Turkish Journal of Mathematics

Volume 43 | Number 5 Article 43

1-1-2019

Radii of starlikeness and convexity of q-Mittag-Leffler functions

EVRİM TOKLU

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2610 – 2630
© TÜBİTAK
doi:10.3906/mat-1907-54

Radii of starlikeness and convexity of q-Mittag-Leffler functions

Evrim TOKLU*
Department of Mathematics, Faculty of Education, Ağrı İbrahim Çeçen University, Ağrı, Turkey

Received: 11.07.2019 • **Accepted/Published Online:** 06.09.2019 • **Final Version:** 28.09.2019

Abstract: In this paper we deal with the radii of starlikeness and convexity of the q-Mittag–Leffler function for three different kinds of normalization by making use of their Hadamard factorization in such a way that the resulting functions are analytic in the unit disk of the complex plane. By applying Euler–Rayleigh inequalities for the first positive zeros of these functions tight lower and upper bounds for the radii of starlikeness of these functions are obtained. The Laguerre–Pólya class of real entire functions plays a pivotal role in this investigation.

Key words: q-Mittag-Leffler functions, univalent, starlike and convex functions, radius of starlikeness and convexity, Laguerre–Pólya class of entire functions

1. Introduction and main results

Frank Hilton Jackson, an English mathematician, studied what is today known as q-calculus. In particular, he investigated some q-functions and the q-analogs of derivatives and integrals [21]. In spite of the fact that Jackson started his studies introducing the q-difference operator, it is possible to say that this q-difference operator goes back to Euler and may go back to Heine, and it was reintroduced by Jackson in [21]. For this reason, the q-difference operator is also called as the Euler–Heine–Jackson operator. After these studies, q-calculus started to appear in the generalization of many subjects, such as hypergeometric series, complex analysis, and particle physics. Because of the vast potential of its applications in solving problems in physical, engineering, and earth sciences, there has been vivid interest in q-calculus from the point of view of geometric function theory. In [19], Ismail et al. introduced and investigated the generalized class of starlike functions by making use of the q-difference operator. Some interesting applications of q-calculus seen in geometric function theory can be found in [1, 5, 24, 27] and the references therein. Recently, in [2], Aktaş and Baricz determined bounds for radii of starlikeness of some q-Bessel functions. Also, in [25], Srivastava and Bansal studied the close-to-convexity of a certain family of q-Mittag-Leffler functions.

The aim of the present investigation is to determine, by using the method of Baricz et al. (see [9, 12, 13]), the radii of starlikeness and convexity of q-Mittag-Leffler functions. Some intriguing applications of the technique of Baricz, which gives us a much simpler approach to determining some geometric properties of special functions, can be found in [3, 4, 11, 14, 28] and the references therein.

Before starting to present our main results, we would like to draw attention to some basic concepts needed for building our main results. For $r > 0$ we denote by $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ the open disk of radius r centered

*Correspondence: evrimtoklu@gmail.com
2010 *AMS Mathematics Subject Classification:* 30C45, 30C15,33D05, 33D15

at the origin. Let $f:\mathbb{D}_r\to\mathbb{C}$ be the function defined by

$$f(z)=z+\sum_{n\geq 2}a_nz^n, \tag{1.1}$$

where r is less than or equal to the radius of convergence of the above power series. Denote by $\mathcal{A}$ the class of all analytic functions of the form of Eq. (1.1), i.e. normalized by the conditions $f(0)=f'(0)-1=0$. We say that the function f, defined by Eq. (1.1), is a starlike function in $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and the image domain $f(\mathbb{D}_r)$ is a starlike domain in $\mathbb{C}$ with respect to the origin (see [16] for more details). Analytically, the function f is starlike in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0 \quad \text{for all} \quad z\in\mathbb{D}_r.$$

For $\alpha\in[0,1)$ we say that the function f is starlike of order α in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \quad \text{for all} \quad z\in\mathbb{D}_r.$$

The radius of starlikeness of order α of function f is defined as the real number

$$r_\alpha^\star(f)=\sup\left\{r>0\,\middle|\,\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha \text{ for all } z\in\mathbb{D}_r\right\}.$$

Note that $r^\star(f)=r_0^\star(f)$ is in fact the largest radius such that the image region $f(\mathbb{D}_{r^\star(f)})$ is a starlike domain with respect to the origin. The function f, defined by Eq. (1.1), is convex in disk $\mathbb{D}_r$ if f is univalent in $\mathbb{D}_r$, and the image domain $f(\mathbb{D}_r)$ is a convex domain in $\mathbb{C}$. Analytically, function f is convex in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \quad \text{for all} \quad z\in\mathbb{D}_r.$$

For $\alpha\in[0,1)$ we say that function f is convex of order α in $\mathbb{D}_r$ if and only if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad \text{for all} \quad z\in\mathbb{D}_r.$$

We shall denote the radius of convexity of order α of the function f by the real number

$$r_\alpha^c(f)=\sup\left\{r>0\,\middle|\,\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \text{ for all } z\in\mathbb{D}_r\right\}.$$

Note that $r^c(f)=r_0^c(f)$ is the largest radius such that the image region $f(\mathbb{D}_{r^c(f)})$ is a convex domain.

We recall that a real entire function q belongs to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented in the form

$$q(x)=cx^me^{-ax^2+bx}\prod_{n\geq 1}\left(1+\frac{x}{x_n}\right)e^{-\frac{x}{x_n}},$$

with $c,b,x_n\in\mathbb{R}, a\geq 0, m\in\mathbb{N}_0$, and $\sum\frac{1}{{x_n}^2}<\infty$. We note that the class $\mathcal{LP}$ is the complement of the space of polynomials whose zeros are all real in the topology induced by the uniform convergence on the compact sets of the complex plane of polynomials with only real zeros. For more details on the class $\mathcal{LP}$ we refer to [15, p. 703] and to the references therein.

1.1. The three-parameter generalization of the q-Mittag-Leffler function

First of all, we note that throughout of this paper, unless otherwise stated, q is a positive number less than 1 and by the word "basic" we mean a q-analog. Now let us consider the function $E_{\alpha,\beta}(z;q)$, which is called a q-Mittag-Leffler function, defined by

$$E_{\alpha,\beta}(z;q)=\sum_{n\geq 0}\frac{q^{\alpha n(n-1)/2}}{\Gamma_q(n\alpha+\beta)}z^n, \quad (z\in\mathbb{C},\alpha>0,\beta\in\mathbb{C}),$$

where Γ_q is the q-gamma function defined for $z\in\mathbb{C}-\{-n:n\in\mathbb{N}_0\}$ by

$$\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad 0<q<1 \tag{1.2}$$

and

$$(a;q)_0=1,(a;q)_n=\prod_{k=1}^{n}(1-aq^{k-1}),(a;q)_\infty=\prod_{k\geq 1}(1-aq^{k-1}). \tag{1.3}$$

It is worth mentioning that the$q$-gamma function was introduced by Thomae [26] and later by Jackson [20]. Since $\Gamma_q(z)$ has no zeros, then $1/\Gamma_q(z)$ is an entire function with zeros at $z=-n,n\in\mathbb{N}_0$. It is clear that

$$\Gamma_q(n)=\frac{(q;q)_{n-1}}{(1-q)^{n-1}} \quad n\in\mathbb{N}.$$

Moreover, it is well known that for $x>0$, $\Gamma_q(x)$ is the unique logarithmically convex function that satisfies the following relations:

$$\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x), \quad \Gamma_q(1)=1.$$

For more historical remarks about the q-gamma function and its intriguing applications, one can refer to [6, 7, 17] and [18].

We know that the function $z\mapsto E_{\alpha,\beta}(z;q)$ has infinitely many zeros. In [7] the authors proved that for specific values of α and β, $E_{\alpha,\beta}(z;q)$, $0<q<1$, may have only a finite number of nonreal zeros. Moreover, if q satisfies additional conditions then the zeros of the function $z\mapsto E_{\alpha,\beta}(z;q)$ are all real. For more details one can refer to [22] and [23].

Let us consider the function

$$\mathscr{E}_{\gamma,\sigma}(z;q)=E_{2,\gamma+1}(-\sigma^2z;q), \quad (z\in\mathbb{C}),$$

where σ is a fixed positive number and $0\leq\gamma<2$. It is obvious that the function $z\mapsto\mathscr{E}_{\gamma,\sigma}(z;q)$ is of the form

$$\mathscr{E}_{\gamma,\sigma}(z;q)=\sum_{n\geq 0}\frac{(-1)^n\sigma^{2n}q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^n. \tag{1.4}$$

It is worth mentioning that we have the following relations:

$$\mathscr{E}_{0,\sigma}(z^2;q)=\cos(q^{-\frac{1}{2}}\sigma z;q) \text{ and } \mathscr{E}_{1,\sigma}(z^2;q)=\frac{\sin(q^{-1}\sigma z;q)}{q^{-1}\sigma z},$$

where $\sin(.;q)$ and $\cos(.;q)$ stand for the q-trigonometric functions. For some interesting applications of q-trigonometric functions, one can consult [7] and [17]. Also, we note that Annabay and Mansour [7, see Chapter 2] proved that the zeros of the functions $\cos(.;q)$ and $\sin(.;q)$ are real and simple.

We note that throughout this investigation, we shall focus on the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(z;q)$ defined by (1.4). It is easy to check that the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(z^2;q)$ is not of class $\mathcal{A}$. Thus, first we shall perform some natural normalization. We define three functions originating from $\mathscr{E}_{\gamma,\sigma}(.;q)$:

$$f_{\gamma,\sigma}(z;q) = \left(z^{\gamma+1}\Gamma_q(\gamma+1)\mathscr{E}_{\gamma,\sigma}(z^2;q)\right)^{\frac{1}{\gamma+1}},$$
$$g_{\gamma,\sigma}(z;q = z\Gamma_q(\gamma+1)\mathscr{E}_{\gamma,\sigma}(z^2;q),$$
$$h_{\gamma,\sigma}(z;q) = z\Gamma_q(\gamma+1)\mathscr{E}_{\gamma,\sigma}(z;q).$$

It is obvious that each of these functions are of class $\mathcal{A}$. Of course, it infinitely many other normalizations can be written; the main motivation for considering the above ones is the studied normalization in the works on Bessel, q-Bessel, Mittag-Leffler, Struve, Lommel, and Wright functions. Moreover, it is worth mentioning here that in fact

$$f_{\gamma,\sigma}(z;q) = \exp\left[\frac{1}{\gamma+1}\operatorname{Log}(z^{\gamma+1}\Gamma_q(\gamma+1)\mathscr{E}_{\gamma,\sigma}(z^2;q))\right],$$

where Log represents the principle branch of the logarithm function and every many-valued function considered in this paper is taken with the principal branch.

The following lemma, which characterizes the reality of zeros of the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(.;q)$, takes a leading part in building our main results. For some results about the zeros of some q-functions, one can refer to [7, 8] and the references therein.

Lemma 1.1 *[7, p. 220] Let $\varepsilon_{\gamma,\sigma,n}(q)$ be the nth positive zero of the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(z^2;q)$ and $0 \le \gamma < 2$. Then:*

1. *If q satisfies the condition*

$$q^{-1}(1-q)(1-q^{\gamma+1})(1-q^{\gamma+2}) > 1, \quad \gamma \in (0,2), \quad \gamma \neq 1,$$

then the zeros of $z \mapsto \mathscr{E}_{\gamma,\sigma}(z^2;q)$ are all real, simple, and symmetric and its positive zeros lie in the intervals, for $n \in \mathbb{N}$,

$$\varepsilon_{\gamma,\sigma,n}(q) \in \begin{cases} \left(\frac{q^{-n+\frac{3}{2}}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}, \frac{q^{-n+\frac{1}{2}}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}\right), & \gamma \in (0,1), \\ \left(\frac{q^{-n+\frac{5}{2}}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}, \frac{q^{-n+\frac{3}{2}}\sqrt{(1-q^{\gamma+1})(1-q^{\gamma+2})}}{\sigma(1-q)}\right), & \gamma \in (1,2), \end{cases}$$

with one zero in each interval.

2. *If $\gamma \in \{0,1\}$ then $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ has only real, simple, and symmetric zeros such that*

$$\varepsilon_{0,\sigma,n}(q) = \frac{q^{\frac{1}{2}}x_m}{\sigma} \quad \text{and} \quad \varepsilon_{1,\sigma,n}(q) = \frac{qy_m}{\sigma} \quad (m \in \mathbb{N}),$$

where x_m and y_m are, respectively, the positive zeros of the functions $\cos(z;q)$ and $\sin(z;q)$.

The following lemma, which we believe is of independent interest, plays a pivotal role in proving our main results, which are related to radii of starlikeness and convexity of functions $f_{\gamma,\sigma}(z;q)$, $g_{\gamma,\sigma}(z;q)$, and $h_{\gamma,\sigma}(z;q)$.

Lemma 1.2 *Let σ be a fixed positive real number, $0 \leq \gamma < 2$. Moreover, under the conditions of Lemma 1.1 the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(z^2;q)$ has infinitely many zeros, which are all real. Denoting by $\varepsilon_{\gamma,\sigma,n}(q)$ the nth positive zero of $z \mapsto \mathscr{E}_{\nu,c}(z^2;q)$, under the same conditions the Weierstrassian decomposition*

$$\mathscr{E}_{\gamma,\sigma}(z^2;q) = \frac{1}{\Gamma_q(\gamma+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)}\right) \tag{1.5}$$

is fulfilled, and this product is uniformly convergent on compact subsets of the complex plane. Moreover, if we denote by $\xi_{\gamma,\sigma,n}(q)$ the nth positive zero of $\Psi'_{\gamma,\sigma}(z;q)$, where $\Psi_{\gamma,\sigma}(z;q) = z^{\gamma+1}\mathscr{E}_{\gamma,\sigma}(z^2;q)$, then positive zeros of $\varepsilon_{\gamma,\sigma,n}(q)$ and $\xi_{\gamma,\sigma,n}(q)$ are interlaced. In other words, the zeros satisfy the following chain of inequalities:

$$\xi_{\gamma,\sigma,1}(q) < \varepsilon_{\gamma,\sigma,1}(q) < \xi_{\gamma,\sigma,2}(q) < \varepsilon_{\gamma,\sigma,2}(q) < \dots.$$

Proof As clearly stated in Lemma 1.1 the function $z \mapsto \mathscr{E}_{\gamma,\sigma}(z^2;q)$ has only real zeros under the condition of the same lemma. Next, we need to calculate the growth order of the function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$. We have

$$\rho(\mathscr{E}_{\gamma,\sigma}(z^2;q)) = \limsup_{n\to\infty} \frac{n\log n}{-\log|c_n|},$$

where c_n stands for the coefficient of z^{2n} stated in (1.4), i.e.

$$c_n = \frac{(-\sigma^2)^n q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}, \quad (n \in \mathbb{N}_0).$$

Hence,

$$-\log|c_n| = -n(n-1)\log q - 2n\log\sigma + \log|\Gamma_q(2n+\gamma+1)|.$$

Making use of (1.3) and the definition of the q-gamma function, c.f. (1.2), we get

$$\begin{aligned} \log|\Gamma_q(2n+\gamma+1)| &= \log\left|\frac{(q;q)_\infty}{(q^{2n+\gamma+1};q)_\infty}(1-q)^{-2n-\gamma}\right| \\ &= \log(q;q)_\infty - (2n+\gamma)\log(1-q) - \log\left|(q^{2n+\gamma+1};q)_\infty\right|, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \log\left|(q^{2n+\gamma+1};q)_\infty\right| &= \log\left(\prod_{k\geq 0}\left|1-q^{2n+\gamma+k+1}\right|\right) = \log\left(\lim_{m\to\infty}\prod_{k=0}^{m}\left|1-q^{2n+\gamma+k+1}\right|\right) \\ &= \lim_{m\to\infty}\sum_{k=0}^{m}\log\left|1-q^{2n+\gamma+k+1}\right| = \sum_{k\geq 0}\log\left|1-q^{2n+\gamma+k+1}\right|. \end{aligned}$$

Since

$$\log\left|1-q^{2n+\gamma+k+1}\right| \leq \log\left(1+\left|q^{2n+\gamma+k+1}\right|\right) \leq \left|q^{2n+\gamma+k+1}\right| = q^{2n+\gamma+k+1},$$

we obtain

$$\sum_{k\geq 0}\log\left|1-q^{2n+\gamma+k+1}\right|\leq\sum_{k\geq 0}q^{2n+\gamma+k+1}=\frac{q^{2n+\gamma+1}}{1-q}.$$

Consequently, we get

$$\lim_{n\to\infty}\frac{\log\left|(q^{2n+\gamma+1};q)_\infty\right|}{n\log n}=0,$$

which implies that

$$\lim_{n\to\infty}\frac{\log\left|\Gamma_q(2n+\gamma+1)\right|}{n\log n}=0.$$

Taking into account that

$$\lim_{n\to\infty}\frac{n-1}{\log n}=\infty,$$

we obtain

$$\lim_{n\to\infty}\frac{n\log n}{-\log|c_n|}=0,$$

which implies that $\rho(\mathscr{E}_{\gamma,\sigma}(z^2;q))=0$. Moreover, it is well known that the finite growth order ρ of an entire function is not equal to a positive integer and then the function has infinitely many zeros. That is to say, the function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ given in (1.4) has infinite zeros, which are all real and simple. In this case, by virtue of the Hadamard theorem on growth order of the entire function, it follows that its infinite product representation is exactly what we have in Lemma 1.2. This means that the function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$ of entire functions. As a natural consequence of this, we deduce that the function $z\mapsto\Psi_{\gamma,\sigma}(z;q)$ belongs also to the Laguerre–Pólya class $\mathcal{LP}$. Since $\mathcal{LP}$ is closed differentiation the function $z\mapsto\Psi'_{\gamma,\sigma}(z;q)$ belongs also to the class $\mathcal{LP}$. Hence, the function $z\mapsto\Psi'_{\gamma,\sigma}(z;q)$ has only real zeros under the same conditions. It is important to mention that throughout this paper, for the sake of simplicity, we use the notation $\lambda_{\gamma,\sigma}(z;q)=\mathscr{E}_{\gamma,\sigma}(z^2;q)$. Now, with the aid of the infinite product representation, we get

$$\frac{\Psi'_{\nu,c}(z;q)}{\Psi_{\nu,c}(z;q)}=\frac{\gamma+1}{z}+\frac{\lambda'_{\gamma,\sigma}(z;q)}{\lambda_{\gamma,\sigma}(z;q)}=\frac{\gamma+1}{z}+\sum_{n\geq 1}\frac{2z}{z^2-\varepsilon^2_{\gamma,\sigma,n}(q)}. \tag{1.7}$$

Differentiating both sides of Eq. (1.7), we arrive at

$$\frac{d}{dz}\left(\frac{\Psi'_{\nu,c}(z;q)}{\Psi_{\nu,c}(z;q)}\right)=-\frac{\gamma+1}{z^2}-2\sum_{n\geq 1}\frac{z^2+\varepsilon^2_{\gamma,\sigma,n}(q)}{\left(z^2-\varepsilon^2_{\gamma,\sigma,n}(q)\right)^2},\quad z\neq\varepsilon_{\gamma,\sigma,n}(q).$$

It is clear that the expression on the right-hand side is real and negative for the same assumptions of the lemma. That is to say, for each real z, $\frac{\Psi'_{\nu,c}(z;q)}{\Psi_{\nu,c}(z;q)}<0$, which implies that the quotient $\frac{\Psi'_{\nu,c}}{\Psi_{\nu,c}}$ is a strictly decreasing function from $+\infty$ to $-\infty$ as z increases through real values over the open interval $(\varepsilon_{\gamma,\sigma,n}(q),\varepsilon_{\gamma,\sigma,n+1}(q))$, $n\in\mathbb{N}$. Hence, between any two zeros of the function $\lambda_{\gamma,\sigma}(z;q)$ there must be precisely one of $\Psi'_{\gamma,\sigma}(z;q)$. □

1.2. Radii of starlikeness of the q-Mittag-Leffler functions

This section is devoted to determining the radii of starlikeness of the normalized forms of the q-Mittag-Leffler functions, i.e. of $f_{\gamma,\sigma}(z;q)$, $g_{\gamma,\sigma}(z;q)$, and $h_{\gamma,\sigma}(z;q)$. In addition, in this section we aim to give some tight lower and upper bounds for the radii of starlikeness of order zero for these functions.

Theorem 1.3 *Let $\alpha \in [0,1)$, and with the conditions of Lemma 1.2 the following assertions hold true:*

a. *The radius of starlikeness of order α of the function $f_{\gamma,\sigma}$ is $r_\alpha^\star(f_{\gamma,\sigma}(z;q)) = x_{\gamma,\sigma,1}(q)$, where $x_{\gamma,\sigma,1}(q)$ stands for the smallest positive zero of the equation*

$$r\lambda'_{\gamma,\sigma}(r;q) - (\gamma+1)(\alpha-1)\lambda_{\gamma,\sigma}(r;q) = 0.$$

b. *The radius of starlikeness of order α of the function $g_{\gamma,\sigma}$ is $r_\alpha^\star(g_{\gamma,\sigma}(z;q)) = y_{\gamma,\sigma,1}(q)$, where $y_{\gamma,\sigma,1}(q)$ stands for the smallest positive zero of the equation*

$$r\lambda'_{\gamma,\sigma}(r;q) - (\alpha-1)\lambda_{\gamma,\sigma}(r;q) = 0.$$

c. *The radius of starlikeness of order α of the function $h_{\gamma,\sigma}$ is $r_\alpha^\star(h_{\gamma,\sigma}(z;q)) = z_{\gamma,\sigma,1}(q)$, where $z_{\gamma,\sigma,1}$ stands for the smallest positive zero of the equation*

$$\sqrt{r}\lambda'_{\gamma,\sigma}(\sqrt{r};q) - 2(\alpha-1)\lambda_{\gamma,\sigma}(\sqrt{r};q) = 0.$$

Proof We need to show that the inequalities

$$\operatorname{Re}\left(\frac{zf'_{\gamma,\sigma}(z;q)}{f_{\gamma,\sigma}(z;q)}\right) \geq \alpha, \quad \operatorname{Re}\left(\frac{zg'_{\gamma,\sigma}(z;q)}{g_{\gamma,\sigma}(z;q)}\right) \geq \alpha, \quad \text{and} \quad \operatorname{Re}\left(\frac{zh'_{\gamma,\sigma}(z;q)}{h_{\gamma,\sigma}(z;q)}\right) \geq \alpha \tag{1.8}$$

hold for $z \in \mathbb{D}_{r_\alpha^\star(f_{\gamma,\sigma})}$, $z \in \mathbb{D}_{r_\alpha^\star(g_{\gamma,\sigma})}$, and $z \in \mathbb{D}_{r_\alpha^\star(h_{\gamma,\sigma})}$, respectively, and each of the above inequalities does not hold in any larger disk. Recall that under the corresponding conditions the zeros of the q-Mittag-Leffler function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ are all real and simple. Hence, in light of Lemma 1.2, the q-Mittag-Leffler function has the infinite product representation given by

$$\mathscr{E}_{\gamma,\sigma}(z^2;q) = \frac{1}{\Gamma_q(\gamma+1)}\prod_{n\geq1}\left(1-\frac{z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)}\right)$$

and this infinite product is uniformly convergent on each compact subset of $\mathbb{C}$. Taking into account the fact that we use the notation $\lambda_{\gamma,\sigma}(z;q) = \mathscr{E}_{\gamma,\sigma}(z^2;q)$, and by logarithmic differentiation, we get

$$\frac{\lambda'_{\gamma,\sigma}(z;q)}{\lambda_{\gamma,\sigma}(z;q)} = -\sum_{n\geq1}\frac{2z}{\varepsilon^2_{\gamma,\sigma,n}(q)-z^2},$$

which implies that

$$\frac{zf'_{\gamma,\sigma}(z;q)}{f_{\gamma,\sigma}(z;q)} = 1 - \frac{1}{\gamma+1}\sum_{n\geq1}\frac{2z}{\varepsilon^2_{\gamma,\sigma,n}(q)-z^2}, \quad \frac{zg'_{\gamma,\sigma}(z;q)}{g_{\gamma,\sigma}(z;q)} = 1 - \sum_{n\geq1}\frac{2z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)-z^2}$$

and

$$\frac{zh'_{\gamma,\sigma}(z;q)}{h_{\gamma,\sigma}(z;q)} = 1 - \sum_{n\geq 1} \frac{2z}{\varepsilon^2_{\gamma,\sigma,n}(q) - z}.$$

From [12], we know that if $z \in \mathbb{C}$ and $\theta \in \mathbb{R}$ are such that $\theta > |z|$, then

$$\frac{|z|}{\theta - |z|} \geq \operatorname{Re}\left(\frac{z}{\theta} - z\right). \tag{1.9}$$

By virtue of the inequality (1.9) we deduce that the inequality

$$\frac{|z|^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - |z|^2} \geq \operatorname{Re}\left(\frac{z^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - z^2}\right)$$

holds under the conditions of Lemma 1.2, $n \in \mathbb{N}$, and $|z| < \varepsilon_{\gamma,\sigma,1}(q)$, and therefore under the same conditions we get

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf'_{\gamma,\sigma}(z;q)}{f_{\gamma,\sigma}(z;q)}\right) &= 1 - \frac{1}{\gamma+1}\operatorname{Re}\left(\sum_{n\geq 1}\frac{2z^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - z^2}\right) \\
&\geq 1 - \frac{1}{\gamma+1}\sum_{n\geq 1}\frac{2|z|^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - |z|^2} = \frac{|z|\, f_{\gamma,\sigma}(|z|;q)}{f_{\gamma,\sigma}(|z|;q)}, \\
\operatorname{Re}\left(\frac{zg'_{\gamma,\sigma}(z;q)}{g_{\gamma,\sigma}(z;q)}\right) &= 1 - \operatorname{Re}\left(\sum_{n\geq 1}\frac{2z^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - z^2}\right) \geq 1 - \sum_{n\geq 1}\frac{2|z|^2}{\varepsilon^2_{\gamma,\sigma,n}(q) - |z|^2} = \frac{|z|\, g_{\gamma,\sigma}(|z|;q)}{g_{\gamma,\sigma}(|z|;q)}, \\
\operatorname{Re}\left(\frac{zh'_{\gamma,\sigma}(z;q)}{h_{\gamma,\sigma}(z;q)}\right) &= 1 - \operatorname{Re}\left(\sum_{n\geq 1}\frac{z}{\varepsilon^2_{\gamma,\sigma,n}(q) - z}\right) \geq 1 - \sum_{n\geq 1}\frac{|z|}{\varepsilon^2_{\gamma,\sigma,n}(q) - |z|} = \frac{|z|\, h_{\gamma,\sigma}(|z|;q)}{h_{\gamma,\sigma}(|z|;q)},
\end{aligned}$$

where equalities occur only when $z = |z| = r$. The minimum principle for harmonic functions and the previous inequalities imply that the corresponding inequalities given in (1.8) are valid if and only if we have $|z| < x_{\gamma,\sigma,1}(q)$, $|z| < y_{\gamma,\sigma,1}(q)$, and $|z| < z_{\gamma,\sigma,1}(q)$, respectively, where $x_{\gamma,\sigma,1}(q)$, $y_{\gamma,\sigma,1}(q)$, and $z_{\gamma,\sigma,1}(q)$ are the smallest positive roots of the following equalities:

$$\operatorname{Re}\left(\frac{rf'_{\gamma,\sigma}(r;q)}{f_{\gamma,\sigma}(r;q)}\right) = \alpha, \quad \operatorname{Re}\left(\frac{rg'_{\gamma,\sigma}(r;q)}{g_{\gamma,\sigma}(r;q)}\right) = \alpha, \quad \text{and} \quad \operatorname{Re}\left(\frac{rh'_{\gamma,\sigma}(r;q)}{h_{\gamma,\sigma}(r;q)}\right) = \alpha,$$

which imply that

$$r\lambda'_{\gamma,\sigma}(r;q) - (\gamma+1)(\alpha-1)\lambda_{\gamma,\sigma}(r;q) = 0, \quad r\lambda'_{\gamma,\sigma}(r;q) - (\alpha-1)\lambda_{\gamma,\sigma}(r;q) = 0,$$

and

$$\sqrt{r}\lambda'_{\gamma,\sigma}(\sqrt{r};q) - 2(\alpha-1)\lambda_{\gamma,\sigma}(\sqrt{r};q) = 0.$$

□

The following theorem gives some tight lower and upper bounds for the radii of starlikeness of the functions seen in the above theorem.

Theorem 1.4 *Let the conditions of Lemma 1.2 remain valid.*

a. *The radius of starlikeness* $r^{\star}(f_{\gamma,\sigma}(z;q))$ *satisfies the inequalities*

$$\frac{\sigma^2(\gamma+3)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+3)}-\frac{2\sigma^2q^2(\gamma+5)\Gamma_q(\gamma+3)}{(\gamma+3)\Gamma_q(\gamma+5)}<(r^{\star}(f_{\gamma,\sigma}(z;q)))^{-2}<\frac{\sigma^2(\gamma+3)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+3)}.$$

b. *The radius of starlikeness* $r^{\star}(g_{\gamma,\sigma}(z;q))$ *satisfies the inequalities*

$$\frac{\Gamma_q(\gamma+3)}{3\sigma^2\Gamma_q(\gamma+1)}<(r^{\star}(g_{\gamma,\sigma}(z;q)))^2<\frac{3\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(9\Gamma_q(\gamma+1)\Gamma_q(\gamma+5)-10q^2\Gamma_q^2(\gamma+3)\right)}.$$

c. *The radius of starlikeness* $r^{\star}(h_{\gamma,\sigma}(z;q))$ *satisfies the inequalities*

$$\frac{\Gamma_q(\gamma+3)}{2\sigma^2\Gamma_q(\gamma+1)}<r^{\star}(h_{\gamma,\sigma}(z;q))<\frac{\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(2\Gamma_q(\gamma+1)\Gamma_q(\gamma+5)-3q^2\Gamma_q^2(\gamma+3)\right)}.$$

Proof

a. The radius of starlikeness of the normalized q-Mittag-Leffler function $f_{\gamma,\sigma}(z;q)$ coincides with the radius of starlikeness of the function $\Psi_{\gamma,\sigma}(z;q)=z^{\gamma+1}\lambda_{\gamma,\sigma}(z;q)$. The infinite series representation of the function $\Psi'_{\gamma,\sigma}(z;q)$ and its derivative are as follows:

$$\Psi'_{\gamma,\sigma}(z;q)=\sum_{n\geq 0}\frac{(-1)^n\sigma^{2n}(2n+\gamma+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^{2n+\gamma} \tag{1.10}$$

and

$$\Psi''_{\gamma,\sigma}(z;q)=\sum_{n\geq 0}\frac{(-1)^n\sigma^{2n}(2n+\gamma+1)(2n+\gamma)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^{2n+\gamma-1}. \tag{1.11}$$

By means of Lemma 1.2 the function $z\mapsto\Psi_{\gamma,\sigma}(z;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$. Because of the fact that this class of functions is closed under differentiation, $z\mapsto\Psi'_{\gamma,\sigma}(z;q)$ also belongs to the Laguerre–Pólya class $\mathcal{LP}$. This means that the zeros of the function $z\mapsto\Psi'_{\gamma,\sigma}(z;q)$ are all real, and in fact due to Lemma 1.2 they are interlaced with the zeros of $z\mapsto\Psi_{\gamma,\sigma}(z;q)$. Therefore, $\Psi'_{\gamma,\sigma}(z;q)$ can be represented by the following infinite product form:

$$\Psi'_{\gamma,\sigma}(z;q)=\frac{(\gamma+1)z^{\gamma}}{\Gamma_q(\gamma+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\xi^2_{\gamma,\sigma,n}(q)}\right). \tag{1.12}$$

If we take the logarithmic derivative of both sides of (1.12) for $|z|<\xi_{\gamma,\sigma,1}$, we obtain

$$\frac{z\Psi''_{\gamma,\sigma}(z;q)}{\Psi'_{\gamma,\sigma}(z;q)}-\gamma=-\sum_{n\geq 1}\frac{2z^2}{\xi^2_{\gamma,\sigma,n}(q)-z^2}=-2\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^{2k+2}}{\xi^{2k+2}_{\gamma,\sigma,n}(q)}=-2\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^{2k+2}}{\xi^{2k+2}_{\gamma,\sigma,n}(q)}=-2\sum_{k\geq 0}\kappa_{k+1}z^{2k+2}, \tag{1.13}$$

where $\kappa_k = \sum_{n\geq 1} \xi_{\gamma,\sigma,n}^{-2k}(q)$. On the other hand, by making use of (1.10) and (1.11), we obtain

$$\frac{z\Psi''_{\gamma,\sigma}(z;q)}{\Psi'_{\gamma,\sigma}(z;q)} = \sum_{n\geq 0} a_n z^{2n} \Bigg/ \sum_{n\geq 0} b_n z^{2n}, \tag{1.14}$$

where

$$a_n = \frac{(-1)^n \sigma^{2n}(2n+\gamma+1)(2n+\gamma)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)} \quad \text{and} \quad b_n = \frac{(-1)^n \sigma^{2n}(2n+\gamma+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}.$$

By comparing the coefficients of (1.13) and (1.14) we get

$$a_0 = \gamma b_0, \quad a_1 = \gamma b_1 - 2b_0\kappa_1, \quad a_2 = \gamma b_2 - 2b_1\kappa_1 - 2b_0\kappa_2,$$

which gives the following Rayleigh sums:

$$\kappa_1 = \frac{\sigma^2(\gamma+3)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+3)} \quad \text{and} \quad \kappa_2 = \frac{\sigma^4(\gamma+3)^2\Gamma_q^2(\gamma+1)}{(\gamma+1)^2\Gamma_q^2(\gamma+3)} - \frac{2\sigma^4 q^2(\gamma+5)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+5)}.$$

By using the Euler–Rayleigh inequalities

$$\kappa_k^{\frac{-1}{k}} < \xi_{\gamma,\sigma,1}^2 < \frac{\kappa_k}{\kappa_{k+1}}$$

for $k=1$, we have

$$\frac{\sigma^2(\gamma+3)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+3)} - \frac{2\sigma^2 q^2(\gamma+5)\Gamma_q(\gamma+3)}{(\gamma+3)\Gamma_q(\gamma+5)} < (r^\star(f_{\gamma,\sigma}(z;q)))^{-2} < \frac{\sigma^2(\gamma+3)\Gamma_q(\gamma+1)}{(\gamma+1)\Gamma_q(\gamma+3)},$$

which is the desired result.

b. If we take $\alpha = 0$ in the second part of Lemma 1.4, then we have that the radius of starlikeness of order zero of the function $g_{\gamma,\sigma}(z;q)$ is the smallest positive root of the equation $(z\lambda_{\gamma,\sigma}(z;q))' = 0$. Therefore, we shall study the first positive zero of

$$\varphi_{\gamma,\sigma}(z;q) = (z\lambda_{\gamma,\sigma}(z;q))' = \sum_{n\geq 0} \frac{(-1)^n \sigma^{2n}(2n+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)} z^{2n}. \tag{1.15}$$

We know that the function $z \mapsto \lambda_{\gamma,\sigma}(z;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$, which is closed under differentiation. Therefore, we get that the function $z \mapsto \varphi_{\gamma,\sigma}(z;q)$ belongs to the Laguerre–Pólya class, and hence all its zeros are real. Let us denote $\theta_{\gamma,\sigma,n}(q)$ as the nth positive zero of $z \mapsto \varphi_{\gamma,\sigma}(z;q)$. Since the growth order of the function $z \mapsto \varphi_{\gamma,\sigma}(z;q)$ coincides with the growth order of the q-Mittag-Leffler function itself, it can be written as

$$\varphi_{\gamma,\sigma}(z;q) = \frac{1}{\Gamma_q(\gamma+1)} \prod_{n\geq 1} \left(1 - \frac{z^2}{\theta_{\gamma,\sigma,n}^2(q)}\right). \tag{1.16}$$

Logarithmic differentiation of both sides of (1.16) for $|z| < \theta_{\gamma,\sigma,1}(q)$ gives

$$\frac{\varphi'_{\gamma,\sigma}(z;q)}{\varphi_{\gamma,\sigma}(z;q)} = \sum_{n\geq 1} \frac{-2z}{\theta^2_{\gamma,\sigma,n}(q) - z^2} = \sum_{n\geq 1}\sum_{k\geq 0} \frac{-2z^{2k+1}}{\theta^{2k+2}_{\gamma,\sigma,n}(q)} = \sum_{k\geq 0}\sum_{n\geq 1} \frac{-2z^{2k+1}}{\theta^{2k+2}_{\gamma,\sigma,n}(q)} = -2\sum_{k\geq 0} \chi_{k+1} z^{2k+1}, \tag{1.17}$$

where $\chi_k = \sum_{n\geq 1} \theta^{-2k}_{\gamma,\sigma,n}(q)$. Moreover, with the aid of (1.15), we get

$$\frac{\varphi'_{\gamma,\sigma}(z;q)}{\varphi_{\gamma,\sigma}(z;q)} = -2\sum_{n\geq 0} c_n z^{2n+1} \Big/ \sum_{n\geq 0} d_n z^{2n}, \tag{1.18}$$

where

$$c_n = \frac{(-1)^n \sigma^{2n+2}(n+1)(2n+3)q^{n(n+1)}}{\Gamma_q(2n+\gamma+3)} \quad \text{and} \quad d_n = \frac{(-1)^n \sigma^{2n}(2n+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}.$$

Comparing the coefficients in (1.17) and (1.18) we have that

$$d_0\chi_1 = c_0 \quad \text{and} \quad d_0\chi_2 + d_1\chi_1 = c_1,$$

which yields the following Rayleigh sums:

$$\chi_1 = \frac{3\sigma^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+3)} \quad \text{and} \quad \chi_2 = \frac{9\sigma^4\Gamma_q^2(\gamma+1)}{\Gamma_q^2(\gamma+3)} - \frac{10\sigma^4 q^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+5)}.$$

By making use of the Euler–Rayleigh inequalities

$$\chi_k^{-\frac{1}{k}} < \theta^2_{\gamma,\sigma,1}(q) < \frac{\chi_k}{\chi_{k+1}}$$

for $k = 1$, we obtain

$$\frac{\Gamma_q(\gamma+3)}{3\sigma^2\Gamma_q(\gamma+1)} < (r^\star(g_{\gamma,\sigma}(z;q)))^2 < \frac{3\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(9\Gamma_q(\gamma+1)\Gamma_q(\gamma+5) - 10q^2\Gamma_q^2(\gamma+3)\right)},$$

which is the desired result.

c. If we take $\alpha = 0$ in the third part of Theorem 1.4 we obtain that the radius of starlikeness of order zero of the function $h_{\gamma,\sigma}$ is the smallest positive root of the equation $(z\lambda_{\gamma,\sigma}(\sqrt{z};q))' = 0$. Therefore, we shall focus on the first positive zero of

$$\varphi_{\gamma,\sigma}(z;q) = (z\lambda_{\gamma,\sigma}(\sqrt{z};q))' = \sum_{n\geq 0} \frac{(-1)^n\sigma^{2n}(n+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)} z^n. \tag{1.19}$$

We know that the function $z \mapsto \lambda_{\gamma,\sigma}(z;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$, and consequently we conclude that $z \mapsto \varphi_{\gamma,\sigma}(z;q)$ belongs also to the Laguerre–Pólya class. This means that the zeros of the function $z \mapsto \varphi_{\gamma,\sigma}(z;q)$ are all real. Suppose that $\varsigma_{\gamma,\sigma,n}(q)$ is the nth positive zero of the function $z \mapsto \varphi_{\gamma,\sigma}(z;q)$. Then the infinite product representation of the function $z \mapsto \varphi_{\gamma,\sigma}(z;q)$ can be written as

$$\varphi_{\gamma,\sigma}(z;q) = \frac{1}{\Gamma_q(\gamma+1)} \prod_{n\geq 1}\left(1 - \frac{z}{\varsigma_{\gamma,\sigma,n}(q)}\right). \tag{1.20}$$

Logarithmic differentiation of both sides of (1.20) for $|z| < \varsigma_{\gamma,\sigma,1}(q)$ yields

$$\frac{\varphi'_{\gamma,\sigma}(z;q)}{\varphi_{\gamma,\sigma}(z;q)} = -\sum_{n\geq 1}\frac{1}{\varsigma_{\gamma,\sigma,n}(q)-z} = -\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^k}{\varsigma^{k+1}_{\gamma,\sigma,n}(q)} = -\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^k}{\varsigma^{k+1}_{\gamma,\sigma,n}(q)} = -\sum_{k\geq 0}\delta_{k+1}z^k, \tag{1.21}$$

where $\delta_k = \sum_{n\geq 1}\varsigma^{-k}_{\gamma,\sigma,n}(q)$. On the other hand, with the aid of (1.19), we obtain

$$\frac{\varphi'_{\gamma,\sigma}(z;q)}{\varphi_{\gamma,\sigma}(z;q)} = -\sum_{n\geq 0}u_n z^n \Big/ \sum_{n\geq 0}v_n z^n, \tag{1.22}$$

where

$$u_n = \frac{(-1)^n\sigma^{2n+2}(n+1)(n+2)q^{n(n+1)}}{\Gamma_q(2n+\gamma+3)} \quad\text{and}\quad v_n = \frac{(-1)^n\sigma^{2n}(n+1)q^{n(n-1)}}{\Gamma_q(2n+\gamma+3)}.$$

By comparing the coefficients of (1.21) and (1.22), we arrive at

$$u_0 = v_0\delta_1 \quad\text{and}\quad u_1 = v_0\delta_2 + v_1\delta_1,$$

which gives the following Rayleigh sums:

$$\delta_1 = \frac{2\sigma^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+3)} \quad\text{and}\quad \delta_2 = \frac{4\sigma^4\Gamma_q^2(\gamma+1)}{\Gamma_q^2(\gamma+3)} - \frac{6q^2\sigma^4\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+5)}.$$

By using the Euler–Rayleigh inequalities

$$\delta_k^{-\frac{1}{k}} < \varsigma_{\gamma,\sigma,1}(q) < \frac{\delta_k}{\delta_{k+1}}$$

for $k=1$, we obtain

$$\frac{\Gamma_q(\gamma+3)}{2\sigma^2\Gamma_q(\gamma+1)} < r^\star(h_{\gamma,\sigma}(z;q)) < \frac{\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(2\Gamma_q(\gamma+1)\Gamma_q(\gamma+5) - 3q^2\Gamma_q^2(\gamma+3)\right)}.$$

□

1.3. Radii of convexity of the q-Mittag-Leffler functions

This section is devoted to determining the radii of convexity of order α of the functions $f_{\gamma,\sigma}(z;q)$, $g_{\gamma,\sigma}(z;q)$, and $h_{\gamma,\sigma}(z;q)$. In addition, we find tight lower and upper bounds for the radii of convexity of order zero for the functions $g_{\gamma,\sigma}(z;q)$ and $h_{\gamma,\sigma}(z;q)$.

Theorem 1.5 *Let $\alpha \in [0,1)$, and with the conditions of Lemma 1.2, the following assertions are valid:*

a. *The radius of convexity $r^c_\alpha(f_{\gamma,\sigma}(z;q))$ is the smallest positive root of the transcendental equation*

$$\left(rf'_{\gamma,\sigma}(r;q)\right)' = \alpha f'_{\gamma,\sigma}(r;q).$$

b. *The radius of convexity* $r_\alpha^c\left(g_{\gamma,\sigma}(z;q)\right)$ *is the smallest positive root of the transcendental equation*

$$\left(rg'_{\gamma,\sigma}(r;q)\right)' = \alpha g'_{\gamma,\sigma}(r;q).$$

c. *The radius of convexity* $r_\alpha^c\left(h_{\gamma,\sigma}(z;q)\right)$ *is the smallest positive root of the transcendental equation*

$$\left(rh'_{\gamma,\sigma}(r;q)\right)' = \alpha h'_{\gamma,\sigma}(r;q).$$

Proof

a. It is easy to check that

$$1+\frac{zf''_{\gamma,\sigma}(z;q)}{f'_{\gamma,\sigma}(z;q)} = 1+\frac{z\Psi''_{\gamma,\sigma}(z;q)}{\Psi'_{\gamma,\sigma}(z;q)}+\left(\frac{1}{\gamma+1}-1\right)\frac{z\Psi'_{\gamma,\sigma}(z;q)}{\Psi_{\gamma,\sigma}(z;q)}.$$

Moreover, from the proof of Theorem 1.3, we conclude the following infinite product representations:

$$\Psi_{\gamma,\sigma}(z;q)=\frac{z^{\gamma+1}}{\Gamma_q(\gamma+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)}\right) \quad \text{and} \quad \Psi'_{\gamma,\sigma}(z;q)=\frac{(\gamma+1)z^{\gamma}}{\Gamma_q(\gamma+1)}\prod_{n\geq 1}\left(1-\frac{z^2}{\xi^2_{\gamma,\sigma,n}(q)}\right),$$

where $\varepsilon_{\gamma,\sigma,n}(q)$ and $\xi_{\gamma,\sigma,n}(q)$ stand for the nth positive roots of $z\mapsto \Psi_{\gamma,\sigma}(z;q)$ and $z\mapsto \Psi'_{\gamma,\sigma}(z;q)$, respectively, as in Lemma 1.2. Logarithmic differentiation of both sides of the above infinite representations yields

$$\frac{z\Psi'_{\gamma,\sigma}(z;q)}{\Psi_{\gamma,\sigma}(z;q)}=\gamma+1-\sum_{n\geq 1}\frac{2z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)-z^2} \quad \text{and} \quad \frac{z\Psi''_{\gamma,\sigma}(z;q)}{\Psi'_{\gamma,\sigma}(z;q)}=\gamma-\sum_{n\geq 1}\frac{2z^2}{\xi^2_{\gamma,\sigma,n}(q)-z^2},$$

which gives

$$1+\frac{zf''_{\gamma,\sigma}(z;q)}{f'_{\gamma,\sigma}(z;q)}=1-\left(\frac{1}{\gamma+1}-1\right)\sum_{n\geq 1}\frac{2z^2}{\xi^2_{\gamma,\sigma,n}(q)-z^2}-\sum_{n\geq 1}\frac{2z^2}{\varepsilon^2_{\gamma,\sigma,n}(q)-z^2}.$$

With the aid of the following inequlaity [12, Lemma 2.1],

$$\alpha\,\mathrm{Re}\left(\frac{z}{a-z}\right)-\mathrm{Re}\left(\frac{z}{b-z}\right)\geq \alpha\frac{|z|}{a-|z|}-\frac{|z|}{b-|z|},$$

where $a>b>0,\ \alpha\in[0,1],\ z\in\mathbb{C}$, we obtain for $\gamma\in(0,2)$

$$\mathrm{Re}\left(1+\frac{zf''_{\gamma,\sigma}(z;q)}{f'_{\gamma,\sigma}(z;q)}\right)\geq 1-\left(\frac{1}{\gamma+1}-1\right)\sum_{n\geq 1}\frac{2r^2}{\varepsilon^2_{\gamma,\sigma,n}(q)-r^2}-\sum_{n\geq 1}\frac{2r^2}{\xi^2_{\gamma,\sigma,n}(q)-r^2}, \tag{1.23}$$

for all $z\in\mathbb{D}_{\xi_{\gamma,\sigma,1}}$. It is important to mention that we tacitly used that the zeros of $\varepsilon_{\gamma,\sigma,n}(q)$ and $\xi_{\gamma,\sigma,n}(q)$ interlace, due to Lemma 1.2. In addition, the above deduced inequalities imply for $r\in(0,\xi_{\gamma,\sigma,1}(q))$

$$\inf_{z\in\mathbb{D}_r}\left\{\mathrm{Re}\left(1+\frac{zf''_{\gamma,\sigma}(z;q)}{f'_{\gamma,\sigma}(z;q)}\right)\right\}=1+\frac{rf''_{\gamma,\sigma}(r;q)}{f'_{\gamma,\sigma}(r;q)}.$$

The function $u_{\gamma,\sigma}:(0,\xi_{\gamma,\sigma,1})\to\mathbb{R}$, given by

$$u_{\gamma,\sigma}(r;q)=1+\frac{rf''_{\gamma,\sigma}(r;q)}{f'_{\gamma,\sigma}(r;q)},$$

is strictly decreasing since

$$\begin{aligned}u'_{\gamma,\sigma}(r;q)&=-\left(\frac{1}{\gamma+1}-1\right)\sum_{n\geq1}\frac{4r\varepsilon^2_{\gamma,\sigma,n}(q)}{\left(\varepsilon^2_{\gamma,\sigma,n}(q)-r^2\right)^2}-\sum_{n\geq1}\frac{4r\xi^2_{\gamma,\sigma,n}(q)}{\left(\xi^2_{\gamma,\sigma,n}(q)-r^2\right)^2}\\&<\sum_{n\geq1}\frac{4r\varepsilon^2_{\gamma,\sigma,n}(q)}{\left(\varepsilon^2_{\gamma,\sigma,n}(q)-r^2\right)^2}-\sum_{n\geq1}\frac{4r\xi^2_{\gamma,\sigma,n}(q)}{\left(\xi^2_{\gamma,\sigma,n}(q)-r^2\right)^2}<0\end{aligned}$$

for $z\in(0,\xi_{\gamma,\sigma,1}(q))$, where we used again the interlacing property of the zeros stated in Lemma 1.2. Observe also that $\lim_{r\searrow0}u_{\gamma,\sigma}(r;q)=1$ and $\lim_{r\nearrow\xi_{\gamma,\sigma,1}}=-\infty$, which means that for $z\in\mathbb{D}_{r_1}$ we get

$$\operatorname{Re}\left(1+\frac{zf''_{\gamma,\sigma}(z;q)}{f'_{\gamma,\sigma}(z;q)}\right)>\alpha$$

if and only if r_1 is the unique root of

$$1+\frac{zf''_{\gamma,\sigma}(r;q)}{f'_{\gamma,\sigma}(r;q)}=\alpha$$

situated in $(0,\xi_{\gamma,\sigma,1})$.

b. By virtue of (1.16) we have

$$g'_{\gamma,\sigma}(z;q)=\prod_{n\geq1}\left(1-\frac{z^2}{\theta^2_{\gamma,\sigma,n}(q)}\right).$$

Now, taking logarithmic derivatives on both sides, we obtain

$$1+\frac{zg''_{\gamma,\sigma}(z;q)}{g'_{\gamma,\sigma}(z;q)}=1-\sum_{n\geq1}\frac{2z^2}{\theta^2_{\gamma,\sigma,n}(q)-z^2}.$$

In light of inequality (1.9) we get

$$\operatorname{Re}\left(1+\frac{zg''_{\gamma,\sigma}(z;q)}{g'_{\gamma,\sigma}(z;q)}\right)\geq1-\sum_{n\geq1}\frac{2r^2}{\theta^2_{\gamma,\sigma,n}(q)-r^2},$$

where $|z|=r$. Hence, for $r\in(0,\theta_{\gamma,\sigma,1}(q))$, we obtain

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zg''_{\gamma,\sigma}(z;q)}{g'_{\gamma,\sigma}(z;q)}\right)\right\}=1+\frac{rg''_{\gamma,\sigma}(r;q)}{g'_{\gamma,\sigma}(r;q)}.$$

The function $v_{\gamma,\sigma}:(0,\theta_{\gamma,\sigma,1}(q))\to\mathbb{R}$, defined by

$$v_{\gamma,\sigma}(r;q)=1+\frac{rg''_{\gamma,\sigma}(r;q)}{g'_{\gamma,\sigma}(r;q)},$$

is strictly decreasing and takes limits $\lim_{r\searrow 0} v_{\gamma,\sigma}(r;q)=1$ and $\lim_{r\nearrow\theta_{\gamma,\sigma,1}} v_{\gamma,\sigma}(r;q)=-\infty$. That means that for $z\in\mathbb{D}_{r_2}$ we get

$$\operatorname{Re}\left(1+\frac{zg''_{\gamma,\sigma}(z;q)}{g'_{\gamma,\sigma}(z;q)}\right)>\alpha$$

if and only if r_2 is the unique root of

$$1+\frac{zg''_{\gamma,\sigma}(z;q)}{g'_{\gamma,\sigma}(z;q)}=\alpha$$

situated in $(0,\theta_{\gamma,\sigma,1}(q))$.

c. By virtue of (1.20) we have

$$h'_{\gamma,\sigma}(z;q)=\prod_{n\geq 1}\left(1-\frac{z}{\varsigma_{\gamma,\sigma,n}(q)}\right).$$

If we take logarithmic derivatives on both sides, we obtain

$$1+\frac{zh''_{\gamma,\sigma}(z;q)}{h'_{\gamma,\sigma}(z;q)}=1-\sum_{n\geq 1}\frac{z}{\varsigma_{\gamma,\sigma,n}-z}.$$

Let $r\in(0,\varsigma_{\gamma,\sigma,1})$ be a fixed number. The minimum principle for harmonic functions and inequality (1.9) imply that for $z\in\mathbb{D}_r$ we have

$$\begin{aligned}\operatorname{Re}\left(1+\frac{zh''_{\gamma,\sigma}(z;q)}{h'_{\gamma,\sigma}(z;q)}\right)&=\operatorname{Re}\left(1-\sum_{n\geq 1}\frac{z}{\varsigma_{\gamma,\sigma,n}-z}\right)\geq\min_{|z|=r}\operatorname{Re}\left(1-\sum_{n\geq 1}\frac{z}{\varsigma_{\gamma,\sigma,n}-z}\right)\\&=\min_{|z|=r}\left(1-\sum_{n\geq 1}\operatorname{Re}\frac{z}{\varsigma_{\gamma,\sigma,n}-z}\right)\geq 1-\sum_{n\geq 1}\frac{r}{\varsigma_{\gamma,\sigma,n}(q)-r}\\&=1+\frac{rh''_{\gamma,\sigma}(r;q)}{h'_{\gamma,\sigma}(r;q)}.\end{aligned}$$

Consequently, it follows that

$$\inf_{z\in\mathbb{D}_r}\left\{\operatorname{Re}\left(1+\frac{zh''_{\gamma,\sigma}(z;q)}{h'_{\gamma,\sigma}(z;q)}\right)\right\}=1+\frac{rh''_{\gamma,\sigma}(r;q)}{h'_{\gamma,\sigma}(r;q)}.$$

Now, let r_3 be the smallest positive root of the equation

$$1+\frac{rh''_{\gamma,\sigma}(r;q)}{h'_{\gamma,\sigma}(r;q)}=\alpha. \tag{1.24}$$

For $z\in\mathbb{D}_{r_3}$, we have

$$\operatorname{Re}\left(1+\frac{zh''_{\gamma,\sigma}(z;q)}{h'_{\gamma,\sigma}(z;q)}\right)>\alpha.$$

In order to finish the proof, we need to show that equation (1.24) has a unique root in $(0,\varsigma_{\gamma,\sigma,1}(q))$, but equation (1.24) is equivalent to

$$w_{\gamma,\sigma}(r;q)=1-\alpha-\sum_{n\geq 1}\frac{r}{\varsigma_{\gamma,\sigma,n}(q)-r}=0,$$

and we have

$$\lim_{r\searrow 0}w_{\gamma,\sigma}(r;q)=1-\alpha>0,\qquad \lim_{r\nearrow\varsigma_{\gamma,\sigma,1}}w_{\gamma,\sigma}(r;q)=-\infty.$$

Since the function $w_{\gamma,\sigma}(r;q)$ is strictly decreasing on $(0,\varsigma_{\gamma,\sigma,1}(q))$, it follows that the equation $w_{\gamma,\sigma}(r;q)=0$ has a unique root.

□

The following theorem gives some tight lower and upper bounds for the radii of convexity of the functions seen in the above theorem, i.e. of $g_{\gamma,\sigma}(z;q)$ and $h_{\gamma,\sigma}(z;q)$. It is important to note that in order to prove our main results we are going to make use of Alexander's duality theorem, which has a very simple proof based on the characterization of starlike and convex functions in the unit disc. By means of this theorem one can deduce that the function $f(z)$ is convex if and only if $z\mapsto(zf(z))'$ is starlike. Moreover, from the studies in [9] and [10] we know that the smallest positive zero of $z\mapsto(zf'(z))'$ is the radius of starlikeness of $zf'(z)$. That is why the radius of convexity $r^c(f(z))$ is the smallest positive root of the equation $(zf'(z))'=0$.

Theorem 1.6 *With the same conditions of Lemma 1.2 the following inequalities are valid:*

a. *The radius of convexity $r^c\left(g_{\gamma,\sigma}(z;q)\right)$ satisfies the inequalities*

$$\frac{\Gamma_q(\gamma+3)}{9\sigma^2\Gamma_q(\gamma+1)}<\left(r^c\left(g_{\gamma,\sigma}(z;q)\right)\right)^2<\frac{9\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(81\Gamma_q(\gamma+1)\Gamma_q(\gamma+5)-50q^2\Gamma_q^2(\gamma+3)\right)}.$$

b. *The radius of convexity $r^c\left(h_{\gamma,\sigma}(z;q)\right)$ satisfies the inequalities*

$$\frac{\Gamma_q(\gamma+3)}{4\sigma^2\Gamma_q(\gamma+1)}<r^c(h_{\gamma,\sigma}(z;q))<\frac{2\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(8\Gamma_q(\gamma+1)\Gamma_q(\gamma+5)-9q^2\Gamma_q^2(\gamma+3)\right)}.$$

Proof

a. By using the infinite series representations of the function $\mathscr{E}_{\gamma,\sigma}(z^2;q)$ (see (1.4)) and its derivative we obtain

$$\Phi_{\gamma,\sigma}(z;q)=\left(zg'_{\gamma,\sigma}(z;q)\right)'=\Gamma_q(\gamma+1)\sum_{n\geq 0}\frac{(-1)^n\sigma^{2n}(2n+1)^2q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^{2n}$$

$$=1+\sum_{n\geq 1}\frac{(-1)^n\sigma^{2n}(2n+1)^2q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^{2n}.$$

We know that $g_{\gamma,\sigma}(z;q) \in \mathcal{LP}$ and this in turn implies that $z \mapsto \Phi_{\gamma,\sigma}(z;q)$ belongs also to the Laguerre–Pólya class; consequently, all its zeros are real. Suppose that $\ell_{\gamma,\sigma,n}(q)$ is the nth positive zero of the function $z \mapsto \Phi_{\gamma,\sigma}(z;q)$. Then we deduce that

$$\Phi_{\gamma,\sigma}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z^2}{\ell^2_{\gamma,\sigma,n}(q)}\right). \tag{1.25}$$

Logarithmic differentiation of both sides of (1.25) implies for $|z| < \ell_{\gamma,\sigma,1}(q)$

$$\frac{\Phi'_{\gamma,\sigma}(z;q)}{\Phi_{\gamma,\sigma}(z;q)} = \sum_{n\geq 1}\frac{-2z}{\ell^2_{\gamma,\sigma,n}(q) - z^2} = \sum_{n\geq 1}\sum_{k\geq 0}\frac{-2z^{2k+1}}{\ell^{2k+2}_{\gamma,\sigma,n}(q)} = \sum_{k\geq 0}\sum_{n\geq 1}\frac{-2z^{2k+1}}{\ell^{2k+2}_{\gamma,\sigma,n}(q)} = -2\sum_{k\geq 0}\mu_{k+1}z^{2k+1}, \tag{1.26}$$

where $\mu_k = \sum_{n\geq 1}\ell^{-2k}_{\gamma,\sigma,n}(q)$. On the other hand, we have

$$\frac{\Phi'_{\gamma,\sigma}(z;q)}{\Phi_{\gamma,\sigma}(z;q)} = -2\sum_{n\geq 0} r_n z^{2n+1} \Big/ \sum_{n\geq 0} s_n z^{2n} \tag{1.27}$$

where

$$r_n = \frac{(-1)^n\sigma^{2n+2}(n+1)(2n+3)^2 q^{n(n+1)}}{\Gamma_q(2n+\gamma+3)} \quad \text{and} \quad s_n = \frac{(-1)^n\sigma^{2n}(2n+1)^2 q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}.$$

By comparing the coefficients of (1.26) and (1.27) we have

$$r_0 = \mu_1 s_0 \quad \text{and} \quad r_1 = s_0\mu_2 + s_1\mu_1,$$

which give us the following Rayleigh sums:

$$\mu_1 = \frac{9\sigma^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+3)} \quad \text{and} \quad \mu_2 = \frac{81\sigma^4\Gamma_q^2(\gamma+1)}{\Gamma_q^2(\gamma+3)} - \frac{50\sigma^4 q^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+5)}.$$

By using the Euler–Rayleigh inequalities

$$\mu_k^{-\frac{1}{k}} < \ell^2_{\gamma,\sigma,1}(q) < \frac{\mu_k}{\mu_{k+1}}$$

for $k = 1$, we obtain the following inequalities:

$$\frac{\Gamma_q(\gamma+3)}{9\sigma^2\Gamma_q(\gamma+1)} < \left(r^c\left(g_{\gamma,\sigma}(z;q)\right)\right)^2 < \frac{9\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(81\Gamma_q(\gamma+1)\Gamma_q(\gamma+5) - 50q^2\Gamma_q^2(\gamma+3)\right)},$$

which is desired result.

b. By means of the definition of the function $\mathscr{E}_{\gamma,\sigma}(z;q)$ (see (1.4)) we have

$$\psi_{\gamma,\sigma}(z;q) = \left(zh'_{\gamma,\sigma}(z;q)\right)' = 1 + \sum_{n\geq 1}\frac{(-1)^n\sigma^{2n}(n+1)^2 q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}z^n$$

and consequently

$$\frac{\psi'_{\gamma,\sigma}(z;q)}{\psi_{\gamma,\sigma}(z;q)} = -\sum_{n\geq 0} t_n z^n \Big/ \sum_{n\geq 0} p_n z^n, \tag{1.28}$$

where

$$t_n = \frac{(-1)^n \sigma^{2n+2}(n+1)(n+2)^2 q^{n(n+1)}}{\Gamma_q(2n+\gamma+3)} \quad \text{and} \quad p_n = \frac{(-1)^n \sigma^{2n}(n+1)^2 q^{n(n-1)}}{\Gamma_q(2n+\gamma+1)}.$$

Because of the fact that $h_{\gamma,\sigma}(z;q)$ belongs to the Laguerre–Pólya class $\mathcal{LP}$, it follows that $h'_{\gamma,\sigma}(z;q) \in \mathcal{LP}$, and consequently the function $z \mapsto \psi_{\gamma,\sigma}(z;q)$ belongs also to the Laguerre–Pólya class $\mathcal{LP}$. Hence, all its zeros are real. Assume that $\nu_{\gamma,\sigma,n}(q)$ is the nth positive zero of the function $z \mapsto \psi_{\gamma,\sigma}(z;q)$, and then the following infinite product representation takes place:

$$\psi_{\gamma,\sigma}(z;q) = \prod_{n\geq 1}\left(1 - \frac{z}{\nu_{\gamma,\sigma,n}(q)}\right). \tag{1.29}$$

Logarithmic differentiation of both sides of (1.29) implies for $|z| < \nu_{\gamma,\sigma,1}(q)$

$$\frac{\psi'_{\gamma,\sigma}(z;q)}{\psi_{\gamma,\sigma}(z;q)} = -\sum_{n\geq 1}\frac{1}{\nu_{\gamma,\sigma,n}(q)-z} = -\sum_{n\geq 1}\sum_{k\geq 0}\frac{z^k}{\nu^{k+1}_{\gamma,\sigma,n}(q)} = -\sum_{k\geq 0}\sum_{n\geq 1}\frac{z^k}{\nu^{k+1}_{\gamma,\sigma,n}(q)} = -\sum_{k\geq 0}\varrho_{k+1}z^k, \tag{1.30}$$

where $\varrho_k = \sum_{n\geq 1}\nu^{-k}_{\gamma,\sigma,n}(q)$. By comparing the coefficients of (1.28) and (1.30) we obtain the following Rayleigh sums:

$$\varrho_1 = \frac{4\sigma^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+3)} \quad \text{and} \quad \varrho_2 = \frac{16\sigma^4\Gamma_q^2(\gamma+1)}{\Gamma_q^2(\gamma+3)} - \frac{18\sigma^4 q^2\Gamma_q(\gamma+1)}{\Gamma_q(\gamma+5)}.$$

By making use of the Euler–Rayleigh inequalities

$$\varrho_k^{-\frac{1}{k}} < \nu_{\gamma,\sigma,1}(q) < \frac{\varrho_k}{\varrho_{k+1}}$$

for $k=1$, the following inequalities immediately take place:

$$\frac{\Gamma_q(\gamma+3)}{4\sigma^2\Gamma_q(\gamma+1)} < r^c(h_{\gamma,\sigma}(z;q)) < \frac{2\Gamma_q(\gamma+3)\Gamma_q(\gamma+5)}{\sigma^2\left(8\Gamma_q(\gamma+1)\Gamma_q(\gamma+5) - 9q^2\Gamma_q^2(\gamma+3)\right)},$$

which is desired result.

□

1.4. Some particular cases of the main results

This section is devoted to giving some interesting results corresponding to the main results for some particular cases, in particular for $\gamma = 0$. Of course it is possible to get new results for the different values of $0 \leq \gamma < 2$; however, we omitted them due to their complicated form.

If we take $\gamma = 0$ in Theorem 1.3, we arrive at the following results. It is important to note that for $\gamma = 0$, the radii of starlikeness of the functions $f_{0,\sigma}(z;q)$ and $g_{0,\sigma}(z;q)$ coincide with each other.

Corollary 1.7 *Let $\alpha \in [0,1)$, and with the conditions of Lemma 1.2, the following assertions hold true:*

a. *The radius of starlikeness of order α of the function $f_{0,\sigma}(z;q) = g_{0,\sigma}(z;q) = z\cos(q^{-\frac{1}{2}}\sigma z;q)$ is $r^{\star}_{\alpha}(f_{0,\sigma}(z;q)) = x_{0,\sigma,1}(q)$, where $x_{0,\sigma,1}(q)$ stands for the smallest positive zero of the equation*

$$rq^{-\frac{1}{2}}\sigma\sin(q^{-\frac{1}{2}}\sigma z;q) + (\alpha-1)\cos(q^{-\frac{1}{2}}\sigma z;q) = 0.$$

b. *The radius of starlikeness of order α of the function $h_{0,\sigma} = z\cos(q^{-\frac{1}{2}}\sigma\sqrt{z};q)$ is $r^{\star}_{\alpha}(h_{0,\sigma}(z;q)) = z_{0,\sigma,1}(q)$, where $z_{0,\sigma,1}$ stands for the smallest positive zero of the equation*

$$\frac{\sigma}{1+\sqrt{q}}\sin(q^{-\frac{1}{2}}\sigma\sqrt{z};q) + 2(\alpha-1)\cos(q^{-\frac{1}{2}}\sigma\sqrt{z};q) = 0.$$

Putting $\gamma = 0$ in Theorem 1.4 we have the following results.

Corollary 1.8 *Let the conditions of Lemma 1.2 remain valid.*

a. *The radii of starlikeness $r^{\star}(f_{0,\sigma}(z;q))$ and $r^{\star}(g_{0,\sigma}(z;q))$ satisfy the inequalities*

$$\frac{1+q}{3\sigma^2} < \left(r^{\star}(f_{0,\sigma}(z;q))\right)^2 < \frac{3(1+q)(1+q^2)(1+q+q^2)}{\sigma^2(9q^4+9q^3+8q^2+9q+9)}.$$

b. *The radius of starlikeness $r^{\star}(h_{\gamma,\sigma}(z;q))$ satisfies the inequalities*

$$\frac{1+q}{2\sigma^2} < r^{\star}(h_{\gamma,\sigma}(z;q)) < \frac{(1+q)(1+q^2)(1+q+q^2)}{\sigma^2(2q^4+2q^3+q^2+2q+2)}.$$

Setting $\gamma = 0$ in Theorem 1.5 we get the following results. It is important to note that for $\gamma = 0$, the radii of convexity of the functions $f_{0,\sigma}(z;q)$ and $g_{0,\sigma}(z;q)$ coincide with each other.

Corollary 1.9 *With the same conditions of Lemma 1.2 the following inequalities are valid:*

a. *The radius of convexity $r^{c}(g_{0,\sigma}(z;q))$ satisfies the inequalities*

$$\frac{1+q}{9\sigma^2} < \left(r^{c}(g_{0,\sigma}(z;q))\right)^2 < \frac{9(1+q)(1+q^2)(1+q+q^2)}{\sigma^2(81q^4+81q^3+112q^2+81q+81)}.$$

b. *The radius of convexity $r^{c}(h_{0,\sigma}(z;q))$ satisfies the inequalities*

$$\frac{1+q}{4\sigma^2} < r^{c}(h_{0,\sigma}(z;q)) < \frac{2(1+q)(1+q^2)(1+q+q^2)}{\sigma^2(8q^4+8q^3+7q^2+8q+8)}.$$

References

[1] Agrawal S, Sahoo SK. On a generalization of close-to-convex functions. Annales Polonici Mathematici 2015; 113: 93-108. doi: 10.4064/ap113-1-6

[2] Aktaş İ, Baricz Á. Bounds for radii of starlikeness of some $q-$Bessel functions. Results in Mathematics 2017; 72: 947-963. doi: 10.1007/s00025-017-0668-6

[3] Aktaş İ, Baricz Á, Orhan H. Bounds for radii of starlikeness and convexity of some special functions. Turkish Journal of Mathematics 2018; 42: 211-226. doi: 10.3906/mat-1610-41

[4] Aktaş İ, Toklu E, Orhan H. Radii of uniform convexity of some special functions. Turkish Journal of Mathematics 2018; 42: 3010-3024. doi: 10.3906/mat-1806-43

[5] Aldweby H, Darus M. Some subordination results on $q-$analogue of Ruscheweyh differential operator. Abstract and Applied Analysis 2014; 2014: 958563.

[6] Andrews GE, Askey R, Roy R. Special Functions. Cambridge, UK: Cambridge University Press, 1999.

[7] Annaby MN, Mansour ZS. q-Fractional Calculus and Equations. Lecture Notes in Mathematics, Vol. 2056. Berlin, Germany: Springer-Verlag, 2012.

[8] Annaby MN, Mansour ZS. On the zeros of basic finite Hankel transforms. Journal of Mathematical Analysis and Applications 2006; 323: 1091-1103.

[9] Baricz Á, Kupán PA, Szász R. The radius of starlikeness of normalized Bessel functions of the first kind. Proceedings of the American Mathematical Society 2014; 142: 2019-2025.

[10] Baricz Á, Orhan H, Szász R. The radius of $\alpha-$convexity of normalized Bessel functions of the first kind. Computational Methods and Function Theory 2016; 16: 93-103. doi: 10.1007/s40315-015-0123-1

[11] Baricz Á, Prajapati A. Radii of starlikeness and convexity of generalized Mittag-Leffler functions. arXiv: 1901.04333.

[12] Baricz Á, Szász R. The radius of convexity of normalized Bessel functions of the first kind. Analysis and Applications 2014; 15: 485-509.

[13] Baricz Á, Szász R. Close-to-convexity of some special functions. Bulletin of the Malaysian Mathematical Sciences Society 2016; 39: 427-437. doi: 10.1007/s40840-015-0180-7

[14] Baricz Á, Toklu E, Kadıoğlu E. Radii of starlikeness and convexity of Wright functions. Mathematical Communications 2018; 23: 97-117.

[15] Dimitrov DK, Cheikh YB. Laguerre polynomials as Jensen polynomials of Laguerre-Pólya entire functions. Journal of Computational and Applied Mathematics 2009; 233: 703-707. doi: 10.1016/j.cam.2009.02.039

[16] Duren PL. Univalent Functions. Grundlehren der mathematischen Wissenschaften 259. New York, NY, USA: Springer, 1983.

[17] Ernst T. A Comprehensive Treatment of q-Calculus. New York, NY, USA: Springer Science and Business Media, 2012.

[18] Gasper G, Rahman M. Basic Hypergeometric Series. 2nd ed. Cambridge, UK: Cambridge University Press, 2004.

[19] Ismail ME, Merkes E, Styer D. A generalization of starlike functions. Complex Variables, Theory and Application 1990; 14: 77-84.

[20] Jackson FH. A generalization of the function $\Gamma(n)$ and x^n. Proceedings of the Royal Society of London 1904; 74: 64-72.

[21] Jackson FH. On q-functions and a certain difference operators. Transactions of the Royal Society of Edinburgh 1909 46: 253-281.

[22] Mansour ZSI. On q-difference and fractional equations and their solutions. PhD, Cairo University, Giza, Egypt, 2006.

[23] Osler TJ. Fractional derivatives and Leibniz rule. American Mathematical Monthly 1971; 78: 645-649.

[24] Raghavendar K, Swaminathan A. Close-to-convexity properties of basic hypergeometric functions using their Taylor coefficients. Journal of Mathematic and Applications 2012; 35: 111-125.

[25] Srivastava HM, Bansal D. Close-to-convexity of certain family of q-Mittag-Leffler functions. Journal of Nonlinear and Variational Analysis 2017; 1: 61-69.

[26] Thomae J. Beitrage zur Theorie der durch die Heinesche Reihe. Journal für die Reine und Angewandte Mathematik 1869; 70: 258-281 (in German).

[27] Toklu E. A new subclass of bi-univalent functions defined by q-derivative. Journal of Applied and Engineering Mathematics 2019; 9: 84-90.

[28] Toklu E, Aktaş İ, Orhan H. Radii problems for normalized q-Bessel and Wright functions. Acta Universitatis Sapientiae Mathematica 2019; 11: 203-223. doi: 10.2478/ausm-2019-0016