y] + [x, D(y)] \tag{3.2}$$

for all $x, y \in L(A)$. Nonetheless, it should be noted that there may exist an associative algebra A and a derivation of the corresponding Lie algebra $L(A)$, which is not a derivation of A. In this paper, we deal mainly with A-derivations, where by A we simply mean the quaternion algebra $H_{\alpha,\beta}$. Once we determine the algebra $Der(H_{\alpha,\beta})$ we will be in a position to obtain derivations of some quaternions by attributing either ± 1 or 0 to α and/or β.

A particular class of derivations are the so-called inner derivations.

Definition 3.2 *(Inner derivation) Given $x \in A$, by an inner derivation associated to x we mean the map*

$$D = ad(x) : A \longrightarrow A, \qquad y \longmapsto [x, y],$$

for every $y \in A$.

Let $ad(H_{\alpha,\beta})$ denote the set of all inner derivations of $H_{\alpha,\beta}$ as a subset of $Der(H_{\alpha,\beta})$. By bilinearity, $ad(H_{\alpha,\beta})$ can be generated by the maps

$$ad(e_i) : H_{\alpha,\beta} \longrightarrow H_{\alpha,\beta},$$

where $e_i \in \mathfrak{B}(H_{\alpha,\beta})$. That is, any inner derivation $D = ad(x)$, $x \in H_{\alpha,\beta}$, is a linear combination of $ad(e_i)$s. It should be noted that for any $e_i \in \mathfrak{B}(H_{\alpha,\beta})$ the map $ad(e_i)$ is the zero-map if and only if $e_i \in Z(H_{\alpha,\beta})$. This simply means that for any quaternion algebra we always have $ad(e_0) = 0$ since e_0 acts for all as a global identity.

Also note that we do not have $x \cdot (y \cdot z) + y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0$ for all $x, y, z \in H_{\alpha,\beta}$, which seems at first glance necessary to guarantee $ad(x) \in Der(H_{\alpha,\beta})$. This is mainly because one needs the Jacobi identity for a given Lie algebra $\mathfrak{g}$ to say $ad(x) \in Der(\mathfrak{g})$. Thanks to the associativity of $H_{\alpha,\beta}$, it is easy to see for every $y, z \in H_{\alpha,\beta}$ and every $D = ad(x)$, $x \in H_{\alpha,\beta}$, that $D(y \cdot z) = D(y) \cdot z + y \cdot D(z)$. Hence, every inner derivation is indeed a derivation in the sense of Definition 3.1.

The set $ad(H_{\alpha,\beta})$ can be determined from the equation $ad(H_{\alpha,\beta}) \simeq \frac{H_{\alpha,\beta}}{Z(H_{\alpha,\beta})}$ and thus it is easy to determine its dimension. From $Der(H_{\alpha,\beta})/ad(H_{\alpha,\beta})$ we obtain outer (=noninner) derivations. We also define for further purposes central derivations as follows:

Definition 3.3 *(Central derivation) A derivation D of an algebra A mapping A into its center $Z(A)$ is called a central derivation.*

Once we explicitly have all the derivations we are able to determine (if existing) the inner and/or central derivations. Let D denote a derivation of $H_{\alpha,\beta}$. Thus, D admits a matrix representation with respect to the basis $\mathfrak{B}(H_{\alpha,\beta})$, which is the 4×4 matrix $[D] = (d_{ij})^T$ whose entries are defined by the following equations:

$$D(e_{i-1}) = \sum_{j=1}^{4} d_{ij} e_{j-1}, \quad 1 \leq i \leq 4.$$

Each column of the matrix $[D]$ is, of course, an element of $H_{\alpha,\beta}$. In order to obtain D in its matrix form it suffices to know the Leibnitz rule in (3.1) for the products $e_i \cdot e_j$ with $1 \leq i \leq j \leq 3$. First, we state for later purposes the following simple remark.

Remark 3.4 *Since e_0 is a central idempotent, it follows that $D(e_0) = 0$ for every $D \in Der(H_{\alpha,\beta})$. Moreover, if we are given a derivation D, then the first column of $[D]$ consists of only zeros. In fact,*

$$\begin{aligned} D(e_0 \cdot e_i) &= D(e_0) \cdot e_i + e_0 \cdot D(e_i), \forall i = 1,2,3 \\ &\Updownarrow \\ D(e_i) &= D(e_0) \cdot e_i + D(e_i), \end{aligned}$$

which occurs if and only if

$$D(e_0) \cdot e_i = 0, \quad \forall i = 1,2,3.$$

Hence, one obtains $d_{11} = d_{12} = d_{13} = d_{14} = 0$ only by evaluating, for instance, $D(e_0) \cdot e_0 = 0$.

Let us apply the Leibnitz rule to the quaternionic units:

$$\begin{aligned} D(e_1^2) &= D(e_1) \cdot e_1 + e_1 \cdot D(e_1), \\ -\alpha D(e_0) &= (d_{21} e_0 \cdot e_1 + d_{22} e_1^2 + d_{23} e_2 \cdot e_1 + d_{24} e_3 \cdot e_1) + \\ &\quad (d_{21} e_1 \cdot e_0 + d_{22} e_1^2 + d_{23} e_1 \cdot e_2 + d_{24} e_1 \cdot e_3) \\ &= -2\alpha d_{22} e_0 + 2 d_{21} e_1 + 0 e_2 + 0 e_3. \end{aligned}$$

That is, $-\alpha d_{11} e_0 - \alpha d_{12} e_1 - \alpha d_{13} e_2 - \alpha d_{14} e_3 = -2\alpha d_{22} e_0 + 2 d_{21} e_1 + 0 e_2 + 0 e_3$ implies $d_{11} = 2 d_{22} = 0$ and $-\alpha d_{12} = 2 d_{21}$, and hence $d_{12} = -2 d_{21} = 0$, since we already know $d_{12} = 0$.

Combining all these together we obtain $d_{11} = d_{12} = d_{13} = d_{14} = d_{22} = d_{21} = 0$, and continuing this way we have

$$\begin{aligned} D(e_2^2) &= D(e_2) \cdot e_2 + e_2 \cdot D(e_2), \\ -\beta D(e_0) &= (d_{31} e_0 \cdot e_2 + d_{32} e_1 \cdot e_2 + d_{33} e_2^2 + d_{34} e_3 \cdot e_2) + \\ &\quad (d_{31} e_2 \cdot e_0 + d_{32} e_2 \cdot e_1 + d_{33} e_2^2 + d_{34} e_2 \cdot e_3) \\ &= -2\beta d_{33} e_0 + 0 e_1 + 2 d_{31} e_2 + 0 e_3, \end{aligned}$$

from which we obtain $-\beta d_{11} = -2\beta d_{33}$ and $-\beta d_{13} = 2d_{31}$ and read $d_{31} = 0$, and

$$d_{33} = \begin{cases} 0 \text{ if } \beta \neq 0 \\ 0 \neq d_{33} \text{ if } \beta = 0 \end{cases}.$$

Also,

$$\begin{aligned} D(e_3^2) &= D(e_3) \cdot e_3 + e_3 \cdot D(e_3) \\ -\alpha\beta D(e_0) &= (d_{41}e_0 \cdot e_3 + d_{42}e_1 \cdot e_3 + d_{43}e_2 \cdot e_3 + d_{44}e_3^2) + \\ &\quad (d_{41}e_3 \cdot e_0 + d_{42}e_3 \cdot e_1 + d_{43}e_3 \cdot e_2 + d_{44}e_3^2) \\ &= -2\alpha\beta d_{44}e_0 + 0e_1 + 0e_2 + 2d_{41}e_3, \end{aligned}$$

which gives $-\alpha\beta d_{11} = -2\alpha\beta d_{44}$ and $-\alpha\beta d_{14} = 2d_{41}$. That is, $d_{41} = 0$ (since $d_{14} = 0$) and

$$d_{44} = \begin{cases} 0 \text{ if } \beta \neq 0 \\ 0 \neq d_{44} \text{ if } \beta = 0 \end{cases}.$$

We are now going to check the same procedure for $e_1e_2 = -e_2e_1$:

$$\begin{aligned} D(e_1 \cdot e_2) &= D(e_1) \cdot e_2 + e_1 \cdot D(e_2), \\ D(e_3) &= (d_{21}e_0 \cdot e_2 + d_{22}e_1 \cdot e_2 + d_{23}e_2^2 + d_{24}e_3 \cdot e_2) + \\ &\quad (d_{31}e_1 \cdot e_0 + d_{32}e_1^2 + d_{33}e_1 \cdot e_2 + d_{34}e_1 \cdot e_3) \\ &= d_{21}e_2 + d_{22}e_3 - \beta d_{23}e_0 - \beta d_{24}e_1 + d_{31}e_1 - \alpha d_{32}e_0 + d_{33}e_3 - \alpha d_{34}e_2 \\ &= (-\beta d_{23} - \alpha d_{32})e_0 + (d_{31} - \beta d_{24})e_1 + (d_{21} - \alpha d_{34})e_2 + (d_{22} + d_{33})e_3. \end{aligned}$$

Thus, from $D(e_1 \cdot e_2) = D(e_1) \cdot e_2 + e_1 \cdot D(e_2)$ we obtain $d_{41} = (-\beta d_{23} - \alpha d_{32})$, $d_{42} = (d_{31} - \beta d_{24})$, $d_{43} = (d_{21} - \alpha d_{34})$, and $d_{44} = (d_{22} + d_{33})$. Therefore, we have

$$\begin{aligned} -\frac{\beta}{\alpha} d_{23} &= d_{32}, \\ -\beta d_{24} &= d_{42}, \\ d_{43} &= -\alpha d_{34}, \\ d_{33} &= d_{44}. \end{aligned}$$

Finally, we obtain a typical derivation of $H_{\alpha,\beta}$ in its matrix form as follows:

Theorem 3.5 *The algebra $Der(H_{\alpha,\beta})$ of derivations for $H_{\alpha,\beta}$ is generated by the following matrices:*

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha}a & -\beta b \\ 0 & a & d & -\alpha c \\ 0 & b & c & d \end{pmatrix} \in Der(H_{\alpha,\beta}), \tag{3.3}$$

where $a, b, c, d \in \mathbb{R}$ such that $d = d(\beta) \neq 0$ if $\beta = 0$ and $d = 0$ otherwise.

It follows at once from the preceding theorem that the dimension of $Der(H_{\alpha,\beta})$ is at most 4. Moreover, there exists a unique noninner derivation and no central derivation arises since $Z(H_{\alpha,\beta}) = \mathbb{R} \cdot e_0$. One might also conclude from $ad(H_{\alpha,\beta}) \simeq H_{\alpha,\beta}/Z(H_{\alpha,\beta})$ that the algebra $ad(H_{\alpha,\beta})$ of inner derivations is generated by the following matrices:

$$ad(e_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\alpha \\ 0 & 0 & 2 & 0 \end{pmatrix}, \qquad ad(e_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix},$$

$$ad(e_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2\beta & 0 \\ 0 & 2\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If $a = b = c = \beta = 0$, then $D = \begin{pmatrix} O & O \\ O & I_2 \end{pmatrix}$, where O and I_2 are 2×2 zero and identity matrices, respectively, is the only noninner derivation of $H_{\alpha,\beta}$, i.e. $D \in Der(H_{\alpha,\beta})/ad(H_{\alpha,\beta})$. It is clear that D does not commute with any inner derivation. If we put $\alpha = \beta = 1$ then $H_{\alpha,\beta}$ is the classical real quaternion algebra, say H_R, for which the automorphism group $Aut(H_R)$ of H_R consists entirely of inner automorphisms

$$i_x : y \to x \cdot y \cdot x^{-1}$$

for invertible $x \in H_R$. It is well known that $Aut(H_R)$ is isomorphic to the group of rotations

$$SO(3) = \{A \in GL(3, \mathbb{R}) : A^T A = 1\},$$

where $GL(3, \mathbb{R})$ denotes the general linear group of 3×3 invertible matrices with real entries. Hence, H_R has only inner derivations (as antisymmetric matrices), which might also be seen from (3.3) if we pick $\alpha = \beta = 1$:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a & -b \\ 0 & a & 0 & -c \\ 0 & b & c & 0 \end{pmatrix}.$$

That is, the algebra $Der(H_R)$ is generated by $D_i = ad(e_i)$ for $i = 1, 2, 3$ and

$$Der(H_R) = ad(H_R)$$

since $Der(H_R)$ is centerless.

Corollary 3.6 *(Semiquaternions) Let $\alpha = 1$ and $\beta = 0$, for which $H_{\alpha,\beta}$ is the algebra of semiquaternions H_s. Then $\dim Der(H_s) = 4$ and any $D \in Der(H_s)$ is of the form*

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & d & -c \\ 0 & b & c & d \end{pmatrix}.$$

There are 3 inner derivations and only one noninner derivation.

Proof Since D does not commute with any inner derivation $ad(e_i)$, $i = 1, 2, 3$, there is no central derivation. This might be also seen from the fact that $D_4(H_s) \subset \mathbb{R}e_2 + \mathbb{R}e_3$ while the center $Z(H_s) = \mathbb{R}e_0$. □

Corollary 3.7 *(Split quaternions) If $\alpha = -\beta = 1$ then $H_{\alpha,\beta}$ is the algebra of split-quaternions H_{sp} and any $D \in Der(H_{sp})$ is of the form*

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & a & 0 & -c \\ 0 & b & c & 0 \end{pmatrix}.$$

Moreover, $\dim Der(H_{sp}) = 3$ *and*

$$Der(H_{sp}) = ad(H_{sp})$$

since there is no nontrivial inner derivation.

Remark 3.8 *It is seen that the algebras H_R and H_{sp} of real and split quaternions, respectively, consist of entirely inner derivations. Recall that H_R possesses derivations that are purely antisymmetric matrices and the exponential of an antisymmetric matrix gives an orthogonal matrix. It is well known that rotations in quaternions are represented by conjugations (i.e. inner automorphisms). However, we have for split quaternions both symmetric and antisymmetric derivations. For example, if $a = b = 0$ we have only one antisymmetric derivation (as a generator), while for $c = 0$ we get two symmetric derivations. Thus, for D_1 with $a = 1$ and $b = c = 0$ or D_2 with $b = 1$ and $a = c = 0$ the exponential* $\exp(D_i)$ *is a symmetric matrix since* $\exp(A^T) = (\exp A)^T$ *for any square matrix A.*

Corollary 3.9 *(Split semiquaternions) If $\alpha = -1$ and $\beta = 0$ then $H_{\alpha,\beta}$ is the algebra of split semiquaternions H_{sps} and any $D \in Der(H_{sps})$ is of the form*

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & a & d & c \\ 0 & b & c & d \end{pmatrix}.$$

Remark 3.10 *Note that here there exists a unique (generic) noninner derivation $D = \begin{pmatrix} O & O \\ O & I_2 \end{pmatrix}$ and the only derivation that makes a difference when we compare $Der(H_s)$ and $Der(H_{sps})$ is the derivation D with c.*

It is known that there exist Lie algebras $\mathfrak{g}$ such that $Der(\mathfrak{g}) = ad(\mathfrak{g})$ and also Lie algebras such that $Der(\mathfrak{g}) = ad(\mathfrak{g}) + C(\mathfrak{g})$, where $C(\mathfrak{g})$ means the set of central derivations of $\mathfrak{g}$. In the latter case, we understand that there exist as few derivations as possible. Hence, our theorem allows us to determine for which quaternion algebra H one might have $Der(H) = ad(H)$ and $Der(H) = C(H)$ or even $Der(H) = ad(H) + C(H)$.

4. Generalized derivations

Generalized derivation is a natural extension of classical derivation and has a wide range of applications in the literature since it is a quite useful tool in algebraic and geometric classification of algebras. For example, a number of papers studied generalized derivations in the context of prime and semiprime rings. In particular, there has been interest regarding the relationship between the commutativity of a ring and the existence of certain derivations. For Lie algebras, a generalized derivation is defined as a linear transformation f of a Lie algebra $\mathfrak{g}$ such that there exists a derivation $d \in Der(\mathfrak{g})$ with the property $f[x,y] = [f(x),y] + [x,d(y)]$ for all $x,y \in \mathfrak{g}$. In a still more general setting, Leger and Lucks considered in [6] an endomorphism f of a Lie algebra $\mathfrak{g}$ as a generalized derivation if there exist $g,h \in$ End$(\mathfrak{g},\mathfrak{g})$ such that $h[x,y] = [f(x),y] + [x,g(y)]$ for all $x,y \in \mathfrak{g}$.

In this final section we deal with the notion of generalized derivations of quaternion algebra. The motivation is that we have already obtained the algebra of derivations for generalized quaternions and hence we can also determine generalized derivations for quaternions of various types.

Definition 4.1 *(Generalized derivation) A linear map F of an algebra A (resp. a ring R) into itself is said to be a generalized derivation of A if there exists a nonzero derivation $D \in Der(A)$ such that*

$$F(x \cdot y) = F(x) \cdot y + x \cdot D(y) \tag{4.1}$$

for all $x,y \in A$.

Clearly, any derivation is a generalized derivation but the converse is, in general, not true. Hence, it is clear by the above definition that a generalized derivation composes a derivation and a left multiplier, which is a linear map f such that $f(xy) = f(x)y$. A simple example of generalized derivations is $F(x) = ax + xb$, where a and b are fixed elements of A. We will call such maps generalized inner derivations since they generalize inner derivations $x \to [a,x] = ax - xa$. On the other hand, since $H_{\alpha,\beta}$ is a ring with the identity e_0, it follows that by putting $x = e_0$ in (4.1) we get

$$F(y) = F(e_0) \cdot y + D(y)$$

for all $y \in H_{\alpha,\beta}$. This means every generalized derivation of $H_{\alpha,\beta}$ is an inner generalized derivation if and only if $Der(H_{\alpha,\beta}) = ad(H_{\alpha,\beta})$, i.e. every derivation of $H_{\alpha,\beta}$ is inner. Hence, we can extract using inner derivations of $H_{\alpha,\beta}$ all generalized inner derivations and determine in which case the quaternion algebra in question possesses a generalized derivation that is not inner.

Generalized inner derivations have been primarily studied on operator algebras. It follows at once from the preceding definition that knowing $Der(A)$ is a crucial step in order to determine generalized derivations. Since we already know $Der(H_{\alpha,\beta})$ we can find linear maps satisfying the equation in (4.1) to reveal generalized derivations.

We would like to note that unlike the procedure for finding derivations of $H_{\alpha,\beta}$ there is no need to verify (4.1) for the products e_ie_j between quaternionic units to determine the matrix representation of F. See Theorem 4.2 below. Hence, generalized derivations for quaternion algebra are easier to obtain once we know derivations of the associative algebra $H_{\alpha,\beta}$.

Theorem 4.2 *Let $F : H_{\alpha,\beta} \to H_{\alpha,\beta}$ be a generalized derivation of $H_{\alpha,\beta}$ and $\mathfrak{B}(H_{\alpha,\beta}) = \{e_0,e_1,e_2,e_3\}$ denote its standard basis. Then the matrix representation $[F]$ of F is as follows:*

$$[F] = \begin{pmatrix} f_{11} & -\alpha f_{12} & -\beta f_{13} & -\alpha\beta f_{14} \\ f_{12} & f_{11} & -\beta f_{14} & \beta f_{13} \\ f_{13} & \alpha f_{14} & f_{11} & -\alpha f_{12} \\ f_{14} & -f_{13} & f_{12} & f_{11} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\beta}{\alpha}a & -\beta b \\ 0 & a & d & -\alpha c \\ 0 & b & c & d \end{pmatrix},$$

where $f'_{ij}s$ and a,b,c,d are all real numbers such that $d = d(\beta) \neq 0$ if $\beta = 0$ and $d = 0$ otherwise.

Proof Let $[F]=(f_{ij})^T$ be the 4×4 matrix whose entries are defined by the following equations:

$$\begin{aligned} F(e_0) &= f_{11}e_0+f_{12}e_1+f_{13}e_2+f_{14}e_3, \\ F(e_1) &= f_{21}e_0+f_{22}e_1+f_{23}e_2+f_{24}e_3, \\ F(e_2) &= f_{31}e_0+f_{32}e_1+f_{33}e_2+f_{34}e_3, \\ F(e_3) &= f_{41}e_0+f_{42}e_1+f_{43}e_2+f_{44}e_3. \end{aligned}$$

It is clear that each column of the above matrix is an element of $H_{\alpha,\beta}$. A simple observation yields that

$$F(e_i)=F(e_0\cdot e_i)=F(e_0)\cdot e_i+e_0\cdot D(e_i) \tag{4.2}$$

for some derivation $D\in Der(H_{\alpha,\beta})$. Hence, we have $F(e_0)=F(e_0)+D(e_0)$, from which we cannot deduce explicitly the entries located in the first column of $[F]$. Nonetheless, we obtain from equation (4.2) with $i=1,2$, and 3 the following relations:

$$\begin{aligned} f_{21} &= -\alpha f_{12}, \quad f_{22}=f_{11}, \quad f_{23}=\alpha f_{14}+d_{23}, \quad f_{24}=-f_{13}+d_{24}, \\ f_{31} &= -\beta f_{13}, \quad f_{32}=-\beta f_{14}-\frac{\beta}{\alpha}d_{23}, \quad f_{33}=f_{11}+d_{33}, \quad f_{34}=f_{12}+d_{34}, \\ f_{41} &= -\alpha\beta f_{14}, \quad f_{42}=\beta f_{13}-\beta d_{24}, \quad f_{43}=-\alpha f_{12}-\alpha d_{34}, \quad f_{44}=f_{11}+d_{33}. \end{aligned}$$

□

Example 4.3 *Let D denote the derivation of H_R such that $\alpha=\beta/\alpha=1$, $d_{23}=-d_{32}=1$, and $d_{ij}=0$ otherwise. Select $f_{11}=1$ and $f_{ij}=0$ otherwise. Then*

$$F(x)=x_0e_0+(x_1-x_2)e_1+(x_1+x_2)e_2+x_3e_3$$

gives a generalized derivation of H_R.

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