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On factorials in Perrin and Padovan sequences

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Abstract: Assume that w_n is the n th term of either Padovan or Perrin sequence. In this paper, we solve the equation $w_n = m!$ completely.

Key words: Factorials, Perrin numbers, Padovan numbers

1. Introduction

A number of mathematicians have been interested in Diophantine equations including both factorials and elements of linear recurrences such as Fibonacci, Tribonacci, and balancing numbers, etc. For example, Luca [6] proved that F_n is a product of factorials only when $n = 1, 2, 3, 6, 12$, where F_n is the n th Fibonacci number. Grossman and Luca [3] showed that the equation

$$F_n = m_1! + m_2! + \cdots + m_k!$$

has finitely many positive integers n for fixed k . In the same paper the solutions were determined for $k \leq 2$. The case $k = 3$ was handled by Bollman et al. in [2]. Irmak et al. [5] solved several equations involving balancing numbers and factorials. Recently, Sobolewski [10] gave the 2-adic valuation of generalized Fibonacci sequences.

Marques and Lengyel [8] searched the factorials in Tribonacci sequence. They characterized the 2-adic order of Tribonacci numbers and then solved the equation

$$T_n = m!$$

completely. This was the first paper to find factorials in third-order linear recurrences. In this paper, we present the 2-adic order of Padovan and Perrin numbers. Afterwards, we investigate factorials in Perrin and Padovan sequences.

Before going further, we give the definitions of Perrin and Padovan numbers.

Definition 1.1 For $n \geq 3$, Perrin $\{R_n\}$ and Padovan $\{P_n\}$ numbers are defined by the recurrence relations

$$R_n = R_{n-2} + R_{n-3}, \quad R_0 = 3, \quad R_1 = 0, \quad R_2 = 2 \tag{1.1}$$

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and

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = 1, \quad P_1 = 1, \quad P_2 = 1, \tag{1.2}$$

respectively.

By the recurrence relations of Perrin and Padovan sequences, negative indices of these numbers can be obtained easily. The following is the list of few Padovan and Perrin numbers.

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
Padovan	1	-1	1	0	0	1	0	1	1	1	2	2	3	4	5
Perrin	-1	-2	4	-3	2	1	-1	3	0	2	3	2	5	5	7

Perrin numbers were studied by several authors in the beginning of the nineteenth century (for details, see [9]). The Padovan sequence is named after Richard Padovan, who attributed its discovery to the Dutch architect Hans van der Laan in his 1994 essay “Dom Hans van der Laan: Modern Primitive”. Associated with (1.1) and (1.2) is the characteristic equation

$$x^3 - x - 1 = 0$$

with distinct roots α, β , and $\bar{\beta}$ where $\alpha \approx 1.3247\dots$ (called a plastic constant) is a unique real root. This constant was first defined in 1924 by Gérard Cordonnier. He described applications to architecture; in 1958, he gave a lecture tour that illustrated the use of the plastic constant in many buildings and monuments.

This paper is divided into two parts. In the first part we give several necessary lemmas and 2-adic orders of Perrin and Padovan numbers. In the second part, we solve the Diophantine equations

$$R_n = m!, \quad P_n = m!$$

completely.

Let w_n be the n th term of Padovan or Perrin sequences.

Theorem 1.2 *Assume that $n \geq 1$. The solutions of the equations*

$$w_n = m!$$

are $(n, m) = (1, 1), (2, 1), (3, 2), (4, 2)$ for Padovan numbers and $(n, m) = (2, 2), (4, 2)$ for Perrin numbers.

2. Auxiliary results

Before proceeding further, some lemmas will be needed. The next lemma gives additional formulas for Perrin and Padovan numbers.

Lemma 2.1 *Let n, m be positive integers. Then*

$$P_{n+m} = P_{n-1}P_{m-1} + P_nP_{m-2} + P_{n-2}P_{m-3} \tag{2.1}$$

and

$$R_{n+m} = P_{n-1}R_{m-1} + P_nR_{m-2} + P_{n-2}R_{m-3} \tag{2.2}$$

follow.

Proof These formulas can be found in [11] (Proposition 2.2). □

The following lemma gives a recurrence relation with arithmetic progression for Perrin and Padovan numbers.

Lemma 2.2 *Let $n, s,$ and r be positive integers with $0 \leq s \leq r - 1$. Then we have*

$$w_{rn+s} = (\alpha^r + \beta^r + \bar{\beta}^r) w_{r(n-1)+s} - ((\alpha\beta)^r + (\alpha\bar{\beta})^r + (\beta\bar{\beta})^r) w_{r(n-2)+s} + w_{r(n-3)+s}, \tag{2.3}$$

where $\alpha, \beta, \bar{\beta}$ are the roots of the equation $x^3 - x - 1 = 0$.

Proof The identity can be proven in a way similar to [4]. □

Since the Binet formula of a Perrin number is

$$R_n = \alpha^n + \beta^n + \bar{\beta}^n,$$

then formula (2.3) can be written as

$$w_{rn+s} = R_r w_{r(n-1)+s} - R_{-r} w_{r(n-2)+s} + w_{r(n-3)+s} \tag{2.4}$$

Now we introduce the following matrix notations:

$$T_n = \begin{pmatrix} w_n \\ w_{n+1} \\ w_{n+2} \end{pmatrix}, B_n = \begin{pmatrix} w_n & w_{n+1} & w_{n+2} \\ w_{n+1} & w_{n+2} & w_{n+3} \\ w_{n+2} & w_{n+3} & w_{n+4} \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

By the recurrence relation of the Perrin and Padovan sequences, one can easily check that $CT_n = T_{n+1}$ and $CB_n = B_{n+1}$. Then we obtain that

$$C^n T_m = T_{n+m}, C^n B_m = B_{n+m}.$$

These facts give that

$$T_{n+m} = B_n B_0^{-1} T_m, w_{n+m} = T_n^T B_0^{-1} T_m. \tag{2.5}$$

Lemma 2.3 *For the integers j and $t \geq 1$, we get the following:*

$$\begin{aligned} P_{7 \cdot 2^t + j} &\equiv P_j \pmod{2^{t+2}}, \text{ if } j \equiv 1, 2, 4 \pmod{7}, \\ P_{7 \cdot 2^t + j} &\equiv P_j + 2^{t+1} \pmod{2^{t+2}}, \text{ if } j \equiv 0, 3, 5, 6 \pmod{7}. \end{aligned}$$

Proof Suppose that $j \equiv 1 \pmod{7}$. We use induction on t . If $t = 1$, then $P_{14+j} \equiv P_j \pmod{8}$ holds for $j \equiv 1 \pmod{7}$. To see that we use induction on j again. If $j = 1$, then $P_{15} \equiv 1 \pmod{8}$ follows. Assume that $j = 7k + 1$ and $P_{7(k+2)+1} \equiv P_{7k+1} \pmod{8}$ holds for the integer k . Lemma 2.2 gives that

$$P_{7(k+3)+1} = 7P_{7(k+2)+1} + P_{7(k+1)+1} + P_{7k+1}.$$

Since $7P_{7(k+2)+1} + P_{7k+1} \equiv -P_{7(k+2)+1} + P_{7k+1} \equiv 0 \pmod{8}$, then we deduce that $P_{7(k+3)+1} \equiv P_{7(k+1)+1} \pmod{8}$ as claimed. Other cases for j can be shown similarly.

From now on, we resume the induction on t . Therefore, we can assume that $P_{7 \cdot 2^t + j} \equiv P_j \pmod{2^{t+2}}$ holds for integers t and j . Our aim is to show $P_{7 \cdot 2^{t+1} + j} \equiv P_j \pmod{2^{t+3}}$.

We follow induction on j again. Assume that $j = 1$. We can write

$$P_{7 \cdot 2^t + 1} = 2^{t+2} a_{t,1} + 1$$

where $a_{t,j}$ are positive integers satisfying the recurrence of the sequence $\{P_n\}$. Now define

$$A_{t,j} = \begin{pmatrix} a_{t,j} & a_{t,j+1} & a_{t,j+2} \end{pmatrix}^T.$$

The second formula in (2.5) gives

$$\begin{aligned} P_{7 \cdot 2^{t+1} + 1} &= P_{7 \cdot 2^t + 7 \cdot 2^t + 1} \\ &= \left(T_{7 \cdot 2^t}^{(P)} \right)^T B_0^{-1} T_{7 \cdot 2^t + 1}^{(P)}, \end{aligned}$$

where $T_n^{(P)}$ is the matrix whose entries are $w_n = P_n$ in the matrix T_n .

By the definitions of the vectors $T_n^{(P)}$, $A_{t,j}$,

$$T_{7 \cdot 2^t + 1}^{(P)} = \begin{pmatrix} P_{7 \cdot 2^t + 1} \\ P_{7 \cdot 2^t + 2} \\ P_{7 \cdot 2^t + 3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2^{t+1} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + 2^{t+2} A_{t,1}$$

and

$$\left(T_{7 \cdot 2^t}^{(P)} \right)^T = \begin{pmatrix} P_{7 \cdot 2^t} \\ P_{7 \cdot 2^t + 1} \\ P_{7 \cdot 2^t + 2} \end{pmatrix}^T = \left(\begin{pmatrix} 2^{t+1} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 2^{t+2} A_{t,0} \right)^T$$

follow.

Therefore, the second formula in (2.5) yields that

$$\begin{aligned} P_{7 \cdot 2^{t+1} + 1} &= \left(T_{7 \cdot 2^t}^{(P)} \right)^T B_0^{-1} T_{7 \cdot 2^t + 1}^{(P)} \\ &= 1 + 2^{t+3} a_{t,1}. \end{aligned}$$

Then

$$P_{7 \cdot 2^{t+1} + 1} \equiv 1 \pmod{2^{t+3}}$$

follows as claimed. Assume that $P_{7 \cdot 2^{t+1} + j} \equiv P_j \pmod{2^{t+3}}$ holds for $j \equiv 1 \pmod{7}$. Let $j = 7k + 1$ for $k \in \mathbb{Z}$.

By Lemma 2.2 and induction k , we have

$$\begin{aligned} P_{7 \cdot 2^{t+1} + 7(k+1) + 1} &\equiv 7P_{7(2^{t+1} + k) + 1} + P_{7(2^{t+1} + k - 1) + 1} + P_{7(2^{t+1} + k - 2) + 1} \pmod{2^{t+3}} \\ &\equiv 7P_{7k + 1} + P_{7(k-1) + 1} + P_{7(k-2) + 1} \pmod{2^{t+3}} \\ &\equiv P_{7(k+1) + 1} \pmod{2^{t+3}}. \end{aligned}$$

Finally, we have shown that $P_{7 \cdot 2^t + j} \equiv P_j \pmod{2^{t+3}}$ holds for all integers t and $j \equiv 1 \pmod{7}$.

The other cases for the integer j can be proven similarly. □

Lemma 2.4 For the integers j and $t \geq 1$, we have

$$\begin{aligned} R_{7 \cdot 2^t + j} &\equiv R_j \pmod{2^{t+2}}, \text{ if } j \equiv 0, 2, 6 \pmod{7}, \\ R_{7 \cdot 2^t + j} &\equiv R_j + 2^{t+1} \pmod{2^{t+2}}, \text{ if } j \equiv 1, 3, 4, 5 \pmod{7}. \end{aligned}$$

Proof It can be proven in a way similar to the proof of the previous lemma. Therefore, we do not give the details. \square

The p -adic order $\nu_p(r)$ of r is the exponent of the highest power of a prime p , which divides r . We provide a complete description of the 2-adic order of Perrin and Padovan numbers.

Lemma 2.5 For $n \geq 1$, we get that

$$\nu_2(R_n) = \begin{cases} 0, & n \equiv 0, 3, 5, 6 \pmod{7} \\ 1, & n \equiv 2 \pmod{14} \\ 2, & n \equiv 9 \pmod{14} \\ \nu_2(n-1) + 1, & n \equiv 1 \pmod{7} \\ 1, & n \equiv 4 \pmod{7}. \end{cases}$$

Proof Assume that $n \equiv 9 \pmod{14}$. Obviously $\nu_2(R_9) = \nu_2(R_{23}) = \nu_2(R_{37}) = 2$. Lemma 2.2 gives that

$$R_{14n+9} = 51R_{14(n-1)+9} + 13R_{14(n-2)+9} + R_{14(n-3)+9}. \tag{2.6}$$

Assume that $R_{14(n-i)+9} = 2^2 \cdot k_i$ for odd integer k_i ($i = 1, 2, 3$) and integer n . The recurrence relation (2.6) gives that $\nu_2(R_{14n+9}) = 2$. The other cases can be proven similarly except the case $n \equiv 1 \pmod{7}$. We will not give a proof of the case $n \equiv 1 \pmod{7}$ because the similar situation is proved in the next lemma for Padovan numbers. \square

The following Lemma is about the 2-adic order of a Padovan number.

Lemma 2.6 For $n \geq 1$, we obtain

$$\nu_2(P_n) = \begin{cases} 0, & n \equiv 0, 1, 2, 5 \pmod{7} \\ \nu_2(n+4) + 1, & n \equiv 3 \pmod{7} \\ \nu_2((n+3)(n+17)) + 1, & n \equiv 4 \pmod{7} \\ \nu_2((n+1)(n+8)) + 1, & n \equiv 6 \pmod{7}. \end{cases}$$

Proof Case 1: $n \equiv 0, 1, 2, 5 \pmod{7}$.

We will only prove the case $n \equiv 0 \pmod{7}$. Other cases can be proven similarly. By Lemma 2.2, the recurrence relation

$$P_{7n} = 7P_{7(n-1)} + P_{7(n-2)} + P_{7(n-3)} \tag{2.7}$$

is obtained for $r = 7$. It is obvious that $P_0 = 1$, $P_7 = 5$, and $P_{14} = 37$ are odd integers. We shall use induction on n . Assume that $\nu_2(P_{7(n-1)}) = \nu_2(P_{7(n-2)}) = \nu_2(P_{7(n-3)}) = 0$. The recurrence relation (2.7) yields that $\nu_2(P_{7n}) = 0$ as claimed.

Case 2: $n \equiv 3 \pmod{7}$.

In order to prove $\nu_2(P_n) = \nu_2(n+4) + 1$, it is enough to show that $P_{7 \cdot 2^t k - 4} \equiv 2^{t+1}k \pmod{2^{t+2}}$. We will use induction on k . Assume that $k = 1$. Together with Lemma 2.3 the equivalent

$$P_{7 \cdot 2^t - 4} \equiv 2^{t+1} + P_{-4} \equiv 2^{t+1} \pmod{2^{t+2}} \tag{2.8}$$

holds for integers $t \geq 1$.

Now, by induction hypothesis, we suppose that the congruence holds for all integers k . By the formula (2.4), we deduce

$$P_{7 \cdot 2^t(k+1)-4} = R_{7 \cdot 2^t} P_{7 \cdot 2^t k-4} - R_{-7 \cdot 2^t} P_{7 \cdot 2^t(k-1)-4} + P_{7 \cdot 2^t(k-2)-4}.$$

for fixed integer t . Since $2R_{-n} = R_n^2 - R_{2n}$ (1.33 of Theorem 1 in [1]), together with Lemma 2.1 and Lemma 2.3, we have

$$R_{\pm 7 \cdot 2^t} \equiv 3 \pmod{2^{t+1}}.$$

This yields that

$$R_{\pm 7 \cdot 2^t} \equiv 3 + \epsilon \cdot 2^{t+1} \pmod{2^{t+2}},$$

where $\epsilon \in \{0, 1\}$.

$$\begin{aligned} P_{7 \cdot 2^t(k+1)-4} &\equiv R_{7 \cdot 2^t} P_{7 \cdot 2^t k-4} - R_{-7 \cdot 2^t} P_{7 \cdot 2^t(k-1)-4} + P_{7 \cdot 2^t(k-2)-4} \pmod{2^{t+2}} \\ &\equiv (3 + \epsilon \cdot 2^{t+1})k2^{t+1} - (3 + \epsilon \cdot 2^{t+1})(k-1)2^{t+1} + (k-2)2^{t+1} \pmod{2^{t+2}} \\ &\equiv 2^{t+1}(k+1) \pmod{2^{t+2}} \end{aligned}$$

follows as claimed.

Case 3: $n \equiv 6 \pmod{7}$.

We separate the case into two subcases.

If n is even, then

$$\nu_2((n+1)(n+8)) + 1 = \nu_2(n+8) + 1$$

follows. Thus, it is needed to show

$$P_{7 \cdot 2^t k-8} \equiv 2^{t+1}k \pmod{2^{t+2}}.$$

Since it can be proven similarly to the Case 2, we omit the details.

If n is odd, then

$$\nu_2((n+1)(n+8)) + 1 = \nu_2(n+1) + 1$$

holds. Since $P_{7 \cdot 2^t k-1} \equiv 2^{t+1}k \pmod{2^{t+2}}$ can be proven similarly to case 2, we omit the proof for this case.

Case 4: $n \equiv 4 \pmod{7}$.

The case n even yields that $\nu_2((n+3)(n+17)) + 1 = 1$. We use the same procedure exactly as before. Therefore, we omit the details. If $n \equiv 3 \pmod{4}$, then $\nu_2((n+3)(n+17)) + 1 = \nu_2(n+17) + 2$ follows. The case $n \equiv 1 \pmod{4}$ gives that $\nu_2((n+3)(n+17)) + 1 = \nu_2(n+3) + 2$. We can follow the details as given in case 3.

Finally, the proof of Lemma 2.6 is completed. □

Lemma 2.7 For any integer $k \geq 1$ and p prime, we have

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_2(k!) \leq \frac{k-1}{p-1}.$$

For the proof, see [7].

Lemma 2.8 For $n \geq 3$, we have

$$\alpha^{n-2} \leq w_n$$

where $\alpha = 1,324\dots$

Proof Use the induction method on n . □

3. The proof of Theorem 1.2

If $m \leq 2$, then the solutions are listed in Theorem 1.2. Now assume that $m \geq 3$. By using Lemma 2.7, we deduce that

$$\begin{aligned} m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 1 &\leq \nu_2(m!) = \nu_2(P_n) \\ &\leq \nu_2((n+1)(n+3)(n+4)(n+8)(n+17)) + 3 \\ &\leq 5\nu_2(n+\delta) + 3 \end{aligned}$$

for some $\delta \in \{1, 3, 4, 8, 17\}$. By applying the log function after some calculations, we obtain

$$\frac{m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 4}{5} \leq \frac{\log(n+17)}{\log 2}. \tag{3.1}$$

On the other hand, since $(1.32)^{n-2} \leq P_n = m! < \left(\frac{m}{2}\right)^m$, then

$$n \leq 3.61m \log \frac{m}{2} + 2 \tag{3.2}$$

holds. By the inequality (3.1) together with (3.2), we arrive at

$$m - \left\lfloor \frac{\log m}{\log 2} \right\rfloor - 4 \leq 7.25 \cdot \log \left(3.61m \log \frac{m}{2} + 19 \right).$$

This inequality gives that $3 \leq m \leq 56$ and then $n \leq 3.61m \log \frac{m}{2} + 2 \leq 675$.

By using similar arguments, we obtain that

$$3 \leq m \leq 16 \quad \text{and} \quad n \leq 123$$

for the equation $R_n = m!$. A simple routine written in Mathematica shows that there is no solution for the equations $P_n = m!$ and $R_n = m!$ for the given interval.

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