Turkish Journal of Mathematics

Volume 43 | Number 5 Article 36

1-1-2019

The generalized Picard groups for finite dimensional C^*-Hopf algebra coactions on unital C^*-algebras

KAZUNORI KODAKA

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2511 – 2539
© TÜBİTAK
doi:10.3906/mat-1905-37

The generalized Picard groups for finite dimensional C^*-Hopf algebra coactions on unital C^*-algebras

Kazunori KODAKA*
Department of Mathematical Sciences, Faculty of Science, Ryukyu University, Nishihara-cho, Okinawa, 901-0213, Japan

Received: 10.05.2019 • **Accepted/Published Online:** 26.08.2019 • **Final Version:** 28.09.2019

Abstract: We shall generalize the notion of the strong Morita equivalence for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra and define the Picard groups with respect to the generalized strong Morita equivalence. We call them the generalized Picard groups for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra. We shall investigate basic properties of the generalized Picard groups and clarify the relation between the generalized Picard groups and the Picard groups for unital inclusions of unital C^*-algebras.

Key words: C^*-algebra, C^*-Hopf algebra, inclusion, Picard group, strong Morita equivalence

1. Introduction

In [14], we introduced the notion of the strong Morita equivalence for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra and in [10] we defined the finite dimensional C^*-Hopf algebra coaction equivariant Picard group of a unital C^*-algebra. We note that this equivariant Picard group can be regarded as the Picard group for the coaction of the finite dimensional C^*-Hopf algebra on the unital C^*-algebra. Furthermore, in [15] and [11] we introduced the notion of the strong Morita equivalence for unital inclusions of unital C^*-algebras and defined the Picard groups for them. In this paper, we shall generalize the notion of the strong Morita equivalence for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra and we shall define the Picard groups with respect to the generalized strong Morita equivalence. We call them the generalized Picard groups for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra. Also, we shall give the results similar to [10, 11, 14, 15]. Furthermore, we shall clarify the relation between the generalized Picard groups and the Picard groups for unital inclusions of unital C^*-algebras.

2. Preliminaries

Let H be a finite dimensional C^*-Hopf algebra. We denote its comultiplication, counit, and antipode by Δ, ϵ, and S, respectively. We shall use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for any $h \in H$ which surppresses a possible summation when we write comultiplications. We denote by N the dimension of H. Let H^0 be the dual C^*-Hopf algebra of H. We denote its comultiplication, counit, and antipode by Δ^0, ϵ^0, and S^0, respectively. There is the distinguished projection e in H. We note that e is the Haar trace

*Correspondence: kodaka@math.u-ryukyu.ac.jp
2010 *AMS Mathematics Subject Classification:* 46L05

on H^0. Also, there is the distinguished projection τ in H^0 which is the Haar trace on H. Since H is finite dimensional, $H\cong\oplus_{k=1}^L M_{f_k}(\mathbf{C})$ and $H^0\cong\oplus_{k=1}^K M_{d_k}(\mathbf{C})$ as C^*-algebras, where $M_n(\mathbf{C})$ is the $n\times n$-matrix algebra over $\mathbf{C}$. Let $\{v_{ij}^k\,|\,k=1,2,\dots,L,\ i,j=1,2,\dots,f_k\}$ be a system of matrix units of H. Let $\{w_{ij}^k\,|\,k=1,2\dots,K,\ i,j=1,2,\dots,d_k\}$ be a basis of H satisfying Szymański and Peligrad's [22, Theorem 2.2,2], which is called a system of *comatrix units* of H, that is, the dual basis of a system of matrix units of H^0. Also, let $\{\varphi_{ij}^k\,|\,k=1,2,\dots,K,\ i,j=1,2,\dots,d_k\}$ and $\{\omega_{ij}^k\,|\,k=1,2,\dots,L,\ i,j=1,2,\dots,f_k\}$ be systems of matrix units and comatrix units of H^0, respectively.

Let A be a C^*-algebra and $M(A)$ its multiplier algebra. Let p,q be projections in A. If p and q are Murray–von Neumann equivalent in A, then we denote it by $p\sim q$ in A. We denote by id_A and 1_A the identity map on A and the unit element in A, respectively. We simply denote them by id and 1 if no confusion arises. Let $\mathrm{Aut}(A)$ be the group of all automorphisms of A and for any $\alpha\in\mathrm{Aut}(A)$, let $\underline{\alpha}$ be the automorphism of $M(A)$ induced by α. Also, for any twisted coaction (ρ,u) of H^0 on A, let $(\underline{\rho},u)$ be the twisted coaction of H^0 on $M(A)$ induced by (ρ,u).

Let $\mathrm{Hom}(H,A)$ be the linear space of all linear maps from H to A. Since H is finite dimensional, $\mathrm{Hom}(H,A)$ is isomorphic to $A\otimes H^0$. We identify $\mathrm{Hom}(H,A)$ with $A\otimes H^0$. For any element $x\in A\otimes H^0$, we denote by $\widehat{x}$ the element in $\mathrm{Hom}(H,A)$ induced by x.

For a twisted coaction (ρ,u) of H^0 on a C^*-algebra A, we consider the twisted action of H on A with its cocycle unitary $\widehat{u}$ defined by

$$h\cdot_{\rho,u}a=(\mathrm{id}\otimes h)(\rho(a))$$

for any $a\in A,\ h\in H$. We call it the twisted action of H on A induced by (ρ,u). We sometimes denote the twisted action by $h\cdot_\rho a$ for any $a\in A,\ h\in H$. Let $A\rtimes_{\rho,u}H$ be the crossed product of A by the action of H on A induced by (ρ,u). Let $a\rtimes_{\rho,u}h$ be the element in $A\rtimes_{\rho,u}H$ induced by elements $a\in A,\ h\in H$. Let $\widehat{\rho}$ be the dual coaction of H on $A\rtimes_\rho H$ defined by

$$\widehat{\rho}(a\rtimes_{\rho,u}h)=(a\rtimes_{\rho,u}h_{(1)})\otimes h_{(2)}$$

for any $a\in A,\ h\in H$. Let $E_1^{\rho,u}$ be the canonical conditional expectation from $A\rtimes_{\rho,u}H$ onto A defined by

$$E_1^{\rho,u}(a\rtimes_{\rho,u}h)=\tau(h)a$$

for any $a\in A,\ h\in H$. We note that $E_1^{\rho,u}(a\rtimes_{\rho,u}h)=\tau\cdot_{\widehat{\rho}}(a\rtimes_{\rho,u}h)$.

Let A be a C^*-algebra and (ρ,u) a twisted coaction of H^0 on A. Let f^0 be a C^*-Hopf algebra automorphism of H^0. Let (ρ_{f^0},u_{f^0}) be the twisted coaction of H^0 on A induced by (ρ,u) and f^0, which is defined by

$$\rho_{f^0}=(\mathrm{id}\otimes f^0)\circ\rho,\quad u_{f^0}=(\mathrm{id}\otimes f^0\otimes f^0)(u).$$

Let (σ,v) be a twisted coaction of H^0 on a C^*-algebra B and (σ_{f^0},v_{f^0}) the twisted coaction of H^0 on B induced by (σ,v) and f^0. For any C^*-Hopf algebra automorphism f^0 of H^0, let f be the C^*-Hopf algebra automorphism of H induced by f^0, which is defined by

$$\varphi(f^{-1}(h))=f^0(\varphi)(h)$$

for any $h\in H$, $\varphi\in H^0$.

Lemma 2.1 *With the above notation, we suppose that* (ρ, u) *and* (σ, v) *are strongly Morita equivalent. Then for any* C^**-Hopf algebra automorphism* f^0 *of* H^0*,* (ρ_{f^0}, u_{f^0}) *and* (σ_{f^0}, v_{f^0}) *are strongly Morita equivalent.*

Proof Since (ρ, u) and (σ, v) are strongly Morita equivalent, there are an $A-B$-equivalence bimodule X and a twisted coaction λ of H^0 on X with respect to $(A, B, \rho, u, \sigma, v)$ (See [14]). Let λ_{f^0} be the linear map from X to $X \otimes H^0$ defined by $\lambda_{f^0} = (\mathrm{id} \otimes f^0) \circ \lambda$. By routine computations, we can see that λ_{f^0} is a twisted coaction of H^0 on X with respect to $(A, B, \rho_{f^0}, u_{f^0}, \sigma_{f^0}, v_{f^0})$. Indeed, for any $a \in A$, $x \in X$,

$$\lambda_{f^0}(a \cdot x) = (\mathrm{id} \otimes f^0)(\lambda(a \cdot x)) = (\mathrm{id} \otimes f^0)(\rho(a) \cdot \lambda(x)).$$

Since $\rho(a) \in A \otimes H^0$ and $\lambda(x) \in X \otimes H^0$, we can write that

$$\rho(a) = \sum_i a_i \otimes \varphi_i, \quad \lambda(x) = \sum_j x_j \otimes \psi_j,$$

where $a_i \in A$, $x_j \in X$, $\varphi_i, \psi_j \in H^0$. Hence,

$$\begin{aligned}\lambda_{f^0}(a \cdot x) &= (\mathrm{id} \otimes f^0)(\sum_{i,j} (a_i \cdot x_j) \otimes \varphi_i \psi_j) = \sum_{i,j} (a_i \cdot x_j) \otimes f^0(\varphi_i) f^0(\psi_j) \\ &= (\mathrm{id} \otimes f^0)(\rho(a)) \cdot (\mathrm{id} \otimes f^0)(\lambda(x)) = \rho_{f^0}(a) \cdot \lambda_{f^0}(x).\end{aligned}$$

Similarly $\lambda_{f^0}(x \cdot b) = \lambda(x) \cdot \sigma_{f^0}(b)$ for any $b \in B$, $x \in X$. For any $x, y \in X$,

$$\sigma_{f^0}(\langle x, y \rangle_B) = (\mathrm{id} \otimes f^0)(\sigma(\langle x, y \rangle_B) = (\mathrm{id} \otimes f^0)(\langle \lambda(x), \lambda(y) \rangle_{B \otimes H^0})$$

Since $\lambda(x), \lambda(y) \in X \otimes H^0$, we can write that

$$\lambda(x) = \sum_i x_i \otimes \varphi_i, \quad \lambda(y) = \sum_j y_j \otimes \psi_j,$$

where $x_i, y_j \in X$, $\varphi_i, \psi_j \in H^0$. Hence

$$\begin{aligned}\sigma_{f^0}(\langle x, y \rangle_B) &= (\mathrm{id} \otimes f^0)(\sum_{i,j} \langle x_i \otimes \varphi_i,\, y_j \otimes \psi_j \rangle_{B \otimes H^0}) \\ &= \sum_{i,j} \langle x_i, y_j \rangle_B \otimes f^0(\varphi_i^*) f^0(\psi_j) \\ &= \langle (\mathrm{id} \otimes f^0)(\lambda(x)),\, (\mathrm{id} \otimes f^0)(\lambda(y)) \rangle_{B \otimes H^0} \\ &= \langle \lambda_{f^0}(x),\, \lambda_{f^0}(y) \rangle_{B \otimes H^0}.\end{aligned}$$

Similarly $\rho_{f^0}({}_A\langle x, y \rangle) = {}_{A \otimes H^0}\langle \lambda_{f^0}(x),\, \lambda_{f^0}(y) \rangle$ for any $x, y \in X$. Also, for any $x \in X$,

$$(\mathrm{id} \otimes \epsilon^0)(\lambda_{f^0}(x)) = ((\mathrm{id} \otimes \epsilon^0) \circ (\mathrm{id} \otimes f^0))(\lambda(x)).$$

Since $\lambda(x) \in X \otimes H^0$, we can write that $\lambda(x) = \sum_i x_i \otimes \varphi_i$, where $x_i \in X$, $\varphi_i \in H^0$. Let f be the C^*-Hopf algebra automorphism of H induced by f^0. Then

$$\begin{aligned}((\mathrm{id}\otimes\epsilon^0)\circ\lambda_{f^0})(x)&=\sum_i(\mathrm{id}\otimes\epsilon^0)(x_i\otimes f^0(\varphi_i))=\sum_i x_i f^0(\varphi_i)(1)\\&=\sum_i x_i\varphi_i(f^{-1}(1))=\sum_i x_i\varphi_i(1)\\&=(\mathrm{id}\otimes\epsilon^0)(\sum_i x_i\otimes\varphi_i)=(\mathrm{id}\otimes\epsilon^0)(\lambda(x))=x.\end{aligned}$$

Hence, $((\mathrm{id}\otimes\epsilon^0)\circ\lambda_{f^0})(x)=x$ for any $x\in X$. Furthermore, for any $x\in X$,

$$\begin{aligned}(\lambda_{f^0}\otimes\mathrm{id})(\lambda_{f^0}(x))&=((\mathrm{id}\otimes f^0\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id})\circ(\mathrm{id}\otimes f^0)\circ\lambda)(x)\\&=(\mathrm{id}\otimes f^0\otimes f^0)(((\lambda\otimes\mathrm{id})\circ\lambda)(x))\\&=(\mathrm{id}\otimes f^0\otimes f^0)(u(\mathrm{id}\otimes\Delta^0)(\lambda(x))v^*)\\&=u_{f^0}(\mathrm{id}\otimes\Delta^0)((\mathrm{id}\otimes f^0)(\lambda(x))v^*_{f^0}\\&=u_{f^0}(\mathrm{id}\otimes\Delta^0)(\lambda_{f^0}(x))v^*_{f^0}.\end{aligned}$$

Therefore, (ρ_{f^0},u_{f^0}) and (σ_{f^0},v_{f^0}) are strongly Morita equivalent. □

Let A, B be C^*-algebras and let X an $A-B$-equivalence bimodule. For any $a\in A$, $b\in B$, $x\in X$, we denote by $a\cdot x$ the left A-action on X and by $x\cdot b$ the right B-action X. Let $\widetilde{X}$ be the dual $B-A$-equivalence bimodule of X and $\widetilde{x}$ denotes the element in $\widetilde{X}$ induced by an element $x\in X$.

3. Generalized strong Morita equivalence for coactions

Definition 3.1 *Let (ρ,u) and (σ,v) be twisted coactions of H^0 on A and B, respectively. The twisted coaction (ρ,u) is generalized strongly Morita equivalent to the twisted coaction (σ,v) if there are C^*-Hopf algebra automorphisms f^0,g^0 such that (ρ_{f^0},u_{f^0}) is strongly Morita equivalent to (σ_{g^0},v_{g^0}).*

Lemma 3.2 *The generalized strong Morita equivalence for twisted coactions of a finite dimensional C^*-Hopf algebra on C^*-algebras is an equivalence relation.*

Proof This is immediate by routine computations and Lemma 2.1. Indeed, it suffices to show transitivity since the other conditions clearly hold. Let A,B,C be C^*-algebras and $(\rho,u),(\sigma,v),(\gamma,w)$ twisted coactions of H^0 on A,B,C, respectively. We suppose that (ρ,u) is generalized strongly Morita equivalent to (σ,v) and that (σ,v) is generalized strongly Morita equivalent to (γ,w). Then there are C^*-Hopf algebra automorphisms f_1^0,f_2^0 and g_1^0,g_2^0 of H^0 such that $(\rho_{f_1^0},u_{f_1^0})$ is strongly Morita equivalent to $(\sigma_{f_2^0},v_{f_2^0})$ and such that $(\sigma_{g_1^0},v_{g_1^0})$ is strongly Morita equivalent to $(\gamma_{g_2^0},w_{g_2^0})$. By Lemma 2.1, $(\sigma_{f_2^0},v_{f_2^0})$ is strongly Morita equivalent to $(\gamma_{f_2^0\circ(g_1^0)^{-1}\circ g_2^0},\, w_{f_2^0\circ(g_1^0)^{-1}\circ g_2^0})$. Hence, by [14, Proposition 3.7], $(\rho_{f_1^0},u_{f_1^0})$ is strongly Morita equivalent to $(\gamma_{f_2^0\circ(g_1^0)^{-1}\circ g_2^0},\, w_{f_2^0\circ(g_1^0)^{-1}\circ g_2^0})$. Therefore, (ρ,u) is generalized strongly Morita equivalent to (γ,w). □

Modifying [10], we shall define the generalized Picard group for a twisted coaction of a finite dimensional C^*-Hopf algebra on a C^*-algebra.

Let A be a C^*-algebra and H a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on A. Let (X, λ, f^0) be a triplet of an $A-A$-equivalence bimodule X, a twisted coaction λ of H^0 on X with respect to $(A, A, \rho, u, \rho_{f^0}, u_{f^0})$ and a C^*-Hopf algebra automorphism f^0 of H^0, where (ρ_{f^0}, u_{f^0}) is the twisted coaction of H^0 on A induced by (ρ, u) and f^0. Let $\mathrm{GEqui}_H^{\rho,u}(A)$ be the set of all such triplets (X, λ, f^0) as above. We define an equivalence relation $\sim$ in $\mathrm{GEqui}_H^{\rho,u}(A)$ as follows: For $(X, \lambda, f^0), (Y, \mu, g^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$, $(X, \lambda, f^0) \sim (Y, \mu, g^0)$ if and only if there is an $A-A$-equivalence bimodule isomorphism π of X onto Y such that $\mu\circ\pi = (\pi\otimes \mathrm{id})\circ\lambda$ and $f^0 = g^0$. We denote by $[X, \lambda, f^0]$ the equivalence class of (X, λ, f^0) in $\mathrm{GEqui}_H^{\rho,u}(A)$. Let $\mathrm{GPic}_H^{\rho,u}(A) = \mathrm{GEqui}_H^{\rho,u}(A)/\sim$. We shall define a product in $\mathrm{GPic}_H^{\rho,u}(A)$ as follows: Let $(X, \lambda, f^0), (Y, \mu, g^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Let μ_{f^0} be the twisted coaction of H^0 on Y induced by μ and f^0. Then by the proof of Lemma 2.1, μ_{f^0} is the twisted coaction of H^0 on Y with respect to $(A, A, \rho_{f^0}, u_{f^0}, \rho_{f^0\circ g^0}, u_{f^0\circ g^0})$ and (ρ_{f^0}, u_{f^0}) is strongly Morita equivalence to $(\rho_{f^0\circ g^0}, u_{f^0\circ g^0})$. Hence

$$(X\otimes_A Y, \lambda\otimes\mu_{f^0}, f^0\circ g^0) \in \mathrm{GEqui}_H^{\rho,u}(A),$$

where $\lambda\otimes\mu_{f^0}$ is the twisted coaction of H^0 on $X\otimes_A Y$ induced by the action "$\cdot_{\lambda\otimes\mu_{f^0}}$" on $X\otimes_A Y$ defined in [14, Proposition 3.7]. We define the product of $[X, \lambda, f^0], [Y, \mu, g^0] \in \mathrm{GPic}_H^{\rho,u}(A)$ by

$$[X, \lambda, f^0][Y, \mu, g^0] = [X\otimes_A Y, \lambda\otimes\mu_{f^0}, f^0\circ g^0] \in \mathrm{GPic}_H^{\rho,u}(A).$$

We show that the above product in $\mathrm{GPic}_H^{\rho,u}(A)$ is well-defined. Let $(\rho, u), (\sigma, v), (\gamma, w)$ be twisted coactions of H^0 on C^*-algebras A, B, C, respectively. We suppose that there are a twisted coaction λ of H^0 on an $A-B$-equivalence bimodule X with respect to $(A, B, \rho, u, \sigma, v)$ and a twisted coaction μ of H^0 on a $B-C$-equivalence bimodule Y with respect to $(B, C, \sigma, v, \gamma, w)$, respectively.

Lemma 3.3 *With the above notation, let λ' be a twisted coaction of H^0 on an $A-B$-equivalence bimodule X' with respect to $(A, B, \rho, u, \sigma, v)$. We suppose that there is an $A-B$-equivalence bimodule isomorphism π of X onto X' such that $\lambda'\circ\pi = (\pi\otimes \mathrm{id}_{H^0})\circ\lambda$. Then $\pi\otimes\mathrm{id}_Y$ is an $A-C$-equivalence bimodule isomorphism of $X\otimes_A Y$ onto $X'\otimes_A Y$ such that*

$$(\lambda'\otimes\mu)\circ(\pi\otimes\mathrm{id}_Y) = (\pi\otimes\mathrm{id}_Y\otimes\mathrm{id}_{H^0})\circ(\lambda\otimes\mu).$$

Proof It is clear that $\pi\otimes\mathrm{id}_Y$ is an $A-C$-equivalence bimodule isomorphism of $X\otimes_A Y$ onto $X'\otimes_A Y$. We have only to show that

$$(\lambda'\otimes\mu)\circ(\pi\otimes\mathrm{id}_Y) = (\pi\otimes\mathrm{id}_Y\otimes\mathrm{id}_{H^0})\circ(\lambda\otimes\mu).$$

That is, we show that

$$h\cdot_{\lambda'\otimes\mu}(\pi\otimes\mathrm{id}_Y)(x\otimes y) = (\pi\otimes\mathrm{id}_Y)(h\cdot_{\lambda\otimes\mu}(x\otimes y))$$

for any $x\in X$, $y\in Y$ and $h\in H$. Indeed,

$$\begin{aligned} h\cdot_{\lambda'\otimes\mu}(\pi\otimes\mathrm{id}_Y)(x\otimes y) &= [h_{(1)}\cdot_{\lambda'}\pi(x)]\otimes[h_{(2)}\cdot_\mu y] \\ &= \pi([h_{(1)}\cdot_\lambda x])\otimes[h_{(2)}\cdot_\mu y] \\ &= (\pi\otimes\mathrm{id}_Y)([h_{(1)}\cdot_\lambda x]\otimes[h_{(2)}\cdot_\mu y]) = (\pi\otimes\mathrm{id}_Y)(h\cdot_{\lambda\otimes\mu}(x\otimes y)) \end{aligned}$$

for any $x \in X$, $y \in Y$, $h \in H$. Therefore, we obtain the conclusion. □

Similarly, we obtain the following lemma:

Lemma 3.4 *With the above notation, let μ' be a twisted coaction of H^0 on a $B-C$-equivalence bimodule Y' with respect to $(B, C, \sigma, v, \gamma, w)$. We suppose that there is a $B-C$-equivalence bimodule isomorphism π of Y onto Y' such that $\mu' \circ \pi = (\pi \otimes \mathrm{id}_{H^0}) \circ \mu$. Then $\mathrm{id}_X \otimes \pi$ is an $A-C$-equivalence bimodule isomorphism of $X \otimes_A Y$ onto $X \otimes_A Y'$ such that*

$$(\lambda \otimes \mu') \circ (\mathrm{id}_X \otimes \pi) = (\mathrm{id}_X \otimes \pi \otimes \mathrm{id}_{H^0}) \circ (\lambda \otimes \mu').$$

Let f^0 be a C^*-Hopf algebra automorphism. Let (ρ, u), (σ, v) be as above. Also, let λ' and X' be as Lemma 3.3.

Lemma 3.5 *With the above notation, we suppose that there is an $A-B$-equivalence bimodule isomorphism π of X onto X' such that $\lambda' \circ \pi = (\pi \otimes \mathrm{id}_{H^0}) \circ \lambda$. Then $\lambda'_{f^0} \circ \pi = (\pi \otimes \mathrm{id}_{H^0}) \circ \lambda_{f^0}$.*

Proof We can prove the lemma by routine computations. Indeed,

$$\begin{aligned} \lambda'_{f^0} \circ \pi &= (\mathrm{id} \otimes f^0) \circ \lambda' \circ \pi = (\mathrm{id} \otimes f^0) \circ (\pi \otimes \mathrm{id}_{H^0}) \circ \lambda \\ &= (\pi \otimes \mathrm{id}_{H^0}) \circ (\mathrm{id} \otimes f^0) \circ \lambda = (\pi \otimes \mathrm{id}_{H^0}) \circ \lambda_{f^0}. \end{aligned}$$

□

Lemma 3.6 *The product in $\mathrm{GPic}_H^{\rho,u}(A)$ defined by*

$$[X, \lambda, f^0][Y, \mu, g^0] = [X \otimes_A Y, \lambda \otimes \mu_{f^0}, f^0 \circ g^0]$$

for any $(X, \lambda, f^0), (Y, \mu, g^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$ is well-defined.

Proof This is immediate by Lemmas 3.3, 3.4 and 3.5. □

We regard A as an $A-A$-equivalence bimodule in the usual way. We sometimes denote it by ${}_AA_A$. We also regard a twisted coaction (ρ, u) of H^0 on A as a twisted coaction of H^0 on the $A-A$- equivalence bimodule ${}_AA_A$ with respect to (A, A, ρ, u, ρ, u). Then $[{}_AA_A, \rho, \mathrm{id}_{H^0}]$ is the unit element in $\mathrm{GPic}_H^{\rho,u}(A)$. Let $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Let $\widetilde{\lambda_{(f^0)^{-1}}}$ be the linear map from X to $\widetilde{X \otimes H^0}$ defined by

$$\widetilde{\lambda_{(f^0)^{-1}}}(\widetilde{x}) = [\lambda_{(f^0)^{-1}}(x)]\widetilde{} = ((\mathrm{id} \otimes (f^0)^{-1}) \circ \lambda)(x)\widetilde{}$$

for any $x \in X$. We note that $\widetilde{X \otimes H^0}$ is identified with $\widetilde{X} \otimes H^0$ by the map

$$\widetilde{X \otimes H^0} \to \widetilde{X} \otimes H^0 : \widetilde{x \otimes \varphi} \mapsto \widetilde{x} \otimes \varphi^*$$

as mentioned in [10, Remarks 4.1 and 4.2]. Hence, by easy computations, we can see that

$$\widetilde{\lambda_{(f^0)^{-1}}}(\widetilde{x}) = ((\mathrm{id} \otimes (f^0)^{-1}) \circ \widetilde{\lambda})(\widetilde{x})$$

for any $x \in X$. Thus, by Lemma 2.1, $\widetilde{\lambda_{(f^0)^{-1}}}$ is a coaction of H^0 on $\widetilde{X}$ with respect to $(A, A, \rho, u, \rho_{(f^0)^{-1}}, u_{(f^0)^{-1}})$ and $(\widetilde{X}, \widetilde{\lambda_{(f^0)^{-1}}}, (f^0)^{-1}) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Furthermore, by the product in $\mathrm{GPic}_H^{\rho,u}(A)$,

$$[X, \lambda, f^0][\widetilde{X}, \widetilde{\lambda_{(f^0)^{-1}}}, (f^0)^{-1}] = [X \otimes_A \widetilde{X}, \lambda \otimes \widetilde{\lambda}, \mathrm{id}_{H^0}] = [A, \rho, \mathrm{id}_{H^0}],$$

$$[\widetilde{X}, \widetilde{\lambda_{(f^0)^{-1}}}, (f^0)^{-1}][X, \lambda, f^0] = [\widetilde{X} \otimes_A X, \widetilde{\lambda_{(f^0)^{-1}}} \otimes \lambda_{(f^0)^{-1}}, \mathrm{id}_{H^0}] = [A, \rho, \mathrm{id}_{H^0}]$$

in $\mathrm{GPic}_H^{\rho,u}(A)$. Indeed, we identify $X \otimes_A \widetilde{X}$ and $\widetilde{X} \otimes_A X$ with A by the maps

$$\Phi_1 : X \otimes_A \widetilde{X} \to A : x \otimes \widetilde{y} \mapsto {}_A\langle x, y\rangle,$$

$$\Phi_2 : \widetilde{X} \otimes_A X \to A : \widetilde{x} \otimes y \mapsto \langle x, y\rangle_A,$$

respectively. Hence, for any $h \in H$, $x, y \in X$,

$$\begin{aligned}
\Phi_1(h \cdot_{\lambda \otimes \widetilde{\lambda}} x \otimes \widetilde{y}) &= \Phi_1([h_{(1)} \cdot_\lambda x] \otimes \widetilde{[S(h^*_{(2)}) \cdot_\lambda y]}) \\
&= {}_A\langle [h_{(1)} \cdot_\lambda x] \, , \, [S(h^*_{(2)}) \cdot_\lambda y]\rangle \\
&= h \cdot_\rho {}_A\langle x, y\rangle = h \cdot_\rho \Phi_1(x \otimes \widetilde{y}), \\
\Phi_2(h \cdot_{\widetilde{\lambda}_{(f^0)^{-1}} \otimes \lambda_{(f^0)^{-1}}} \widetilde{x} \otimes y) &= \Phi_2(\widetilde{[S(h^*_{(1)}) \cdot_{\lambda_{(f^0)^{-1}}} x]} \otimes [h_{(2)} \cdot_{\lambda_{(f^0)^{-1}}} y]) \\
&= \langle [S(h^*_{(1)}) \cdot_{\lambda_{(f^0)^{-1}}} x] \, , \, [h_{(2)} \cdot_{\lambda_{(f^0)^{-1}}} y]\rangle_A \\
&= h \cdot_\rho \langle x, y\rangle_A = h \cdot_\rho \Phi_2(\widetilde{x} \otimes y)
\end{aligned}$$

since $\lambda_{(f^0)^{-1}}$ is a coaction of H^0 on X with respect to $(A, A, \rho_{(f^0)^{-1}}, u_{(f^0)^{-1}}, \rho, u)$. Therefore, $[\widetilde{X}, \widetilde{\lambda_{(f^0)^{-1}}}, (f^0)^{-1}]$ is the inverse element of $[X, \lambda, f^0]$ in $\mathrm{GPic}_H^{\rho,u}(A)$. By the above product, $\mathrm{GPic}_H^{\rho,u}(A)$ is a group. We call it the generalized Picard group of a twisted coaction of (ρ, u) of H^0 on A.

Remark 3.7 If (ρ, u) and (σ, v) are generalized strongly Morita equivalent twisted coactions of H^0 on A and B, respectively, then $\mathrm{GPic}_H^{\rho,u}(A) \cong \mathrm{GPic}_H^{\sigma,v}(B)$. Indeed, since (ρ, u) and (σ, v) are generalized strongly Morita equivalent, there are a C^*-Hopf algebra automorphism g^0 of H^0, an $A - B$-equivalence bimodule Y and a twisted coaction μ of H^0 on Y such that $(A, B, \rho, u, \sigma_{g^0}, v_{g^0}, \mu, H^0)$ is a twisted covariant system by Lemma 2.1 and Definition 3.1. We define the map Φ from $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{GPic}_H^{\sigma,v}(B)$ by

$$\Phi([X, \lambda, f^0]) = [\widetilde{Y} \otimes_A X \otimes_A Y, \, \widetilde{\mu_{(g^0)^{-1}}} \otimes \lambda_{(g^0)^{-1}} \otimes \mu_{(g^0)^{-1} \circ f^0}, \, (g^0)^{-1} \circ f^0 \circ g^0]$$

for any $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Then by routine computations, we can see that Φ is an isomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ onto $\mathrm{GPic}_H^{\sigma,v}(B)$.

Let $\mathrm{Aut}(H^0)$ be the group of all C^*-Hopf algebra automorphisms of H^0. For any $f^0 \in \mathrm{Aut}(H^0)$, let $\mathrm{Aut}_H^{\rho,u,f^0}(A)$ be the set of all automorphisms α of A satisfying that

$$\rho \circ \alpha = (\alpha \otimes f^0) \circ \rho = (\alpha \otimes \mathrm{id}) \circ \rho_{f^0}, \quad (\underline{\alpha} \otimes f^0 \otimes f^0)(u) = u.$$

Let $\mathrm{Aut}_H^{\rho,u}(A) = \mathrm{Aut}_H^{\rho,u,\mathrm{id}_{H^0}}(A)$. We note that $\mathrm{Aut}_H^{\rho,u}(A)$ is a subgroup of $\mathrm{Aut}(A)$.

Remark 3.8 (1) Let $f^0, g^0 \in \mathrm{Aut}(H^0)$. For any $\alpha \in \mathrm{Aut}_H^{\rho,u,f^0}(A)$, $\beta \in \mathrm{Aut}_H^{\rho,u,g^0}(A)$, $\alpha \circ \beta \in \mathrm{Aut}_H^{\rho,u,f^0\circ g^0}(A)$ by easy computations.
(2) By the definitions of $\mathrm{Pic}_H^{\rho,u}(A)$ and $\mathrm{GPic}_H^{\rho,u}(A)$, we have the inclusion map $\jmath$:

$$\jmath : \mathrm{Pic}_H^{\rho,u}(A) \to \mathrm{GPic}_H^{\rho,u}(A) : [X, \lambda] \mapsto [X, \lambda, \mathrm{id}_{H^0}].$$

By easy computations, $\jmath$ is a monomorphism of $\mathrm{Pic}_H^{\rho,u}(A)$ into $\mathrm{GPic}_H^{\rho,u}(A)$. We regard $\mathrm{Pic}_H^{\rho,u}(A)$ as a subgroup of $\mathrm{GPic}_H^{\rho,u}(A)$ by $\jmath$. Then $\mathrm{Pic}_H^{\rho,u}(A)$ is a normal subgroup of $\mathrm{GPic}_H^{\rho,u}(A)$ by $\jmath$. Indeed, for any $[X, \lambda] \in \mathrm{Pic}_H^{\rho,u}(A)$ and $[Y, \mu, f^0] \in \mathrm{GPic}_H^{\rho,u}(A)$,

$$\begin{aligned}[Y, \mu, f^0][X, \lambda, \mathrm{id}_{H^0}][Y, \mu, f^0]^{-1} &= [Y \otimes_A X, \mu \otimes \lambda_{f^0}, f^0][\widetilde{Y}, \widetilde{\mu_{(f^0)^{-1}}}, f^0] \\ &= [Y \otimes_A X \otimes_A \widetilde{Y}\,,\ \mu \otimes \lambda_{f^0} \otimes (\widetilde{\mu_{(f^0)^{-1}}})_{f^0}\,,\ \mathrm{id}_{H^0}]\end{aligned}$$

in $\mathrm{GPic}_H^{\rho,u}(A)$. By the definition of $(\widetilde{\mu_{(f^0)^{-1}}})_{f^0}$, for any $y \in Y$,

$$(\widetilde{\mu_{(f^0)^{-1}}})_{f^0}(\widetilde{y}) = ((\mathrm{id} \otimes f^0) \circ (\mathrm{id} \otimes (f^0)^{-1}) \circ \widetilde{\mu})(\widetilde{y}) = \widetilde{\mu}(\widetilde{y}).$$

Thus,

$$[Y, \mu, f^0][X, \lambda, \mathrm{id}_{H^0}][Y, \mu, f^0]^{-1} = [Y \otimes_A X \otimes_A \widetilde{Y},\ \mu \otimes \lambda_{f^0} \otimes \widetilde{\mu}\,, \mathrm{id}_{H^0}]$$

in $\mathrm{GPic}_H^{\rho,u}(A)$. Hence, $\mathrm{Pic}_H^{\rho,u}(A)$ is a normal subgroup of $\mathrm{GPic}_H^{\rho,u}(A)$. Also, let η be the map from $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Aut}(H^0)$ defined by

$$\eta([X, \lambda, f^0]) = f^0$$

for any $[X, \lambda, f^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. Then clearly η is an epimorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ onto $\mathrm{Aut}(H^0)$. By the definition of η, $\mathrm{Im}\,\jmath = \mathrm{Ker}\,\eta$. Thus, we have the exact sequence

$$1 \longrightarrow \mathrm{Pic}_H^{\rho,u}(A) \stackrel{\jmath}{\longrightarrow} \mathrm{GPic}_H^{\rho,u}(A) \stackrel{\eta}{\longrightarrow} \mathrm{Aut}(H^0) \longrightarrow 1.$$

Furthermore, let κ be the map from $\mathrm{Aut}(H^0)$ to $\mathrm{GPic}_H^{\rho,u}(A)$ defined by

$$\kappa(f^0) = [{}_AA_A, \rho, f^0]$$

for any $f^0 \in \mathrm{Aut}(H^0)$. Then κ is a homomorphism of $\mathrm{Aut}(H^0)$ to $\mathrm{GPic}_H^{\rho,u}(A)$ with $\eta \circ \kappa = \mathrm{id}$ on $\mathrm{Aut}(H^0)$. Indeed, for any $f^0, g^0 \in \mathrm{Aut}(H^0)$,

$$\kappa(f^0)\kappa(g^0) = [{}_AA_A, \rho, f^0][{}_AA_A, \rho, g^0] = [A \otimes_A A\,,\ \rho \otimes \rho_{f^0}\,,\ f^0 \circ g^0]$$

in $\mathrm{GPic}_H^{\rho,u}(A)$. Let π be the $A-A$-equivalence bimodule isomorphism of $A \otimes_A A$ onto ${}_AA_A$ defined by $\pi(a \otimes b) = ab$ for any $a, b \in A$. Then for any $h \in H$, $f^0, g^0 \in \mathrm{Aut}(H^0)$,

$$\begin{aligned}\pi(h \cdot_{\rho\otimes\rho_{f^0}} a \otimes b) &= \pi(h \cdot_{\rho\otimes\rho_{f^0}} ab \otimes 1) = \pi([h_{(1)} \cdot_\rho ab] \otimes [h_{(2)} \cdot_{\rho_{f^0}} 1]) \\ &= [h_{(1)} \cdot_\rho ab]\epsilon(h_{(2)}) = h \cdot_\rho ab = h \cdot_\rho \pi(a \otimes b).\end{aligned}$$

Hence, $[A \otimes_A A\,,\ \rho \otimes \rho_{f^0}\,,\ f^0 \circ g^0] = [{}_AA_A\,,\ \rho\,,\ f^0 \circ g^0]$ in $\mathrm{GPic}_H^{\rho,u}(A)$. Thus,

$$\kappa(f^0)\kappa(g^0) = \kappa(f^0 \circ g^0).$$

It follows that κ is a homomorphism of $\mathrm{Aut}(H^0)$ to $\mathrm{GPic}_H^{\rho,u}(A)$. It is clear that $\eta\circ\kappa=\mathrm{id}$ on $\mathrm{Aut}(H^0)$. Therefore, $\mathrm{GPic}_H^{\rho,u}(A)$ is isomorphic to a semi-dierct product of $\mathrm{Pic}_H^{\rho,u}(A)$ by $\mathrm{Aut}(H^0)$.

(3) Generally, $\mathrm{Aut}_H^{\rho,u,f^0}(A)\cap\mathrm{Aut}_H^{\rho,u,g^0}(A)\neq\emptyset$ even if $f^0,g^0\in\mathrm{Aut}(H^0)$ with $f^0\neq g^0$. Indeed, let $\rho_{H^0}^A$ be the trivial coaction of H^0 on A. Then for any $\alpha\in\mathrm{Aut}(A)$, $f^0\in\mathrm{Aut}(H^0)$,

$$(\rho_{H^0}^A\circ\alpha)(a)=\alpha(a)\otimes 1^0, \quad ((\alpha\otimes f^0)\circ\rho_{H^0}^A)(a)=\alpha(a)\otimes 1^0$$

for any $a\in A$. Hence, $\rho_{H^0}^A\circ\alpha=(\alpha\otimes f^0)\circ\rho_{H^0}^A$. Thus, for any $f^0\in\mathrm{Aut}(H^0)$, $\mathrm{Aut}_H^{\rho_{H^0}^A, f^0}(A)=\mathrm{Aut}(A)$.

Modifying [3] and [10], for each $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$, we construct the element $(X_\alpha,\lambda_\alpha,f^0)\in\mathrm{GEqui}_H^{\rho,u,f^0}(A)$ as follows: Let $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$. Let X_α be the vector space A with the obvious left action of A on X_α and the obvious left A-valued inner product, but we define the right action of A on X_α by $x\cdot a=x\alpha(a)$ for any $x\in X_\alpha$, $a\in A$ and define the right A-valued inner product by $\langle x,y\rangle_A=\alpha^{-1}(x^*y)$ for any $x,y\in X_\alpha$. Then by [3], X_α is an $A-A$-equivalence bimodule. Also, ρ can be regarded as a linear map from X_α to an $A\otimes H^0-A\otimes H^0$-equivalence bimodule $X_\alpha\otimes H^0$. We denote it by λ_α. By easy computations, λ_α is a twisted coaction H^0 on X_α with respect to $(A,A,\rho,u,\rho_{f^0},u_{f^0})$. Thus, we obtain the map Φ_{f^0}:

$$\Phi_{f^0}:\mathrm{Aut}_H^{\rho,u,f^0}(A)\to\mathrm{GPic}_H^{\rho,u}(A):\alpha\mapsto[X_\alpha,\lambda_\alpha,f^0].$$

Let $\mathcal{G}=\cup_{f^0\in\mathrm{Aut}(H^0)}\mathrm{Aut}_H^{\rho,u,f^0}(A)$ and let Φ be the map from $\mathcal{G}$ to $\mathrm{GPic}_H^{\rho,u}(A)$ defined by

$$\Phi(\alpha)=\Phi_{f^0}(\alpha)=[X_\alpha,\lambda_\alpha,f^0]$$

for any $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$, where $f^0\in\mathrm{Aut}(H^0)$. We note that by easy computations, $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(A)$.

Lemma 3.9 *With the above notation, the map* $\Phi:\mathcal{G}\to\mathrm{GPic}_H^{\rho,u}(A)$ *is a homomorphism.*

Proof Let $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$ and $\beta\in\mathrm{Aut}_H^{\rho,u,g^0}(A)$. Then $\alpha\circ\beta\in\mathrm{Aut}_H^{\rho,u,f^0\circ g^0}(A)$. Also,

$$\Phi(\alpha)\Phi(\beta)=[X_\alpha,\lambda_\alpha,f^0][X_\beta,\lambda_\beta,g^0]=[X_\alpha\otimes_A X_\beta\, ,\, \lambda_\alpha\otimes(\lambda_\beta)_{f^0}\, ,\, f^0\circ g^0].$$

Let π be the map from $X_\alpha\otimes_A X_\beta$ to $X_{\alpha\circ\beta}$ defined by

$$\pi(x\otimes y)=x\alpha(y)$$

for any $x\in X_\alpha$, $y\in X_\beta$. By easy computations, π is an $A-A$-equivalence bimodule isomorphism of $X_\alpha\otimes_A X_\beta$ onto $X_{\alpha\circ\beta}$. Also, for any $h\in H$, $x,y\in A$,

$$\begin{aligned}\pi(h\cdot_{\lambda_\alpha\otimes(\lambda_\beta)_{f^0}} x\otimes y)&=\pi([h_{(1)}\cdot_{\lambda_\alpha}x]\otimes[h_{(2)}\cdot_{(\lambda_\beta)_{f^0}}y])\\&=\pi([h_{(1)}\cdot_\rho x]\otimes[h_{(2)}\cdot_{\rho_{f^0}}y])\\&=[h_{(1)}\cdot_\rho x]\alpha([h_{(2)}\cdot_{\rho_{f^0}}y]).\end{aligned}$$

Since $\rho\circ\alpha=(\alpha\otimes\mathrm{id})\circ\rho_{f^0}$, we can see that

$$\pi(h\cdot_{\lambda_\alpha\otimes(\lambda_\beta)_{f^0}} x\otimes y)=[h_{(1)}\cdot_\rho x][h_{(2)}\cdot_\rho \alpha(y)]=h\cdot_\rho x\alpha(y)=h\cdot_\rho \pi(x\otimes y).$$

Hence,

$$\Phi(\alpha)\Phi(\beta)=[X_{\alpha\circ\beta}\,,\,\lambda_{\alpha\circ\beta}\,,\,f^0\circ g^0]=\Phi(\alpha\circ\beta).$$

Therefore, we obtain the conclusion. □

Let $\mathrm{Int}_H^{\rho,u}(A)$ be the group of all generalized inner automorphisms $\mathrm{Ad}(v)$ satisfying that v is a unitary element in $M(A)$ with $\underline{\rho}(v)=v\otimes 1^0$ and that $u(v\otimes 1^0\otimes 1^0)=(v\otimes 1^0\otimes 1^0)u$. By easy computations, $\mathrm{Int}_H^{\rho,u}(A)$ is a normal subgroup of $\mathcal{G}$. Indeed, for any $f^0\in\mathrm{Aut}(H^0)$, $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$ and $\mathrm{Ad}(v)\in\mathrm{Int}_H^{\rho,u}(A)$,

$$\alpha\circ\mathrm{Ad}(v)\circ\alpha^{-1}=\mathrm{Ad}(\underline{\alpha}(v))\circ\alpha\circ\alpha^{-1}=\mathrm{Ad}(\underline{\alpha}(v)).$$

Also,

$$\begin{aligned}
\underline{\rho}(\underline{\alpha}(v))&=((\underline{\alpha}\otimes\mathrm{id})\circ\underline{\rho})(v)=(\underline{\alpha}\otimes\mathrm{id})(v\otimes 1^0)=\underline{\alpha}(v)\otimes 1^0,\\
u(\underline{\alpha}(v)\otimes 1^0\otimes 1^0)&=(\underline{\alpha}\otimes f^0\otimes f^0)(u)(\underline{\alpha}\otimes f^0\otimes f^0)(v\otimes 1^0\otimes 1^0)\\
&=(\underline{\alpha}\otimes f^0\otimes f^0)(u(v\otimes 1^0\otimes 1^0))\\
&=(\underline{\alpha}\otimes f^0\otimes f^0)((v\otimes 1^0\otimes 1^0)u)\\
&=(\underline{\alpha}(v)\otimes 1^0\otimes 1^0)u.
\end{aligned}$$

Hence, $\mathrm{Ad}(\underline{\alpha}(v))\in\mathrm{Int}_H^{\rho,u}(A)$. Thus, $\mathrm{Int}_H^{\rho,u}(A)$ is a normal subgroup of $\mathcal{G}$. We denote by $\imath$ the inclusion map from $\mathrm{Int}_H^{\rho,u}(A)$ to $\mathcal{G}$.

Lemma 3.10 *With the above notation, we have the exact sequence*

$$1\longrightarrow\mathrm{Int}_H^{\rho,u}(A)\stackrel{\imath}{\longrightarrow}\mathcal{G}\stackrel{\Phi}{\longrightarrow}\mathrm{GPic}_H^{\rho,u}(A).$$

Proof We have to show that $\mathrm{Im}\,\imath=\mathrm{Ker}\,\Phi$. It is clear that $\mathrm{Im}\,\imath\subset\mathrm{Ker}\,\Phi$ by [10, Proposition 5.1] and Remark 3.8(2). We show that $\mathrm{Ker}\,\Phi\subset\mathrm{Im}\,\imath$. Let $f^0\in\mathrm{Aut}(H^0)$ and $\alpha\in\mathrm{Aut}_H^{\rho,u,f^0}(A)$. We suppose that $\Phi(\alpha)=[X_\alpha\,,\,\lambda_\alpha,\,f^0]=[{}_AA_A\,,\,\rho\,,\,\mathrm{id}_{H^0}]$ in $\mathrm{GPic}_H^{\rho,u}(A)$. Then $f^0=\mathrm{id}_{H^0}$ and by Remark 3.8(2), $[X_\alpha,\lambda_\alpha]=[{}_AA_A,\rho]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. Hence, by [10, Proposition 5.1], there is a unitary element $v\in M(A)$ such that

$$\alpha=\mathrm{Ad}(v),\quad \underline{\rho}(v)=v\otimes 1^0,\quad u(v\otimes 1^0\otimes 1^0)=u(v\otimes 1^0\otimes 1^0).$$

Hence, $\mathrm{Ker}\,\Phi\subset\mathrm{Im}\,\imath$. Therefore, we obtain the conclusion. □

Let A be a unital C^*-algebra and $\mathbf{K}$ the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Let $A^s=A\otimes\mathbf{K}$. Let ρ be a coaction of H^0 on A and ρ^s the coaction of H^0 on A^s induced by ρ, that is, $\rho^s=\rho\otimes\mathrm{id}_{\mathbf{K}}$. Let Φ be the map from $\mathcal{G}$ to $\mathrm{GPic}_H^{\rho^s}(A^s)$ defined in the above.

Modifying the proof of [10, Lemma 5.4], we shall show that Φ is surjective under some assumptions.

We suppose that $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ in $(A \rtimes_\rho H) \otimes H$. Let $f^0 \in \mathrm{Aut}(H^0)$. Then $\widehat{\rho_{f^0}}(1 \rtimes_{\rho_{f^0}} e) \sim (1 \rtimes_{\rho_{f^0}} e) \otimes 1$ in $(A \rtimes_{\rho_{f^0}} H) \otimes H$, where $\widehat{\rho_{f^0}}$ is the dual coaction of ρ_{f^0}. Indeed, by [16, Lemma 6.1], there is an isomorphism π of $A \rtimes_\rho H$ onto $A \rtimes_{\rho_{f^0}} H$ defined by

$$\pi(a \rtimes_\rho h) = a \rtimes_{\rho_{f^0}} f(h)$$

for any $a \in A$, $h \in H$, which satisfies that

$$\widehat{\rho_{f^0}} \circ \pi = (\pi \otimes \mathrm{id}_H) \circ (\mathrm{id}_{A \rtimes_\rho H} \otimes f) \circ \widehat{\rho},$$

where f is the C^*-Hopf algebra automorphism of H induced by f^0, that is

$$f^0(\varphi)(h) = \varphi(f^{-1}(h))$$

for any $h \in H$, $\varphi \in H^0$. Hence, since $\pi(1 \rtimes_\rho e) = 1 \rtimes_{\rho_{f^0}} f(e) = 1 \rtimes_{\rho_{f^0}} e$,

$$\begin{aligned} \widehat{\rho_{f^0}}(1 \rtimes_{\rho_{f^0}} e) &= ((\pi \otimes \mathrm{id}_H) \circ (\mathrm{id}_{A \rtimes_\rho H} \otimes f))(\widehat{\rho}(1 \rtimes_\rho e)) \\ &\sim ((\pi \otimes \mathrm{id}_H) \circ (\mathrm{id}_{A \rtimes_\rho H} \otimes f))((1 \rtimes_\rho e) \otimes 1) \\ &= (\pi \otimes \mathrm{id}_H)((1 \rtimes_\rho e) \otimes 1) = (1 \rtimes_{\rho_{f^0}} e) \otimes 1 \end{aligned}$$

in $(A \rtimes_{\rho_{f^0}} H) \otimes H$. It follows by [10, Lemma 5.2] that

$$\widehat{\rho^s}(1 \rtimes_{\rho^s} e) \sim (1 \rtimes_{\rho^s} e) \otimes 1 \quad \text{in} \quad (M(A^s) \rtimes_{\underline{\rho^s}} H) \otimes H,$$

$$\widehat{\rho^s_{f^0}}(1 \rtimes_{\rho^s_{f^0}} e) \sim (1 \rtimes_{\rho^s_{f^0}} e) \otimes 1 \quad \text{in} \quad (M(A^s) \rtimes_{\underline{\rho^s_{f^0}}} H) \otimes H.$$

Let $[X, \lambda, f^0]$ be any element in $\mathrm{GPic}_H^{\rho^s, f^0}(A^s)$. Let

$$X^\lambda = \{x \in X \mid \lambda(x) = x \otimes 1^0\}.$$

Since X is an $A^s - A^s$-equivalence bimodule, by [10, Lemma 3.9 and Theorem 4.9], X^λ is an $(A^s)^{\rho^s_{f^0}} - (A^s)^{\rho^s_{f^0}}$-equivalence bimodule. On the other hand, $(A^s)^{\rho^s_{f^0}} = (A^s)^{\rho^s}$. Indeed, for any $a \in (A^s)^{\rho^s_{f^0}}$, $(\mathrm{id} \otimes f^0)(\rho^s(a)) = a \otimes 1^0$. Hence, $\rho^s(a) = a \otimes 1^0$, that is, $a \in (A^s)^{\rho^s}$. Also, for any $a \in (A^s)^{\rho^s}$,

$$\rho^s_{f^0}(a) = (\mathrm{id} \otimes f^0)(\rho^s(a)) = (\mathrm{id} \otimes f^0)(a \otimes 1^0) = a \otimes 1^0.$$

Hence, $a \in (A^s)^{\rho^s_{f^0}}$. It follows that X^λ is an $(A^s)^{\rho^s} - (A^s)^{\rho^s}$- equivalence bimodule. Let C be the linking C^*-algebra for X and γ the coaction of H^0 on C induced by ρ^s and λ. Let C^γ be the fixed point C^*-algebra of C for γ. Then by [10, Lemma 4.6], C^γ is isomorphic to C_0, the linking C^*-algebra for X^λ. We identify C^γ with C_0. Let

$$p = \begin{bmatrix} 1_A \otimes 1_{M(\mathbf{K})} & 0 \\ 0 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 & 0 \\ 0 & 1_A \otimes 1_{M(\mathbf{K})} \end{bmatrix}.$$

Since $M(C)^{\underline{\gamma}} = M(C^\gamma)$ by [10, Lemmas 2.14 and 4.7], p and q are full for C^γ such that there is a partial isometry $w \in M(C)^{\underline{\gamma}}$ with $w^* w = p$, $ww^* = q$. Let α be the map on A^s defined by

$$\alpha(a) = w^* a w = w^* \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} w$$

for any $a \in A^s$. By routine computations, α is an automorphism of A^s. Let π be the linear map from X to X_α defined by

$$\pi(x) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w = p \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} wp$$

for any $x \in X$. In the same way as in the proof of [3, Lemma 3.3], we can see that π is an $A^s - A^s$-equivalence bimodule isomorphism of X onto X_α. For any $a \in A^s$,

$$\begin{aligned} (\rho^s \circ \alpha)(a) &= \rho^s(w^* a w) = \gamma(w^* \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} w) \\ &= \underline{\gamma}(w^*) \begin{bmatrix} 0 & 0 \\ 0 & \rho^s_{f^0}(a) \end{bmatrix} \underline{\gamma}(w) \\ &= (w^* \otimes 1^0) \begin{bmatrix} 0 & 0 \\ 0 & \rho^s_{f^0}(a) \end{bmatrix} (w \otimes 1^0) \\ &= (\alpha \otimes \mathrm{id}_{H^0})(\rho^s_{f^0}(a)) \end{aligned}$$

since $w \in M(C)^{\underline{\gamma}}$. Hence, $\alpha \in \mathrm{Aut}_H^{\rho^s, f^0}(A)$. Furthermore,

$$\begin{aligned} (\lambda_\alpha \circ \pi)(x) &= \lambda_\alpha(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w) = \rho^s(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w) \\ &= \gamma(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} w) = \begin{bmatrix} 0 & \lambda(x) \\ 0 & 0 \end{bmatrix} (w \otimes 1^0) \\ &= (\pi \otimes \mathrm{id})(\lambda(x)), \end{aligned}$$

where we identify $\mathbf{K} \otimes H^0$ and $H^0 \otimes \mathbf{K}$. Thus, $\Phi(\alpha) = [X, \lambda, f^0]$. Therefore, we obtain the following lemma:

Lemma 3.11 *With the above notation, we suppose that* $\widehat{\rho}(1 \rtimes_\rho e) \sim (1 \rtimes_\rho e) \otimes 1$ *in* $(A \rtimes_\rho H) \otimes H$. *Then* Φ *is a surjective homomorphism of* $\mathcal{G}$ *onto* $\mathrm{GPic}_H^{\rho^s}(A^s)$.

Proof This is immediate by the above discussions and Lemma 3.9. □

Proposition 3.12 *With the above notation, we have the exact sequence*

$$1 \longrightarrow \mathrm{Int}_H^{\rho^s}(A^s) \xrightarrow{\imath} \mathcal{G} \xrightarrow{\Phi} \mathrm{GPic}_H^{\rho^s}(A^s) \longrightarrow 1.$$

Proof This is immediate by Lemmas 3.10 and 3.11. □

4. Crossed products

Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on a unital C^*-algebra A. Let $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Then λ is a twisted coaction of H^0 on X with respect to $(A, A, \rho, u, \rho_{f^0}, u_{f^0})$. Hence, by [14, Section 4], $\widehat{\lambda}$ is a coaction of H on $X \rtimes_\lambda H$ with respect to $(A \rtimes_{\rho,u} H, A \rtimes_{\rho_{f^0}, u_{f^0}} H, \widehat{\rho}, \widehat{\rho_{f^0}})$, where $\widehat{\rho}$ and $\widehat{\rho_{f^0}}$ are the dual coactions of (ρ, u) and (ρ_{f^0}, u_{f^0}), which are coactions of H on $A \rtimes_{\rho,u} H$ and $A \rtimes_{\rho_{f^0},u_{f^0}} H$, respectively. Also, by [16, Lemma 6.1], there is an isomorphism π of $A \rtimes_{\rho,u} H$ onto $A \rtimes_{\rho_{f^0},u_{f^0}} H$ such that

$$\widehat{\rho_{f^0}} \circ \pi = (\pi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes f) \circ \widehat{\rho}, \quad \pi(a \rtimes_{\rho,u} h) = a \rtimes_{\rho_{f^0},u_{f^0}} f(h),$$

where $a \in A$, $h \in H$ and f is a C^*-Hopf algebra automorphism of H induced by f^0. By the above isomorphism π, we can regard $\widehat{\lambda}$ as a coaction of H on $X \rtimes_\lambda H$ with respect to $(A \rtimes_{\rho,u} H, A \rtimes_{\rho,u} H, \widehat{\rho}, (\widehat{\rho})_f)$, where $(\widehat{\rho})_f = (\mathrm{id} \otimes f) \circ \widehat{\rho}$. Thus, we obtain the element

$$(X \rtimes_\lambda H, \widehat{\lambda}, f) \in \mathrm{GEqui}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H).$$

Let F be the map from $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$ defined by

$$F([X, \lambda, f^0]) = [X \rtimes_\lambda H, \widehat{\lambda}, f]$$

for any $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. We can see that F is well-defined in a straightforward way. In this section, we show that F is an isomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ onto $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$. First, we show that F is a homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$. Let $[X, \lambda, f^0], [Y, \mu, g^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. Then

$$\begin{aligned} F([X, \lambda, f^0][Y, \mu, g^0]) &= F([X \otimes_A Y, \lambda \otimes \mu_{f^0}, f^0 \circ g^0]) \\ &= [(X \otimes_A Y) \rtimes_{\lambda \otimes \mu_{f^0}} H, \widehat{\lambda \otimes \mu_{f^0}}, j], \end{aligned}$$

where j is the C^*-Hopf algebra automorphism of H induced by $f^0 \circ g^0$. Then $j = f \circ g$. Indeed, for any $h \in H$, $\varphi \in H^0$,

$$\varphi(j^{-1}(h)) = (f^0 \circ g^0)(\varphi(h)) = g^0(\varphi)(f^{-1}(h)) = \varphi((g^{-1} \circ f^{-1})(h)).$$

Hence, $j = f \circ g$. Thus,

$$F([X, \lambda, f^0][Y, \mu, g^0]) = [(X \otimes_A Y) \rtimes_{\lambda \otimes \mu_{f^0}} H, \widehat{\lambda \otimes \mu_{f^0}}, f \circ g].$$

By [10, Lemmas, 7.1 and 7.2], there is an $A \rtimes_{\rho,u} H - A \rtimes_{\rho_{f^0 \circ g^0},u_{f^0 \circ g^0}} H$-equivalence bimodule isomorphism Φ of $(X \otimes_A Y) \rtimes_{\lambda \otimes \mu_{f^0}} H$ onto $(X \rtimes_\lambda H) \otimes_{A \rtimes_{\rho_{f^0},u_{f^0}} H} (Y \rtimes_{\mu_{f^0}} H)$ such that

$$\Phi(\varphi \cdot_{\widehat{\lambda \otimes \mu_{f^0}}} (x \otimes y \rtimes_{\lambda \otimes \mu_{f^0}} h)) = \varphi \cdot_{\widehat{\lambda} \otimes \widehat{\mu_{f^0}}} \Phi(x \otimes y \rtimes_{\lambda \otimes \mu_{f^0}} h)$$

for any $x \in X$, $y \in Y$, $h \in H$ and $\varphi \in H^0$. Then Φ is defined by

$$\Phi(x \otimes y \rtimes_{\lambda \otimes \mu_{f^0}} h) = (x \rtimes_\lambda 1) \otimes (y \rtimes_{\mu_{f^0}} h)$$

for any $x \in X$, $y \in Y$ and $h \in H$. Let π_A (resp. $\pi_A^{g^0}$) be the linear map from $A \rtimes_{\rho,u} H$ (resp. $A \rtimes_{\rho_{f^0 \circ g^0},u_{f^0 \circ g^0}} H$) to $A \rtimes_{\rho_{f^0},u_{f^0}} H$ (resp. $A \rtimes_{\rho_{f^0 \circ g^0},u_{f^0 \circ g^0}} H$) defined by

$$\pi_A(a \rtimes_{\rho,u} h) = a \rtimes_{\rho_{f^0},u_{f^0}} f(h)$$

$$(\text{resp.} \quad \pi_A^{g^0}(a \rtimes_{\rho_{g^0},u_{g^0}} h) = a \rtimes_{\rho_{f^0 \circ g^0},u_{f^0 \circ g^0}} f(h))$$

for any $a \in A$, $h \in H$. Then by [16, Lemma 6.1], π_A and $\pi_A^{g^0}$ are isomorphisms of $A \rtimes_{\rho,u} H$ and $A \rtimes_{\rho_{g^0},u_{g^0}} H$ onto $A \rtimes_{\rho_{f^0},u_{f^0}} H$ and $A \rtimes_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}} H$, respectively. Also, let π_Y be the linear map from $Y \rtimes_\mu H$ to $Y \rtimes_{\mu_{f^0}} H$ defined by

$$\pi_Y(y \rtimes_\mu h) = y \rtimes_{\mu_{f^0}} f(h)$$

for any $y \in Y$, $h \in H$. Clearly π_Y is surjective.

Lemma 4.1 *With the above notation, the following conditions hold:*
(1) $\pi_Y((a \rtimes_{\rho,u} h) \cdot (y \rtimes_\mu l)) = \pi_A(a \rtimes_{\rho,u} h) \cdot \pi_Y(y \rtimes_\mu l)$,
(2) $\pi_Y((y \rtimes_\mu l) \cdot (a \rtimes_{\rho_{g^0},u_{g^0}} h)) = \pi_Y(y \rtimes_\mu l) \cdot \pi_A^{g^0}(a \rtimes_{\rho_{g^0},u_{g^0}} h)$,
(3) ${}_{A\rtimes_{\rho_{f^0},u_{f^0}}H}\langle \pi_Y(y \rtimes_\mu h), \pi_Y(z \rtimes_\mu l)\rangle = \pi_A({}_{A\rtimes_{\rho,u}H}\langle y \rtimes_\mu h, z \rtimes_\mu l\rangle)$,
(4) $\langle \pi_Y(y \rtimes_\mu h), \pi_Y(z \rtimes_\mu l)\rangle_{A\rtimes_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}H} = \pi_A^{g^0}(\langle y \rtimes_\mu h, z \rtimes_\mu l\rangle_{A\rtimes_{\rho_{f^0},u_{f^0}}H})$,
(5) $\widehat{\mu_{f^0}} \circ \pi_Y = (\pi_Y \otimes \mathrm{id}_H) \circ (\widehat{\mu})_f$
for any $a \in A$, $y, z \in Y$, $h, l \in H$.

Proof We can prove Conditions (1)–(4) in a straightforward way by the definitions of π_A, $\pi_A^{g^0}$, and π_Y. We prove Condition (5). For any $y \in Y$, $h \in H$,

$$(\widehat{\mu_{f^0}} \circ \pi_Y)(y \rtimes_\mu h) = \widehat{\mu_{f^0}}(y \rtimes_{\mu_{f^0}} f(h)) = (y \rtimes_{\mu_{f^0}} f(h_{(1)})) \otimes f(h_{(2)}).$$

On the other hand,

$$\begin{aligned}((\pi_Y \otimes \mathrm{id}_H) \circ (\widehat{\mu})_f)(y \rtimes_\mu h) &= ((\pi_Y \otimes \mathrm{id}_H) \circ (\mathrm{id}_{Y\rtimes_\mu H} \otimes f) \circ \widehat{\mu})(y \rtimes_\mu h) \\ &= ((\pi_Y \otimes \mathrm{id}_H) \circ (\mathrm{id}_{Y\rtimes_\mu H} \otimes f))((y \rtimes_\mu h_{(1)}) \otimes h_{(2)}) \\ &= (\pi_Y \otimes \mathrm{id}_H)((y \rtimes_\mu h_{(1)}) \otimes f(h_{(2)})) \\ &= (y \rtimes_{\mu_f^0} f(h_{(1)})) \otimes f(h_{(2)}).\end{aligned}$$

Hence, we obtain the conclusion. □

Lemma 4.2 *With the above notation, the map F is a homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A \rtimes_{\rho,u} H)$.*

Proof Let $[X, \lambda, f^0], [Y, \mu, g^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. Then

$$F([X, \lambda, f^0][Y, \mu, g^0]) = [(X \otimes_A Y) \rtimes_{\lambda\otimes\mu_{f^0}} H, \widehat{\lambda \otimes \mu_{f^0}}, f \circ g].$$

Also,

$$\begin{aligned}F([X, \lambda, f^0])F([Y, \mu, g^0]) &= [X \rtimes_\lambda H, \widehat{\lambda}, f][Y \rtimes_\mu H, \widehat{\mu}, g] \\ &= [(X \rtimes_\lambda H) \otimes_{A\rtimes_{\rho_{f^0},u_{f^0}}H} (Y \rtimes_\mu H), \widehat{\lambda} \otimes \widehat{\mu_{f^0}}, f \circ g].\end{aligned}$$

By the discussions before Lemma 4.1, there is an $A\rtimes_{\rho,u}H - A\rtimes_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}H$-equivalence bimodule isomorphism Φ of $(X \otimes_A Y) \rtimes_{\lambda\otimes\mu_{f^0}} H$ onto $(X \rtimes_\lambda H) \otimes_{A\rtimes_{\rho_{f^0},u_{f^0}}H} (Y \rtimes_{\mu_{f^0}} H)$ such that

$$\Phi(\varphi \cdot_{\widehat{\lambda\otimes\mu_{f^0}}} ((x \otimes y) \rtimes_{\lambda\otimes\mu_{f^0}} h)) = \varphi \cdot_{\widehat{\lambda}\otimes\widehat{\mu_{f^0}}} \Phi((x \otimes y) \rtimes_{\lambda\otimes\mu_{f^0}} h)$$

for any $x\in X$, $y\in Y$, $h\in H$ and $\varphi\in H^0$. Since we identify $A\rtimes_{\rho_{f^0}, u_{f^0}} H$ with $A\rtimes_{\rho, u} H$ in the tensor product $(X\rtimes_\lambda H)\otimes_{A\rtimes_{\rho_{f^0}, u_{f^0}} H}(Y\rtimes_\mu H)$ by the isomorphism π_A by Lemma 4.1, we can see that

$$F([X, \lambda, f^0])F([Y, \mu, g^0])=F([X, \lambda, f^0][Y, \mu, g^0])$$

Therefore, we obtain the conclusion. □

Next, we construct the inverse homomorphism of F from $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A\rtimes_{\rho, u} H)$ to $\mathrm{GPic}_H^{\rho, u}(A)$ modifying [10, Section 7]. By the above discussions, there is the homomorphism of $\widehat{F}$ of $\mathrm{GPic}_H^{\widehat{\rho}}(A\rtimes_{\rho, u} H)$ to $\mathrm{GPic}_H^{\widehat{\widehat{\rho}}}(A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0)$ defined by

$$\widehat{F}([Y, \mu, f])=[Y\rtimes_\mu H^0, \widehat{\mu}, f^0]$$

for any $[Y, \mu, f]\in\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A\rtimes_{\rho, u} H)$. By [10, Proposition 2.13], there are an isomorphism Ψ_A of $A\otimes M_N(\mathbf{C})$ onto $A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0$ and a unitary element $U\in(A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0)\otimes H^0$ such that

$$\mathrm{Ad}(U)\circ\widehat{\widehat{\rho}}=(\Psi_A\otimes\mathrm{id}_{H^0})\circ(\rho\otimes\mathrm{id}_{M_N(\mathbf{C})})\circ\Psi^{-1},$$

$$(\Psi_A\otimes\mathrm{id}_{H^0}\otimes\mathrm{id}_{H^0})(u\otimes I_N)=(U\otimes 1^0)(\widehat{\widehat{\rho}}\otimes\mathrm{id}_{H^0})(U)(\mathrm{id}\otimes\Delta^0)(U^*).$$

Let $\overline{\rho}=(\Psi_A^{-1}\otimes\mathrm{id}_{H^0})\circ\widehat{\widehat{\rho}}\circ\Psi_A$. For any $[X, \lambda, f^0]\in\mathrm{GPic}_H^{\widehat{\widehat{\rho}}}(A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0)$, we construct an element $[X_{\Psi_A}, \lambda_{\Psi_A}, f^0]\in\mathrm{GPic}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C}))$ as follows: Let $X_{\Psi_A}=X$ as vector spaces. For any $x, y\in X_{\Psi_A}$ and $a\in A\otimes M_N(\mathbf{C})$,

$$a\cdot_{\Psi_A} x=\Psi_A(a)\cdot x \quad , \quad x\cdot_{\Psi_A} a=x\cdot\Psi_A(a),$$

$${}_{A\otimes M_N(\mathbf{C})}\langle x, y\rangle=\Psi_A^{-1}({}_{A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0}\langle x, y\rangle),$$

$$\langle x, y\rangle_{A\otimes M_N(\mathbf{C})}=\Psi_A^{-1}(\langle x, y\rangle_{A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0}).$$

We regard λ as a linear map from X_{Ψ_A} to $X_{\Psi_A}\otimes H^0$. We denote it by λ_{Ψ_A}. Then $(X_{\Psi_A}, \lambda_{\Psi_A}, f^0)\in\mathrm{GEqui}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C}))$. By easy computations, the map

$$\mathrm{GPic}_H^{\widehat{\widehat{\rho}}}(A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}} H^0)\to\mathrm{GPic}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C})): [X, \lambda, f^0]\mapsto[X_{\Psi_A}, \lambda_{\Psi_A}, f^0]$$

is well-defined and it is an isomorphism of $\mathrm{GPic}_H^{\widehat{\widehat{\rho}}}(A\rtimes_{\rho, u} H\rtimes_{\widehat{\rho}})$ onto $\mathrm{GPic}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C}))$. We denote by G_1 the above isomorphism. Furthermore, the coaction $\overline{\rho}$ of H^0 on $A\otimes M_N(\mathbf{C})$ is exterior equivalent to the twisted coaction $(\rho\otimes\mathrm{id}, u\otimes I_N)$ since

$$\rho\otimes\mathrm{id}_{M_N(\mathbf{C})}=(\Psi_A^{-1}\otimes\mathrm{id}_{H^0})\circ\mathrm{Ad}(U)\circ\widehat{\widehat{\rho}}\circ\Psi_A=\mathrm{Ad}(U_1)\circ\overline{\rho},$$

$$u\otimes I_N=(U_1\otimes 1^0)(\overline{\rho}\otimes\mathrm{id})(U_1)(\mathrm{id}\otimes\Delta^0)(U_1^*),$$

where $U_1=(\Psi_A^{-1}\otimes\mathrm{id}_{H^0})(U)$.

Hence, there is the isomorphism G_2 of $\mathrm{GPic}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C}))$ onto $\mathrm{GPic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})}, u\otimes I_N}(A\otimes M_n(\mathbf{C}))$ defined by

$$G_2([X, \lambda, f^0])=[X, \mathrm{Ad}(U_1)\circ\lambda, f^0]$$

for any $[X,\lambda,f^0]\in \mathrm{GPic}_H^{\overline{\rho}}(A\otimes M_N(\mathbf{C}))$. Since (ρ,u) is strongly Morita equivalent to $(\rho\otimes \mathrm{id}_{M_N(\mathbf{C})}, u\otimes I_N)$, there is the isomorphism of G_3 of $\mathrm{GPic}_H^{\rho,u}(A)$ onto $\mathrm{GPic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},u\otimes I_N}(A\otimes M_N(\mathbf{C}))$ defined by

$$G_3([X,\lambda,f^0])=[X\otimes M_N(\mathbf{C})\,,\,\lambda\otimes\mathrm{id}_{M_N(\mathbf{C})}\,,\,f^0]$$

for any $[X,\lambda,f^0]\in\mathrm{GPic}_H^{\rho,u}(A)$. Let $G=G_3^{-1}\circ G_2\circ G_1\circ\widehat{F}$. Thus, G is a homomorphism of $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A\rtimes_{\rho,u}H)$ to $\mathrm{GPic}_H^{\rho,u}(A)$.

Lemma 4.3 *With the above notation,* $G\circ F=\mathrm{id}$ *on* $\mathrm{GPic}_H^{\rho,u}(A)$.

Proof We prove the lemma modifying [10, Proposition 7.4]. Let $[X,\lambda,f^0]\in\mathrm{GPic}_H^{\rho,u}(A)$. By the definitions of $F,\widehat{F},G_1,G_2$,

$$\begin{aligned}(G_2\circ G_1\circ\widehat{F}\circ F)([X,\lambda,f^0])&=(G_2\circ G_1\circ\widehat{F})([X\rtimes_\lambda H\,,\,\widehat{\lambda}\,,\,f])\\&=(G_2\circ G_1)([X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0\,,\,\widehat{\widehat{\lambda}}\,,\,f^0])\\&=G_2([(X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0)_{\Psi_A}\,,\,(\widehat{\widehat{\lambda}})_{\Psi_A}\,,\,f^0])\\&=[(X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0)_{\Psi_A}\,,\,\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A}\,,\,f^0].\end{aligned}$$

Let Ψ_X be the linear map from $X\otimes M_N(\mathbf{C})$ to $X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0$ defined in [10, Proposition 3.8] and we regard Ψ_X as an $A\otimes M_N(\mathbf{C})-A\otimes M_N(\mathbf{C})$-equivalence bimodule isomorphism of $X\otimes M_N(\mathbf{C})$ onto $(X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0)_{\Psi_A}$. Also, since

$$\mathrm{Ad}(U)\circ\widehat{\widehat{\lambda}}=(\Psi_X\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id})\circ\Psi_X^{-1}$$

by [10, Proposition 3.8], for any $x\in A\otimes M_N(\mathbf{C})$,

$$\begin{aligned}(\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A})(x)&=U_1\cdot_{\Psi_A}(\widehat{\widehat{\lambda}})_{\Psi_A}(x)\cdot_{\Psi_A}U_1^*=U\widehat{\widehat{\lambda}}(x)U^*\\&=((\Psi_X\otimes\mathrm{id})\circ(\lambda\otimes\mathrm{id})\circ\Psi_X^{-1})(x).\end{aligned}$$

Thus,

$$[(X\rtimes_\lambda H\rtimes_{\widehat{\lambda}}H^0)_{\Psi_A}\,,\,\mathrm{Ad}(U_1)\circ(\widehat{\widehat{\lambda}})_{\Psi_A}\,,\,f^0]=[X\otimes M_N(\mathbf{C})\,,\,\lambda\otimes\mathrm{id}\,,\,f^0]$$

in $\mathrm{GPic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},u\otimes I_N}(A\otimes M_N(\mathbf{C}))$. Since

$$G_3([X,\lambda,f^0])=[X\otimes M_N(\mathbf{C})\,,\,\lambda\otimes\mathrm{id}_{M_N(\mathbf{C})}\,,\,f^0]$$

in $\mathrm{GPic}_H^{\rho\otimes\mathrm{id}_{M_N(\mathbf{C})},u\otimes I_N}(A\otimes M_N(\mathbf{C}))$, we obtain the conclusion. □

Theorem 4.4 *Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ,u) be a twisted coaction of H^0 on a unital C^*-algebra A. Then $\mathrm{GPic}_H^{\rho,u}(A)\cong\mathrm{GPic}_{H^0}^{\widehat{\rho}}(A\rtimes_{\rho,u}H)$, where $\widehat{\rho}$ is the dual coaction of (ρ,u).*

Proof This is immediate by Lemma 4.3 in the same way as in the proof of [10, Theorem 7.5]. □

5. Preparation

In this section, we prepare some results for the next section.

Let (ρ, u) and (σ, v) be twisted coactions of H^0 on A and let $C = A \rtimes_{\rho,u} H$ and $D = A \rtimes_{\sigma,v} H$. We suppose that $A' \cap C = \mathbf{C}1$. We also suppose that the unital inclusions $A \subset C$ and $A \subset D$ are strongly Morita equivalent with respect to a $C-D$-equivalence bimodule Y and its closed subspace ${}_A A_A$. Hence, we regard A as a closed subspace of Y. Furthermore, since $A' \cap C = \mathbf{C}1$, by [11, Lemma 4.1], there is the unique conditional expectation F from Y onto A with respect to $E_1^{\rho,u}$ and $E_1^{\sigma,v}$, where $E_1^{\rho,u}$ and $E_1^{\sigma,v}$ are the canonical conditional expectations from C and D onto A, respectively defined by

$$E_1^{\rho,u}(a \rtimes_{\rho,u} h) = \tau(h)a, \quad E_1^{\sigma,v}(a \rtimes_{\sigma,v} h) = \tau(h)a$$

for any $a \in A$, $h \in H$. By the proof of Rieffel [19, Proposition 2.1], there is an isomorphism Ψ of D onto C defined by

$$\Psi(d) = {}_C\langle 1_A \cdot d \, , \, 1_A \rangle$$

for any $d \in D$, where 1_A is the unit element in A and we regard 1_A as an element in Y. Let C_Ψ be the $C-D$-equivalence bimodule induced by C and Ψ, that is, $C_\Psi = C$ as vector spaces and the left C-action and the left C-valued inner product are defined in the usual way. We define the right D-action by $x \cdot d = x\Psi(d)$ for any $x \in C$, $d \in D$ and define the right D-valued inner product by $\langle x, y \rangle_D = \Psi^{-1}(x^* y)$ for any $x, y \in C$.

Lemma 5.1 *With the above notation, $Y \cong C_\Psi$ as $C - D$-equivalence bimodules.*

Proof We note that $y = {}_C\langle y, 1_A \rangle \cdot 1_A$ for any $y \in Y$ since $\langle 1_A, 1_A \rangle_D = 1_A$. Let η be the map from Y to C_Ψ defined by

$$\eta(y) = {}_C\langle y, 1_A \rangle$$

for any $y \in Y$. Then the map $c \in C_\Psi \mapsto c \cdot 1_A \in Y$ is the inverse map of η. Hence, η is bijective. Clearly η is linear. For any $y, z \in Y$,

$$\begin{aligned}
\langle \eta(y), \eta(z) \rangle_D &= \Psi^{-1}({}_C\langle y, 1_A \rangle^* {}_C\langle z, 1_A \rangle) = \Psi^{-1}({}_C\langle 1_A, y \rangle {}_C\langle z, 1_A \rangle) \\
&= \Psi^{-1}({}_C\langle {}_C\langle 1_A, y \rangle \cdot z \, , \, 1_A \rangle) \\
&= \Psi^{-1}({}_C\langle 1_A \cdot \langle y, z \rangle_D \, , \, 1_A \rangle) = \langle y, z \rangle_D, \\
{}_C\langle \eta(y), \eta(z) \rangle &= {}_C\langle {}_C\langle y, 1_A \rangle \, , \, {}_C\langle z, 1_A \rangle \rangle = {}_C\langle y, 1_A \rangle \, {}_C\langle 1_A, z \rangle \\
&= {}_C\langle {}_C\langle y, 1_A \rangle \cdot 1_A \, , \, z \rangle = {}_C\langle y \cdot \langle 1_A, 1_A \rangle_D \, , \, z \rangle = {}_C\langle y, z \rangle.
\end{aligned}$$

Hence, η preserves the left C-valued and the right D-valued inner products. Therefore, we obtain the conclusion. □

Let $\widehat{\rho}$ and $\widehat{\sigma}$ be the dual coactions of (ρ, u) and (σ, v), respectively and let $C_1 = C \rtimes_{\widehat{\rho}} H^0$, $D_1 = D \rtimes_{\widehat{\sigma}} H^0$. Let Y_1 be the upward basic construction of Y for F. Let $\widehat{\Psi}$ be the isomorphism of D_1 onto C_1 defined by

$$\widehat{\Psi}(T) = \Psi \circ T \circ \Psi^{-1}$$

for any $T \in \mathbf{B}_A(D)$, where we regard C and D as a $C_1 - A$-equivalence bimodule and $D_1 - A$- equivalence bimodule using $E_1^{\rho,u}$ and $E_1^{\sigma,v}$, respectively and we identify C_1 and D_1 with the C^*-algebra $\mathbf{B}_A(C)$, the

C^*-algebra of all adjointable right A-module maps on C and $\mathbf{B}_A(D)$, the C^*-algebra of all adjointable right A-module maps on D, respectively. Then by easy computations,

$$\widehat{\Psi}|_D = \Psi, \quad \Psi \circ E_2^{\sigma,v} = E_2^{\rho,u}, \quad \widehat{\Psi}(1 \rtimes_{\widehat{\sigma}} \tau) = 1 \rtimes_{\widehat{\rho}} \tau,$$

where $E_2^{\rho,u}$ and $E_2^{\sigma,v}$ are the conditional expectations from C_1 and D_1 onto C and D, which are defined by

$$E_2^{\rho,u}(c \rtimes_{\widehat{\rho}} \varphi) = \varphi(e)c, \quad E_2^{\sigma,v}(d \rtimes_{\widehat{\sigma}} \varphi) = \varphi(e)d$$

for any $c \in C$, $d \in D$, $\varphi \in H^0$, respectively. Let Ψ_1 be the map from D_1 to C_1 defined by

$$\Psi_1(d_1) = {}_{C_1}\langle 1_A \cdot d_1 \, , \, 1_A \rangle$$

for any $d_1 \in D_1$. Then by routine computations, $\Psi_1(d) = d$ for any $d \in D$ and $E_2^{\rho,u} \circ \Psi_1 = \Psi \circ E_2^{\sigma,v}$. Indeed, we note that $1_A \in A$ is regarded as an element

$$\sum_{i,j} u_i \otimes F(u_i^* \cdot 1 \cdot v_j) \otimes \widetilde{v_j}$$

in Y_1, where $\{(u_i, u_i^*)\}$ is a quasibasis for $E_1^{\rho,u}$ and $\{(v_j, v_j^*)\}$ is a quasibasis for $E_1^{\sigma,v}$. Let $c, d \in D$. Then

$$\begin{aligned}
&\Psi_1((c \rtimes_{\widehat{\sigma}} 1^0)(1 \rtimes_{\sigma,v} 1 \rtimes_{\widehat{\sigma}} \tau)(d \rtimes_{\widehat{\sigma}} 1^0)) \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \otimes F(u_i^* \cdot 1 \cdot v_j) \otimes [d^* E_1^{\sigma,v}(c^* v_j)]\widetilde{\ }, \, u_{i_1} \otimes F(u_{i_1}^* \cdot 1 \cdot v_{j_1}) \otimes \widetilde{v_{j_1}} \rangle \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot v_j) \otimes [d^* E_1^{\sigma,v}(c^* v_j)]\widetilde{\ }, \, F(u_{i_1}^* \cdot 1 \cdot v_{j_1}) \otimes \widetilde{v_{j_1}} \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot v_j) \cdot \langle d^* E_1^{\sigma,v}(c^* v_j) \, , \, v_{j_1} \rangle_A \, , \, F(u_{i_1}^* \cdot 1 \cdot v_{j_1}) \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot v_j) \cdot E_1^{\sigma,v}(E_1^{\sigma,v}(v_j^* c) d v_{j_1}) \, , \, F(u_{i_1}^* \cdot 1 \cdot v_{j_1}) \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot v_j) \cdot E_1^{\sigma,v}(v_j^* c) E_1^{\sigma,v}(d v_{j_1}) \, , \, F(u_{i_1}^* \cdot 1 \cdot v_{j_1}) \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,j,i_1,j_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot v_j E_1^{\sigma,v}(v_j^* c)) \, , \, F(u_{i_1}^* \cdot 1 \cdot v_{j_1} E_1^{\sigma,v}(v_{j_1}^* d^*)) \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,i_1} {}_{C_1}\langle u_i \cdot {}_A\langle F(u_i^* \cdot 1 \cdot c) \, , \, F(u_{i_1}^* \cdot 1 \cdot d^*) \rangle \, , \, u_{i_1} \rangle \\
&= \sum_{i,i_1} {}_{C_1}\langle u_i \cdot F(u_i^* \cdot 1 \cdot c) F(u_{i_1}^* \cdot 1 \cdot d^*)^* \, , \, u_{i_1} \rangle \\
&= \sum_{i,i_1} [u_i \cdot F(u_i^* \cdot 1 \cdot c) F(u_{i_1}^* \cdot 1 \cdot d^*)^*](1 \rtimes_{\rho,u} 1 \rtimes_{\widehat{\rho}} \tau) u_{i_1}^* .
\end{aligned}$$

Hence, by [15, Lemma 5.4]

$$\begin{aligned}
&(E_2^{\rho,u}\circ\Psi_1)((c\rtimes_{\widehat{\sigma}}1^0)(1\rtimes_{\sigma,v}1\rtimes_{\widehat{\sigma}}\tau)(d\rtimes_{\widehat{\sigma}}1^0)) \\
&=\sum_{i,i_1}\frac{1}{N}u_iF(u_i^*\cdot 1\cdot c)F(u_{i_1}^*\cdot 1\cdot d^*)^*u_{1_1}^* \\
&=\frac{1}{N}\sum_{i_1}(1\cdot c)(u_{i_1}F(u_{i_1}^*\cdot 1\cdot d^*))^*=\frac{1}{N}(1\cdot c)(1\cdot d^*)^*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\Psi\circ E_2^{\sigma,v})((c\rtimes_{\widehat{\sigma}}1^0)(1\rtimes_{\sigma,v}1\rtimes_{\widehat{\sigma}}\tau)(d\rtimes_{\widehat{\sigma}}1^0)) \\
&=\frac{1}{N}\Psi((c\rtimes_{\widehat{\sigma}}1^0)(d\rtimes_{\widehat{\sigma}}1^0))={}_C\langle 1_A\cdot\frac{1}{N}cd\,,\,1_A\rangle \\
&=\frac{1}{N}{}_C\langle 1_A\cdot c\,,\,1_A\cdot d^*\rangle=\frac{1}{N}(1_A\cdot c)(1_A\cdot d^*)^*.
\end{aligned}$$

Therefore, $E_1^{\rho,u}\circ\Psi_1=\Psi\circ E_1^{\sigma,v}$ since D_1 is the linear span of elements $(c\rtimes_{\widehat{\sigma}}1^0)(1\rtimes_{\widehat{\sigma}}\tau)(d\rtimes_{\widehat{\sigma}}1^0)$, where $c,d\in D$.

Lemma 5.2 *With the above notation, $\widehat{\Psi}=\Psi_1$ on D_1.*

Proof Since D_1 is the linear span of elements $(c\rtimes_{\widehat{\sigma}}1^0)(1\rtimes_{\widehat{\sigma}}\tau)(d\rtimes_{\widehat{\sigma}}1^0)$, where $c,d\in D$ and $\widehat{\Psi}=\Psi_1$ on D, we have to show that

$$\Psi_1(1\rtimes_{\widehat{\sigma}}\tau)=\widehat{\Psi}(1\rtimes_{\widehat{\sigma}}\tau)=1\rtimes_{\widehat{\rho}}\tau.$$

Indeed, since 1_A is regarded as an element

$$\sum_{i,j}u_i\otimes F(u_i^*\cdot 1_A\cdot v_j)\otimes\widetilde{v_j}$$

in $Y_1(=C\otimes_A A\otimes_A\widetilde{D})$, where $\{(u_i,u_i^*)\}$ and $\{(v_j,v_j^*)\}$ are as above. Thus,

$$\Psi_1(1\rtimes_{\widehat{\sigma}}\tau)={}_C\langle 1_A\cdot(1\rtimes_{\widehat{\sigma}}\tau)\,,\,1_A\rangle={}_C\langle 1_A\cdot(1\rtimes_{\widehat{\sigma}}\tau)\,,\,1_A\cdot(1\rtimes_{\widehat{\sigma}}\tau)\rangle.$$

Here

$$\begin{aligned}
1_A\cdot(1\rtimes_{\widehat{\sigma}}\tau)&=\sum_{i,j}u_i\otimes F(u_i^*\cdot 1_A\cdot v_j)\otimes\widetilde{v_j}\cdot(1\rtimes_{\widehat{\sigma}}\tau) \\
&=\sum_{i,j}u_i\otimes F(u_i^*\cdot 1_A\cdot v_j)\otimes[(1\rtimes_{\widehat{\sigma}}\tau)\cdot v_j]^{\widetilde{}} \\
&=\sum_{i,j}u_i\otimes F(u_i^*\cdot 1_A\cdot v_jE_1^{\sigma,v}(v_j^*))\otimes\widetilde{1_A} \\
&=\sum_{i}u_i\otimes F(u_i^*\cdot 1_A)\otimes\widetilde{1_A} \\
&=\sum_{i}u_iE_1^{\rho,u}(u_i^*)\otimes 1_A\otimes\widetilde{1_A} \\
&=1_A\otimes 1_A\otimes\widetilde{1_A}.
\end{aligned}$$

Hence,

$$\Psi_1(1 \rtimes_{\widetilde{\sigma}} \tau) = {}_C\langle 1_A \otimes 1_A \otimes \widetilde{1_A} \, , \, 1_A \otimes 1_A \otimes \widetilde{1_A} \rangle = {}_C\langle 1_A \, , \, 1_A \rangle = 1 \rtimes_{\widehat{\rho}} \tau.$$

Therefore, we obtain the conclusion. □

By the above lemma, we obtain the following corollary:

Corollary 5.3 *With the above notation, there is an isomorphism* $\widehat{\Psi}$ *of* D_1 *onto* C_1 *satisfying that*

$$\widehat{\Psi}|_D = \Psi, \quad \Psi \circ E_2^{\sigma,v} = E_2^{\rho,u} \circ \widehat{\Psi},$$
$$\widehat{\Psi}(1 \rtimes_{\widehat{\sigma}} \tau) = 1 \rtimes_{\widehat{\rho}} \tau,$$
$$\widehat{\Psi}(d_1) = {}_{C_1}\langle 1_A \cdot d_1 \, , \, 1_A \rangle \quad \text{for any } d_1 \in D_1.$$

In the same way as Corollary 5.3, we obtain the following lemma:

Lemma 5.4 *With the above notation, let* $\widehat{\widehat{\Psi}}$ *be the isomorphism of* $D_2(= D_1 \rtimes_{\widehat{\widehat{\sigma}}} H)$ *onto* $C_2(= C_1 \rtimes_{\widehat{\widehat{\rho}}} H)$ *defined by*

$$\widehat{\widehat{\Psi}}(T) = \widehat{\Psi} \circ T \circ \widehat{\Psi^{-1}}$$

for any $T \in \mathbf{B}_D(D_1)$*, where we identify* C_2 *and* D_2 *with* $\mathbf{B}_C(C_1)$*, the* C^**-algebra of all adjointable right* C*-module maps on* C_1 *and* $\mathbf{B}_D(D_1)$*, the* C^**-algebra of all adjointable right* D*-module maps on* D_1*, respectively and we regard* C_1 *and* D_1 *as a* $C_2 - C$*- equivalence bimodule and a* $D_2 - D$*-equivalence bimodule using the canonical conditional expectations* $E_2^{\rho,u} : C_1 \to C$ *and* $E_2^{\sigma,v} : D_1 \to D$*, respectively. Then* $\widehat{\widehat{\Psi}}$ *satisfies that*

$$\widehat{\widehat{\Psi}}|_{D_1} = \widehat{\Psi}, \quad \widehat{\Psi} \circ E_3^{\sigma,v} = E_3^{\rho,u} \circ \widehat{\widehat{\Psi}},$$
$$\widehat{\widehat{\Psi}}(1 \rtimes_{\widehat{\widehat{\sigma}}} e) = 1 \rtimes_{\widehat{\widehat{\rho}}} e,$$
$$\widehat{\widehat{\Psi}}(d_2) = {}_{C_2}\langle 1_A \cdot d_2 \, , \, 1_A \rangle \quad \text{for any } d_2 \in D_2,$$

where $E_3^{\rho,u}$ *and* $E_3^{\sigma,v}$ *are the canonical conditional expectations from* C_2 *and* D_2 *onto* C_1 *and* D_1 *defined by*

$$E_3^{\rho,u}(c_1 \rtimes_{\widehat{\widehat{\rho}}} h) = c_1 \tau(h) \, , \quad E_3^{\sigma,v}(d_1 \rtimes_{\widehat{\widehat{\sigma}}} h) = d_1 \tau(h)$$

for any $c_1 \in C_1$*,* $d_1 \in D_1$*,* $h \in H$*, respectively.*

By Lemmas 5.1, 5.4, $Y_1 \cong C_{1\widehat{\Psi}}$ as $C_1 - D_1$-equivalence bimodules and $Y_2 \cong C_{2\widehat{\widehat{\Psi}}}$ as $C_2 - D_2$-equivalence bimodules, where Y_2 is the upward basic construction of Y_1 for F_1, the dual conditional expectation from Y_1 onto Y. Also, by [16, Lemma 5.10], there is a C^*-Hopf algebra automorphism f^0 of H^0 such that

$$\widehat{\widehat{\rho}} \circ \widehat{\Psi} = (\widehat{\Psi} \otimes f^0) \circ \widehat{\widehat{\sigma}}.$$

Lemma 5.5 *With the above notation, let* $\widehat{\widehat{\sigma}}_{f^0} = (\mathrm{id} \otimes f^0) \circ \widehat{\widehat{\sigma}}$*. Then* $\widehat{\widehat{\rho}}$ *and* $\widehat{\widehat{\sigma}}_{f^0}$ *are strongly Morita equivalent.*

Proof Let $\lambda_{\widehat{\widehat{\rho}}}$ be the linear map from $C_{1\widehat{\Psi}}$ to $C_{1\widehat{\Psi}} \otimes H^0$ defined by

$$\lambda_{\widehat{\widehat{\rho}}}(x) = \widehat{\widehat{\rho}}(x)$$

for any $x \in C_{1\widehat{\Psi}}$. Then $\lambda_{\widehat{\widehat{\rho}}}$ is a coaction of H^0 on $C_{1\widehat{\Psi}}$ with respect to $(C_1, D_1, \widehat{\widehat{\rho}}, \widehat{\widehat{\sigma}}_{f^0})$. Indeed, for any $c \in C_1$, $d \in D_1$, $x \in C_{1\widehat{\Psi}}$,

$$\lambda_{\widehat{\widehat{\rho}}}(c \cdot x) = \lambda_{\widehat{\widehat{\rho}}}(cx) = \widehat{\widehat{\rho}}(c)\widehat{\widehat{\rho}}(x) = \widehat{\widehat{\rho}}(c) \cdot \lambda_{\widehat{\widehat{\rho}}}(x),$$

$$\lambda_{\widehat{\widehat{\rho}}}(x \cdot d) = \lambda_{\widehat{\widehat{\rho}}}(x\widehat{\Psi}(d)) = \widehat{\widehat{\rho}}(x)\widehat{\widehat{\rho}}(\widehat{\Psi}(d)) = \widehat{\widehat{\rho}}(x)\widehat{\Psi}(\widehat{\widehat{\sigma}}_{f^0}(d)) = \lambda_{\widehat{\widehat{\rho}}}(x) \cdot \widehat{\widehat{\sigma}}_{f^0}(d)$$

by the equation as above. Also, for any $x, y \in C_{1\widehat{\Psi}}$,

$${}_{C_1 \otimes H^0}\langle \lambda_{\widehat{\widehat{\rho}}}(x) \, , \, \lambda_{\widehat{\widehat{\rho}}}(y) \rangle = \widehat{\widehat{\rho}}(x)\widehat{\widehat{\rho}}(y)^* = \widehat{\widehat{\rho}}(xy^*) = \widehat{\widehat{\rho}}({}_{C_1}\langle x, y \rangle),$$

$$\langle \lambda_{\widehat{\widehat{\rho}}}(x) \, , \, \lambda_{\widehat{\widehat{\rho}}}(y) \rangle_{D_1 \otimes H^0} = (\widehat{\Psi}^{-1} \otimes \mathrm{id}_{H^0})(\widehat{\widehat{\rho}}(x^*y)) = \widehat{\widehat{\sigma}}_{f^0}(\widehat{\Psi}^{-1}(x^*y)) = \widehat{\widehat{\sigma}}_{f^0}(\langle x, y \rangle_{D_1})$$

by the equation as above. Furthermore,

$$(\lambda_{\widehat{\widehat{\rho}}} \otimes \mathrm{id}) \circ \lambda_{\widehat{\widehat{\rho}}} = (\widehat{\widehat{\rho}} \otimes \mathrm{id}) \circ \widehat{\widehat{\rho}} = (\mathrm{id} \otimes \Delta^0) \circ \widehat{\widehat{\rho}} = (\mathrm{id} \otimes \Delta^0) \circ \lambda_{\widehat{\widehat{\rho}}},$$

$$(\mathrm{id} \otimes \epsilon^0) \circ \lambda_{\widehat{\widehat{\rho}}} = (id \otimes \epsilon^0) \circ \widehat{\widehat{\rho}} = \mathrm{id}.$$

Hence, we obtain the conclusion. □

Lemma 5.6 *With the above notation, $C_{2\widehat{\widehat{\Psi}}} \cong C_{1\widehat{\Psi}} \rtimes_{\lambda_{\widehat{\widehat{\rho}}}} H$ as $C_2 - D_2$-equivalence bimodules.*

Proof We note that $C_{2\widehat{\widehat{\Psi}}} = (C_1 \rtimes_{\widehat{\widehat{\rho}}} H)_{\widehat{\widehat{\Psi}}}$. For any $x \in C_1$, $h \in H$, $(x \rtimes_{\widehat{\widehat{\rho}}} h)_{\widehat{\widehat{\Psi}}}$ denotes the element in $C_{2\widehat{\widehat{\Psi}}}$ induced by $x \in C_1$, $h \in H$. Let Θ be the bijective map from $C_{2\widehat{\widehat{\Psi}}}$ onto $C_{1\widehat{\Psi}} \rtimes_{\lambda_{\widehat{\widehat{\rho}}}} H$ defined by

$$\Theta((x \rtimes_{\widehat{\widehat{\rho}}} h)_{\widehat{\widehat{\Psi}}}) = x \rtimes_{\lambda_{\widehat{\widehat{\rho}}}} h$$

for any $x \in C_1$, $h \in H$. By easy computations, Θ is a $C_2 - D_2$-equivalence bimodule isomorphism of $C_{2\widehat{\widehat{\Psi}}}$ onto $C_{1\widehat{\Psi}} \rtimes_{\lambda_{\widehat{\widehat{\rho}}}} H$. Therefore, we obtain the conclusion. □

Since $C_{2\widehat{\widehat{\Psi}}} \cong Y_2$ as $C_2 - D_2$-equivalence bimodules and $C_{1\widehat{\Psi}} \cong Y_1$ as $C_1 - D_1$-equivalence bimodules, by the proofs of Lemmas 5.5, 5.6, there is a coaction λ of H^0 on Y_1 with respect to $(C_1, D_1, \widehat{\widehat{\rho}}, \widehat{\widehat{\sigma}}_{f^0})$ such that

$$Y_2 \cong Y_1 \rtimes_\lambda H$$

as $C_2 - D_2$-equivalence bimodules. Therefore, we obtain the following proposition:

Proposition 5.7 *Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) and (σ, v) be twisted coactions of H^0 on a unital C^*-algebra A. Let $C = A \rtimes_{\rho,u} H$, $D = A \rtimes_{\sigma,v} H$. We suppose that $A' \cap C = \mathbf{C}1$. (Hence, $A' \cap D = \mathbf{C}1$.) Also, we suppose that the unital inclusions $A \subset C$ and $A \subset D$ are strongly Morita equivalent with respect to a $C - D$-equivalence bimodule Y and its closed subspace*

A. Let C_1, C_2, D_1, D_2, $\widehat{\widehat{\rho}}$, $\widehat{\widehat{\sigma}}$, $\widehat{\widehat{\sigma_{f^0}}}$ and Y_1, Y_2 be as above. Then there are a C^*-Hopf algebra automorphism f^0 of H^0 and a coaction λ of H^0 on Y_1 with respect to $(C_1, D_1, \widehat{\widehat{\rho}}, \widehat{\widehat{\sigma_{f^0}}})$ such that

$$Y_2 \cong Y_1 \rtimes_\lambda H$$

as $C_2 - D_2$-equivalence bimodules.

6. The generalized Picard groups for coactions and the Picard groups for inclusions of unital C^*-algebras

In this section, we shall investigate the relation between the generalized Picard groups for coactions of a finite dimensional C^*-Hopf algebra on a unital C^*-algebra and the Picard groups for unital inclusions of unital C^*-algebras induced by the coaction.

Let H be a finite dimensional C^*-Hopf algebra with its dual C^*-Hopf algebra H^0. Let (ρ, u) be a twisted coaction of H^0 on a unital C^*-algebra A. Then we have the unital inclusion of unital C^*-algebras, $A \subset A \rtimes_{\rho,u} H$. Hence, we can obtain the generalized Picard group $\mathrm{GPic}_H^{\rho,u}(A)$ for the twisted coaction (ρ, u) and the Picard group $\mathrm{Pic}(A, A \rtimes_{\rho,u} H)$ for the unital inclusion of unital C^*-algebras $A \subset A \rtimes_{\rho,u} H$.

Let $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$. Then we obtain the element $(X, X \rtimes_\lambda H) \in \mathrm{Equi}(A, A \rtimes_{\rho,u} H)$. Indeed, X is an $A - A$-equivalence and $X \rtimes_\lambda H$ is an $A \rtimes_{\rho,u} H - A \rtimes_{\rho,u} H$-equivalence bimodule. Also, $A \rtimes_{\rho_{f^0}, u_{f^0}} H$ is isomorphic to $A \rtimes_{\rho,u} H$ by the isomorphism π_{f^0} of $A \rtimes_{\rho,u} H$ onto $A \rtimes_{\rho_{f^0}, u_{f^0}} H$ defined by

$$\pi_{f^0}(a \rtimes_{\rho,u} h) = a \rtimes_{\rho_{f^0}, u_{f^0}} f(h)$$

for any $a \in A$, $h \in H$, where f is the C^*-Hopf algebra automorphism of H induced by f^0. Furthermore, by routine computations, we can see that X and $X \rtimes_\lambda H$ satisfy Conditions (1), (2) in [15, Definition 2.1]. Thus, we can define the map θ from $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Pic}(A, A \rtimes_{\rho,u} H)$ by

$$\theta([X, \lambda, f^0]) = [X, X \rtimes_\lambda H]$$

for any $[X, \lambda, f^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. We note that θ is well-defined by routine computations. We show that θ is a homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Pic}(A, A \rtimes_{\rho,u} H)$. Let $[X, \lambda, f^0]$, $[Y, \mu, g^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. Then

$$\begin{aligned} \theta([X, \lambda, f^0][Y, \mu, g^0]) &= \theta([X \otimes_A Y\,,\, \lambda \otimes \mu_{f^0}\,,\, f^0 \circ g^0]) \\ &= [X \otimes_A Y\,,\, (X \otimes_A Y) \rtimes_{\lambda \otimes \mu_{f^0}} H]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \theta([X, \lambda, f^0])\theta([Y, \mu, g^0]) &= [X, X \rtimes_\lambda H][Y, Y \rtimes_\mu H] \\ &= [X \otimes_A Y\,,\, (X \rtimes_\lambda H) \otimes_{A \rtimes_{\rho,u} H} (Y \rtimes_\mu H)] \\ &= [(X \rtimes_\lambda 1) \otimes_A (Y \rtimes_\mu 1)\,,\, (X \rtimes_\lambda H) \otimes_{A \rtimes_{\rho,u} H} (Y \rtimes_\mu H)]. \end{aligned}$$

We note that $A \rtimes_{\rho_{f^0}, u_{f^0}} H$, $A \rtimes_{\rho_{g^0}, u_{g^0}} H$ and $A \rtimes_{\rho_{f^0 \circ g^0}, u_{f^0 \circ g^0}} H$ are identified with $A \rtimes_{\rho,u} H$ by the isomorphisms

π_{f^0}, π_{g^0} and $\pi_{f^0\circ g^0}$ defined respectively as follows:

$$\pi_{f^0}(a\rtimes_{\rho,u}h)=a\rtimes_{\rho_{f^0},u_{f^0}}f(h)$$
$$\pi_{g^0}(a\rtimes_{\rho,u}h)=a\rtimes_{\rho_{g^0},u^{g^0}}g(h)$$
$$\pi_{f^0\circ g^0}(a\rtimes_{\rho,u}h)=a\rtimes_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}(f\circ g)(h)$$

for any $a\in A$, $h\in H$. Let Φ be the linear map from $(X\otimes_A Y)\rtimes_{\lambda\otimes\mu_{f^0}}H$ to $(X\rtimes_\lambda H)\otimes_{A\rtimes_{\rho,u}H}(Y\rtimes_\mu H)$ defined by

$$\Phi((x\otimes y)\rtimes_{\lambda\otimes\mu_{f^0}}h)=(x\rtimes_\lambda 1)\otimes(y\rtimes_\mu f^{-1}(h))$$

for any $x\in X$, $y\in Y$, $h\in H$. For any $x\in X$, $y\in Y$, $h,l\in H$,

$$\begin{aligned}(x\rtimes_\lambda h)\otimes(y\rtimes_\mu l)&=(x\rtimes_\lambda 1)\cdot(1\rtimes_{\rho_{f^0},u_{f^0}}h)\otimes(y\rtimes_\mu l)\\&=(x\rtimes_\lambda 1)\otimes(1\rtimes_{\rho,u}f^{-1}(h))\cdot(y\rtimes_\mu l)\\&=(x\rtimes_\lambda 1)\otimes([f^{-1}(h_{(1)})\cdot_\mu y]\widehat{u_{g^0}}(f^{-1}(h_{(2)}),l_{(1)})\rtimes_\mu f^{-1}(h_{(3)})l_{(2)}).\end{aligned}$$

Hence, $(X\rtimes_\lambda H)\otimes_{A\rtimes_{\rho,u}H}(Y\rtimes_\mu H)$ is the closure of linear spans of elements $(x\rtimes_\lambda 1)\otimes(y\rtimes_\mu h)$ in $(X\rtimes_\lambda H)\otimes_{A\rtimes_{\rho,u}H}(Y\rtimes_\mu H)$, where $x\in X$, $y\in Y$, $h\in H$. Thus, Φ is surjective. Also, its inverse map Φ^{-1} is following:

$$\Phi^{-1}((x\rtimes_\lambda 1)\otimes(y\rtimes_\mu h))=(x\otimes y)\rtimes_{\rho\otimes\mu_{f^0}}f(h)$$

for any $x\in X$, $y\in Y$, $h\in H$. Furthermore, let $x,z\in X$, $y,w\in Y$, $h,l\in H$. Then

$$\begin{aligned}&{}_{A\rtimes_{\rho,u}H}\langle(x\otimes y)\rtimes_{\lambda\otimes\mu_{f^0}}h\,,\,(z\otimes w)\rtimes_{\lambda\otimes\mu_{f^0}}l\rangle\\&={}_A\langle x\otimes y\,,\,[S(h_{(2)}l^*_{(3)})^*\cdot_{\lambda\otimes\mu_{f^0}}(z\otimes w)]\cdot\widehat{u_{f^0\circ g^0}}(S(h_{(1)}l^*_{(2)})^*\,,\,l_{(1)})\rtimes_{\rho,u}h_{(3)}l_{(4)}\\&={}_A\langle x\otimes y\,,\,[S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\otimes[S(h_{(2)}l^*_{(3)})^*\cdot_{\mu_{f^0}}w]\cdot\widehat{u_{f^0\circ g^0}}(S(h_{(1)}l^*_{(2)})^*\,,\,l_{(1)})\\&\rtimes_{\rho,u}h_{(4)}l^*_{(5)}\\&={}_A\langle x\cdot{}_A\langle y\,,\,[S(h_{(2)}l^*_{(3)})^*\cdot_{\mu_{f^0}}w]\cdot\widehat{u_{f^0\circ g^0}}(S(h_{(1)}l^*_{(2)})^*,l_{(1)})\rangle\,,\,[S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\rangle\\&\rtimes_{\rho,u}h_{(4)}l^*_{(5)}.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&{}_{A\rtimes_{\rho,u}H}\langle \Phi((x\otimes y)\rtimes_{\lambda\otimes\mu_{f^0}} h)\,,\ \ \Phi((z\otimes w)\rtimes_{\lambda\otimes\mu_{f^0}} l)\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\otimes(y\rtimes_\mu f^{-1}(h))\,,\,(z\rtimes_\lambda 1)\otimes(w\rtimes_\mu f^{-1}(l))\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\cdot {}_{A\rtimes_{\rho,u}H}\langle y\rtimes_\mu f^{-1}(h)\,,\,w\rtimes_\mu f^{-1}(l)\rangle\,,\,z\rtimes_\lambda 1\rangle \\
&= {}_{A\rtimes_{\rho,u}H}(x\rtimes_\lambda 1)\cdot {}_A\langle y\,,\,[f^{-1}(S(h_{(2)}l^*_{(3)})^*)\cdot_\mu w]\cdot\widehat{u_{g^0}}(f^{-1}(S(h_{(1)}l^*_{(2)})^*)\,,\,f^{-1}(l_{(1)}))\rangle \\
&\rtimes_{\rho,u} f^{-1}(h_{(3)}l^*_{(4)})\,,\,z\rtimes_\lambda 1\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle (x\rtimes_\lambda 1)\cdot {}_A\langle y\,,\,[S(h_{(2)}l^*_{(3)})^*\cdot_{\mu_{f^0}} w]\cdot\widehat{u_{f^0\circ g^0}}(S(h_{(1)}l^*_{(2)})^*)\,,\,l_{(1)}\rangle \\
&\rtimes_{\rho,u} f^{-1}(h_{(3)}l^*_{(4)})\,,\,z\rtimes_\lambda 1\rangle \\
&= {}_{A\rtimes_{\rho,u}H}\langle x\ \cdot\ {}_A\langle y\,,\,[S(h_{(2)}l^*_{(3)})^*\cdot_{\mu_{f^0}} w]\cdot\widehat{u_{f^0\circ g^0}}(Sh_{(1)}l^*_{(2)})^*\,,\,l_{(1)})\rangle \\
&\rtimes_\lambda h_{(3)}l^*_{(4)}\,,\,z\rtimes_\lambda 1\rangle \\
&= {}_A\langle x\cdot {}_A\langle y\,,\,[S(h_{(2)}l^*_{(3)})^*\cdot_{\mu_{f^0}} w]\cdot\widehat{u_{f^0\circ g^0}}(S(h_{(1)}l^*_{(2)})^*\,,\,l_{(1)})\rangle\,,\,[S(h_{(3)}l^*_{(4)})^*\cdot_\lambda z]\rangle \\
&\rtimes_{\rho,u} h_{(4)}l^*_{(5)}.
\end{aligned}
$$

Hence, Φ preserves the left $A\rtimes_{\rho,u}H$-inner products. Also,

$$
\begin{aligned}
&\langle (x\otimes y)\rtimes_{\lambda\otimes\mu_{f^0}} h\,,\,(z\otimes w)\rtimes_{\lambda\otimes\mu_{f^0}} l\rangle_{A\rtimes_{\rho,u}H} \\
&= \widehat{u^*_{f^0\circ g^0}}(h^*_{(2)}\,,\,S(h_{(1)})^*)[h^*_{(3)}\cdot_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}\langle x\otimes y\,,\,z\otimes w\rangle_A]\widehat{u_{f^0\circ g^0}}(h^*_{(4)}\,,\,l_{(1)}) \\
&\rtimes_{\rho,u}(g^{-1}\circ f^{-1})(h^*_{(5)}l_{(2)}) \\
&= \widehat{u^*_{f^0\circ g^0}}(h^*_{(2)}\,,\,S(h^*_{(1)}))[h^*_{(3)}\cdot_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}\langle y\,,\,\langle x,z\rangle_A\cdot w\rangle_A]\widehat{u_{f^0\circ g^0}}(h^*_{(4)}\,,\,l_{(1)}) \\
&\rtimes_{\rho,u}(g^{-1}\circ f^{-1})(h^*_{(5)}l_{(2)}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\langle \Phi((x\otimes y)\rtimes_{\lambda\otimes\mu_{f^0}} h)\,,\,\Phi((z\otimes w)\rtimes_{\lambda\otimes\mu_{f^0}} l)\rangle_{A\rtimes_{\rho,u}H} \\
&\langle (x\rtimes_\lambda 1)\otimes(y\rtimes_\mu f^{-1}(h))\,,\,(z\rtimes_\lambda 1)\otimes(w\rtimes_\mu f^{-1}(l))\rangle_{A\rtimes_{\rho,u}H} \\
&= \langle y\rtimes_\mu f^{-1}(h)\,,\,\langle x\rtimes_\lambda 1\,,\,z\rtimes_\lambda 1\rangle_{A\rtimes_{\rho,u}H}\cdot w\rtimes_\mu f^{-1}(l)\rangle_{A\rtimes_{\rho,u}H} \\
&= \langle y\rtimes_\mu f^{-1}(h)\,,\,\langle x,z\rangle_A\rtimes_{\rho,u}1\cdot w\rtimes_\mu f^{-1}(l)\rangle_{A\rtimes_{\rho,u}H} \\
&= \langle y\rtimes_\mu f^{-1}(h)\,,\,\langle x,y\rangle_A\cdot w\rtimes_\mu f^{-1}(l)\rangle_{A\rtimes_{\rho,u}H} \\
&= \widehat{u^*_{g^0}}(f^{-1}(h^*_{(2)})\,,\,f^{-1}(S(h^*_{(1)})))[f^{-1}(h^*_{(3)})\cdot_{\rho_{g^0},u_{g^0}}\langle y\,,\,\langle x,z\rangle_A\cdot w\rangle_A] \\
&\times\widehat{u_{g^0}}(f^{-1}(h^*_{(4)})\,,\,f^{-1}(l_{(1)}))\rtimes_{\rho,u} g^{-1}(f^{-1}(h^*_{(5)}l_{(2)})) \\
&= \widehat{u^*_{f^0\circ g^0}}(h^*_{(2)}\,,\,S(h^*_{(1)}))[h^*_{(3)}\cdot_{\rho_{f^0\circ g^0},u_{f^0\circ g^0}}\langle y\,,\,\langle x,z\rangle_A\cdot w\rangle_A]\widehat{u_{f^0\circ g^0}}(h^*_{(4)}\,,\,l_{(1)}) \\
&\rtimes_{\rho,u}(g^{-1}\circ f^{-1})(h^*_{(5)}l_{(2)}).
\end{aligned}
$$

Hence, Φ preserves the right $A\rtimes_{\rho,u}H$-inner products. Therefore, by the remark after Jensen Thomsen [6, Definition 1.1.18], Φ is an $A\rtimes_{\rho,u}H-A\rtimes_{\rho,u}H$-equivalence bimodule isomorphism of $(X\otimes_A Y)\rtimes_{\lambda\otimes\mu_{f^0}}H$ onto

$(X \rtimes_\lambda H) \otimes_{A \rtimes_{\rho,u} H} (Y \rtimes_\mu H)$. Furthermore,

$$\Phi((X \otimes_A Y) \rtimes_{\lambda \otimes \mu_{f^0}} 1) = (X \rtimes_\lambda 1) \otimes_{A \rtimes_{\rho,u} H} (Y \rtimes_\mu 1)$$

by the definition of Φ. Therefore, θ is a homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Pic}(A, A \rtimes_{\rho,u} H)$.

We shall show that θ is surjective if $A' \cap (A \rtimes_{\rho,u} H) = \mathbf{C}1$. Let $C = A \rtimes_{\rho,u} H$ and we suppose that $A' \cap C = \mathbf{C}1$. Let $[X, Y] \in \mathrm{Pic}(A, C)$. Then $[Y, Y_1] \in \mathrm{Pic}(C, C_1)$, where $C_1 = C \rtimes_{\widehat{\rho}} H^0$ and Y_1 is the upward basic construction of Y for E^X, the unique conditional expectation from Y onto X with respect to $E_1^{\rho,u}$ and $E_1^{\rho,u}$ (See [15, Definition 6.5]). Also, by [16, Section 4], there are a coaction β of H on C and a coaction μ of H on Y such that $\widehat{\rho}$ and β are strongly Morita equivalent with respect to the coaction μ, that is, μ is a coaction of H on Y with respect to $(C, C, \widehat{\rho}, \beta)$. Hence, the unital inclusions $C \subset C_1$ and $C \subset D_1$ are strongly Morita equivalent with respect to the $C_1 - D_1$-equivalence bimodule $Y \rtimes_\mu H^0$ and its closed subspace Y, where $D_1 = C \rtimes_\beta H^0$. Since $[Y, Y_1] \in \mathrm{Pic}(C, C_1)$, the unital inclusions $C \subset C_1$ and $C \subset D_1$ are strongly Morita equivalent with respect to the $C_1 - D_1$-equivalence bimodule $\widetilde{Y_1} \otimes_{C_1} (Y \rtimes_\mu H^0)$ and its closed subspace $\widetilde{Y} \otimes_C Y \cong {}_C C_C$. We note that $C' \cap C_1 = \mathbf{C}1$ since $A' \cap C = \mathbf{C}1$. Let $Z = \widetilde{Y_1} \otimes_{C_1} (Y \rtimes_\mu H^0)$. Then there is the unique conditional expectation F from Z onto ${}_C C_C$ with respect to $E_2^{\rho,u}$ and $E_2^{\rho,u}$ by [11, Lemma 4.1] since $A' \cap C = \mathbf{C}1$. Let Z_1 be the upward basic construction of Z for F and Z_2 the second upward basic construction of Z for F. By Proposition 5.7, there are a C^*-Hopf algebra automorphism f of H and a coaction λ of H on Z_1 with respect to $(C_2, D_2, \widehat{\widehat{\rho}}, \widehat{\widehat{\beta_f}})$ such that

$$Z_2 \cong Z_1 \rtimes_\lambda H^0$$

as $C_3 - D_3$-equivalence bimodules, where $C_2 = C_1 \rtimes_{\widehat{\widehat{\rho}}} H$, $D_2 = D_1 \rtimes_{\widehat{\beta}} H$, $C_3 = C_2 \rtimes_{\widehat{\widehat{\widehat{\rho}}}} H^0$, $D_3 = D_2 \rtimes_{\widehat{\widehat{\beta}}} H^0$, $\beta_f = (\mathrm{id} \otimes f) \circ \beta$. Let Y_2 and Y_3 be the upward and the second upward basic constructions of Y_1 for E^Y, respectively. Then $[Y_2, Y_3] \in \mathrm{Pic}(C_2, C_3)$. Let $\widehat{\widehat{\widehat{\theta}}}$ be the homomorphism of $\mathrm{GPic}_{H^0}^{\widehat{\widehat{\widehat{\rho}}}}(C_2)$ to $\mathrm{Pic}(C_2, C_3)$ in the same way as in the beginning of this section. We show that there is an element $x \in \mathrm{GPic}_{H^0}^{\widehat{\widehat{\widehat{\rho}}}}(C_2)$ such that

$$\widehat{\widehat{\widehat{\theta}}}(x) = [Y_2, Y_3]$$

in $\mathrm{Pic}(C_2, C_3)$. In order to do this, we prepare some lemmas: Let (ρ, u) and (σ, v) be twisted coactions of H^0 on unital C^*-algebras A and B, respectively. We suppose that (ρ, u) and (σ, v) are strongly Morita equivalent. Let λ be the twisted coaction of H^0 on an $A - B$-equivalence bimodule X with respect to $(A, B, \rho, u, \sigma, v)$. Let $C = A \rtimes_{\rho,u} H$, $C_1 = C \rtimes_{\widehat{\rho}} H^0$, $D = B \rtimes_{\sigma,v} H$, $D_1 = D \rtimes_{\widehat{\sigma}} H^0$, respectively. Let E^X be the conditional expectation from $X \rtimes_\lambda H$ onto X for $E_1^{\rho,u}$ and $E_1^{\sigma,v}$, which is defined by

$$E^X(x \rtimes_\lambda h) = x\tau(h)$$

for any $x \in X$, $h \in H$.

Lemma 6.1 *With the above notation and assumptions, let Y_1 be the upward basic construction of $X \rtimes_\lambda H$ for E^X. Then $Y_1 \cong X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0$ as $C_1 - D_1$-equivalence bimodules.*

Proof Let $E^{X\rtimes_\lambda H}$ be the conditional expectation from $X\rtimes_\lambda H\rtimes_{\widehat{\lambda}} H^0$ onto $X\rtimes_\lambda H$ with respect to $E_2^{\rho,u}$ and $E_2^{\sigma,v}$, which is defined by

$$E^{X\rtimes_\lambda H}(x\rtimes_\lambda h\rtimes_{\widehat{\lambda}}\varphi)=(x\rtimes_\lambda h)\varphi(e)$$

for any $x\in X$, $h\in H$. Hence, by [15, Proposition 6.12], we obtain the conclusion. □

Lemma 6.2 *With the above notation and assumptions, $(X\rtimes_\lambda H)\widetilde{}$ is isomorphic to $\widetilde{X}\rtimes_{\widetilde{\lambda}} H$ as $B\rtimes_{\sigma,v} H-A\rtimes_{\rho,u} H$- equivalence bimodules*

Proof We can prove the lemma modifying the proof of [10, Lemma 4.3]. Let π be the bijective linear map from $(X\rtimes_\lambda H)\widetilde{}$ onto $\widetilde{X}\rtimes_{\widetilde{\lambda}} H$ defined by

$$\pi((x\rtimes_\lambda h)\widetilde{})=\{[S(h_{(3)})\cdot_\lambda x]\widehat{v}(S(h_{(2)}),h_{(1)})\}\widetilde{}\rtimes_\lambda h_{(4)}^*$$

for any $x\in X$, $h\in H$. Then by routine computations, π is a $B\rtimes_{\sigma,v} H-A\rtimes_{\rho,u} H$ -equivalence bimodule isomorphism of $(X\rtimes_\lambda H)\widetilde{}$ onto $\widetilde{X}\rtimes_{\widetilde{\lambda}} H$. □

We recall that $Z=\widetilde{Y_1}\otimes_{C_1}(Y\rtimes_\mu H^0)$. Hence, by Lemmas 6.1 and [11, Lemma 4.3],

$$Z_1\cong\widetilde{Y_2}\otimes_{C_2}(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)$$

as C_2-D_2-equivalence bimodules and

$$Z_2\cong\widetilde{Y_3}\otimes_{C_3}(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H\rtimes_{\widehat{\widehat{\mu}}} H^0)$$

as C_3-D_3-equivalence bimodules. Furthermore, by [11, Lemmas 4.4 and 4.5] the inclusions $Z_1\subset Z_2$ and

$$\widetilde{Y_2}\otimes_{C_2}(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)\subset\widetilde{Y_3}\otimes_{C_3}(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H\rtimes_{\widehat{\widehat{\mu}}} H^0)$$

are strongly Morita equivalent as unital inclusions of unital C^*-algebras. Also, we recall that by Proposition 5.7, there are the C^*-Hopf algebra automorphism f of H and the coaction λ of H on Z_1 with respect to $(C_2,\, D_2,\, \widehat{\widehat{\rho}},\, \widehat{\widehat{\beta_f}})$ such that

$$Z_2\cong Z_1\rtimes_\lambda H^0$$

as C_3-D_3-equivalence bimodules. Hence

$$Z_1\rtimes_\lambda H^0\cong\widetilde{Y_3}\otimes_{C_3}(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H\rtimes_{\widehat{\widehat{\mu}}} H^0).$$

as C_3-D_3-equivalence bimodules. Thus, by Lemma 6.2,

$$\begin{aligned}Y_3&\cong(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H\rtimes_{\widehat{\widehat{\mu}}} H^0)\otimes_{D_3}(Z_1\rtimes_\lambda H^0)\widetilde{}\\&\cong[(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)\otimes_{D_2}\widetilde{Z_1}]\rtimes_{\widehat{\widehat{\mu}}\otimes\widetilde{\lambda}} H^0\end{aligned}$$

as C_3-C_3-equivalence bimodules. Also, $Y_2\cong(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)\otimes_{D_2}\widetilde{Z_1}$ as C_2-C_2-equivalence bimodules. It follows that

$$[Y_2,Y_3]=[(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)\otimes_{D_2}\widetilde{Z_1}\,,\,[(Y\rtimes_\mu H^0\rtimes_{\widehat{\mu}} H)\otimes_{D_2}\widetilde{Z_1}]\rtimes_{\widehat{\widehat{\mu}}\otimes\widetilde{\lambda}} H^0]$$

in $\mathrm{Pic}(C_2, C_3)$. Then

$$[(Y \rtimes_\mu H^0 \rtimes_{\widehat{\mu}} H) \otimes_{D_2} \widetilde{Z_1}\,,\, \widehat{\widehat{\mu}} \otimes \widetilde{\lambda}\,,\, f^{-1}] \in \mathrm{Pic}_{H^0}^{\widehat{\widehat{\widehat{\rho}}}}(C_2)$$

and

$$\widehat{\widehat{\widehat{\theta}}}([(Y \rtimes_\mu H^0 \rtimes_{\widehat{\mu}} H) \otimes_{D_2} \widetilde{Z_1}\,,\, \widehat{\widehat{\mu}} \otimes \widetilde{\lambda}\,,\, f^{-1}]) = [Y_2, Y_3].$$

Furthermore, we have the following lemma:

Lemma 6.3 *Let (ρ, u) be a twisted coaction of H^0 on a unital C^*-algebra A. Let θ be the homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Pic}(A, C)$ defined by*

$$\theta([X, \lambda, f^0]) = [X, X \rtimes_\lambda H]$$

for any $(X, \lambda, f^0) \in \mathrm{GEqui}_H^{\rho,u}(A)$, where $C = A \rtimes_{\rho,u} H$. Also, let $\widehat{\theta}$ be the homomorphism of $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(C)$ to $\mathrm{Pic}(C, C_1)$ defined by

$$\widehat{\theta}([Y, \mu, g]) = [Y, Y \rtimes_\mu H^0]$$

for any $(Y, \mu, g) \in \mathrm{GEqui}_{H^0}^{\widehat{\rho}}(C)$, where $C_1 = C \rtimes_{\widehat{\rho}} H^0$. Furthermore, let F and G be the isomorphisms of $\mathrm{GPic}_H^{\rho,u}(A)$ and $\mathrm{Pic}(A, C)$ onto $\mathrm{GPic}_{H^0}^{\widehat{\rho}}(C)$ and $\mathrm{Pic}(C, C_1)$ defined by

$$F([X, \lambda, f^0]) = [X \rtimes_\lambda H, \widehat{\lambda}, f],$$
$$G([X, Y]) = [Y, Y_1]$$

for any $(X, \lambda, f^0) \in \mathrm{GEqui}_{H^0}^{\widehat{\rho}}(C)$ and $(X, Y) \in \mathrm{Equi}(A, C)$, respectively, where f is the C^*-Hopf algebra automorphism of H induced by f^0 and Y_1 is the upward basic construction of Y for the conditional expectation $E^X : Y \to X$ with respect to the canonical conditional expectation $E_1^{\rho,u} : C \to A$ and $E_1^{\rho,u}$. Then $G \circ \theta = \widehat{\theta} \circ F$.

Proof Let $[X, \lambda, f^0]$ be any element in $\mathrm{GPic}_H^{\rho,u}(A)$. Then

$$(G \circ \theta)([X, \lambda, f^0]) = [X \rtimes_\lambda H\,,\, C \otimes_A X \otimes_A \widetilde{C}],$$
$$(\widehat{\theta} \circ F)([X, \lambda, f^0]) = [X \rtimes_\lambda H\,,\, X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0].$$

By Lemma 6.1 and [15, Proposition 6.11], we can see that

$$[X \rtimes_\lambda H\,,\, C \otimes_A X \otimes_A \widetilde{C}] = [X \rtimes_\lambda H\,,\, X \rtimes_\lambda H \rtimes_{\widehat{\lambda}} H^0]$$

in $\mathrm{Pic}(C, C_1)$. Therefore, we obtain the conclusion. □

Theorem 6.4 *Let θ be the homomorphism of $\mathrm{GPic}_H^{\rho,u}(A)$ to $\mathrm{Pic}(A, C)$ defined in the above. We suppose that $A' \cap C = \mathbf{C}1$. Then θ is surjective.*

Proof Let $[X, Y]$ be any element in $\mathrm{Pic}(A, C)$. Then by the discussions before Lemma 6.3,

$$\widehat{\widehat{\widehat{\theta}}}([(Y \rtimes_\mu H^0 \rtimes_{\widehat{\mu}} H) \otimes_{D_2} \widetilde{Z_1}\,,\, \widehat{\widehat{\mu}} \otimes \widetilde{\lambda}\,,\, f^{-1}]) = [Y_2, Y_3].$$

Hence, by Lemma 6.3, θ is surjective. □

Next, we shall show that

$$\mathrm{Ker}\,\theta = \{[{}_A A_A\,,\,\rho, f^0] \in \mathrm{GPic}_H^{\rho,u}(A) \mid f^0 \in \mathrm{Aut}(H^0)\}.$$

Clearly, for any $f^0 \in \mathrm{Aut}(H^0)$, $[{}_A A_A\,,\,\rho\,,\,f^0] \in \mathrm{Ker}\,\theta$. We show that

$$\mathrm{Ker}\,\theta \subset \{[{}_A A_A\,,\,\rho, f^0] \in \mathrm{GPic}_H^{\rho,u}(A) \mid f^0 \in \mathrm{Aut}(H^0)\}.$$

Let $[X, \lambda, f^0] \in \mathrm{GPic}_H^{\rho,u}(A)$. We suppose that $\theta([X, \lambda, f^0]) = [A, C]$ in $\mathrm{Pic}_H^{\rho,u}(A)$. Then there is a $C-C$-equivalence bimodule isomorphism π of C onto $X \rtimes_\lambda H$ such that $\pi|_A$ is an $A-A$-equivalence bimodule isomorphism of A onto X. Let $\pi_0 = \pi|_A$.

Lemma 6.5 *With the above notation and assumptions,* $[X, \lambda, f^0] = [{}_A A_A\,,\,\rho, f^0]$ *in* $\mathrm{GPic}_H^{\rho,u}(A)$.

Proof For any $a \in A$, $h \in H$,

$$\begin{aligned}
\pi_0(h \cdot_{\rho,u} a) &= \pi((1 \rtimes_{\rho,u} h_{(1)})(a \rtimes_{\rho,u} 1)(1 \rtimes_{\rho,u} S(h_{(2)})^*)^*) \\
&= \pi((1 \rtimes_{\rho,u} h_{(1)}) \cdot (a \rtimes_{\rho,u} 1) \cdot (1 \rtimes_{\rho,u} S(h_{(2)})^*)^*) \\
&= (1 \rtimes_{\rho,u} h_{(1)}) \cdot (\pi_0(a) \rtimes_\lambda 1) \cdot (1 \rtimes_{\rho,u} S(h_{(2)})^*)^* \\
&= h \cdot_\lambda \pi_0(a)
\end{aligned}$$

by [12, Lemma 3.12] and [14, Lemma 5.6]. Hence, $[X, \lambda, f^0] = [{}_A A_A\,,\,\rho, f^0]$ in $\mathrm{GPic}_H^{\rho,u}(A)$. □

By Lemma 6.5, we have the following corollary:

Corollary 6.6 *With the above notation, we suppose that* $A' \cap C = \mathbf{C}1$. *Then*

$$\mathrm{Ker}\,\theta = \{[{}_A A_A\,,\,\rho, f^0] \in \mathrm{GPic}_H^{\rho,u}(A) \mid f^0 \in \mathrm{Aut}(H^0)\}.$$

Furthermore, $\mathrm{Ker}\,\theta \cong \mathrm{Aut}(H^0)$. Indeed, let κ be the homomorphism of $\mathrm{Aut}(H^0)$ to $\mathrm{GPic}_H^{\rho,u}(A)$ defined in Remark 3.8(2). Then clearly $\mathrm{Im}\kappa = \mathrm{Ker}\theta$. Hence, $\mathrm{Ker}\theta \cong \mathrm{Aut}(H^0)$.

Corollary 6.7 *With the above notation, we suppose that* $A' \cap C = \mathbf{C}1$. *Then we have the exact sequence*

$$1 \longrightarrow \mathrm{Aut}(H^0) \xrightarrow{\kappa} \mathrm{GPic}_H^{\rho,u}(A) \xrightarrow{\theta} \mathrm{Pic}(A, C) \longrightarrow 1,$$

where $C = A \rtimes_{\rho,u} H$.

References

[1] Blattner RJ, Cohen M, Montgomery S. Crossed products and inner actions of Hopf algebras. Transactions of the American Mathematical Society 1986; 298: 671-711.

[2] Brown LG. Stable isomorphism of hereditary subalgebra of C^*-algebras. Pacific Journal of Mathematics 1977; 71: 335-348.

[3] Brown LG, Green P, Rieffel MA. Stable isomorphism and strong Morita equivalence of C^*-algebras. Pacific Journal of Mathematics 1977; 71: 349-363.

[4] Combes F. Crossed products and Morita equivalence. Proceedings of the London Mathematical Society 1984; 49: 289-306.

[5] Curto RE, Muhly PS, Williams DP. Cross products of strong Morita equivalent C^*-algebras. Proceedings of the American Mathematical Society 1984; 90: 528-530.

[6] Jensen KK, Thomsen K. Elements of KK-theory. Boston, MA, USA: Birkhäuser, 1991.

[7] Izumi M. Inclusions of simple C^*-algebras. Journal für die reine und angewandte Mathematik 2002; 547: 97-138.

[8] Kajiwara T, Watatani Y. Jones index theory by Hilbert C^*-bimodules and K-Thorey. Transactions of the American Mathematical Society 2000; 352: 3429-3472.

[9] Kajiwara T, Watatani Y. Crossed products of Hilbert C^*-bimodules by countable discrete groups. Proceedings of the American Mathematical Society 1998; 126: 841-851.

[10] Kodaka K. Equivariant Picard groups of C^*-algebras with finite dimensional C^*-Hopf algebra coactions. Rocky Mountain Journal of Mathematics 2017; 47: 1565-1615.

[11] Kodaka K. The Picard groups for unital inclusions of unital C^*-algebras. preprint, arXiv: 1712.09499v1, Acta Scientiarum Mathematicarum, to appear.

[12] Kodaka K, Teruya T. Inclusions of unital C^*-algebras of index-finite type with depth 2 induced by saturated actions of finite dimensional C^*-Hopf algebras. Mathematica Scandinavica 2009; 104: 221-248.

[13] Kodaka K, Teruya T. The Rohlin property for coactions of finite dimensional C^*-Hopf algebras on unital C^*-algebras. Journal of Operator Theory 2015; 74: 329-369.

[14] Kodaka K, Teruya T. The strong Morita equivalence for coactions of a finite dimensional C^*-Hopf algebra on unital C^*-algebras. Studia Mathematica 2015; 228: 259-294.

[15] Kodaka K, Teruya T. The strong Morita equivalence for inclusions of C^*-algebras and conditional expectations for equivalence bimodules. Journal of the Australian Mathematical Society 2018; 105: 103-144.

[16] Kodaka K, Teruya T. Coactions of a finite dimensional C^*-Hopf algebra on unital C^*-algebras, unital inclusions of unital C^*-algebras and the strong Morita equivalence. preprint, arXiv:1706.09530.

[17] Osaka H, Kodaka K, Teruya T. The Rohlin property for inclusions of C^*-algebras with a finite Watatani index. In: de Jeu M, Sivestrov S, Skau C, Tomiyama J. (editors) Operator structures and dynamical systems: Contemporary Mathematics. Providence, USA: American Mathematical Society, 2009.

[18] Raeburn I, Williams DP. Morita equivalence and continuous -trace C^*-algebras. American Mathematical Society, 1998.

[19] Rieffel MA. C^*-algebras associated with irrational rotations. Pacific Journal of Mathematics 1981; 93: 415-429.

[20] Sweedler ME. Hopf Algebras. New York, NY, USA: Benjamin, 1969.

[21] Szymański W. Finite index subfactors and Hopf algebra crossed products. Proceedings of the American Mathematical Society 1994; 120: 519-528.

[22] Szymański W, Peligrad C. Saturated actions of finite dimensional Hopf *-algebras on C^*-algebras. Mathematica Scandinavica 1994;75: 217-239.

[23] Watatani Y. Index for C^*-subalgebras. Memoirs of the American Mathematical Society. Providence, USA: American Mathematical Society, 1990.