Turkish Journal of Mathematics

Volume 43 | Number 5 Article 34

1-1-2019

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2494 – 2498
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doi:10.3906/mat-1905-90

A remark on a paper of P. B. Djakov and M. S. Ramanujan*

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Received: 23.05.2019 • **Accepted/Published Online:** 24.08.2019 • **Final Version:** 28.09.2019

Abstract: Let ℓ be a Banach sequence space with a monotone norm in which the canonical system (e_n) is an unconditional basis. We show that if there exists a continuous linear unbounded operator between ℓ-Köthe spaces, then there exists a continuous unbounded quasidiagonal operator between them. Using this result, we study the corresponding Köthe matrices when every continuous linear operator between ℓ-Köthe spaces is bounded. As an application, we observe that the existence of an unbounded operator between ℓ-Köthe spaces, under a splitting condition, causes the existence of a common basic subspace.

Key words: Bounded operators, unbounded operators, ℓ-Köthe spaces

1. Introduction

Following [2], we denote by ℓ a Banach sequence space in which the canonical system (e_n) is an unconditional basis. The norm $\|.\|$ is called monotone if $\|x\| \leq \|y\|$ whenever $|x_n| \leq |y_n|$, $x = (x_n)$, $y = (y_n) \in \ell$, $n \in \mathbb{N}$. Let Λ be the class of such spaces with monotone norm. In particular, $l_p \in \Lambda$ and $c_0 \in \Lambda$. It is known that every Banach space with an unconditional basis (e_n) has a monotone norm which is equivalent to its original norm. Indeed, it is enough to put

$$\|x\| = \sup_{|\beta_n| \leq 1} \left| \sum_n {e_n}'(x) \beta_n e_n \right|$$

where $|.|$ denotes the original norm, $({e_n}')$ denote the sequence of coefficient functionals.

Let $\ell \in \Lambda$ and $\|.\|$ be a monotone norm in ℓ. If $A = (a_n^k)$ is a Köthe matrix, the ℓ-Köthe space $\lambda^\ell(A)$ is the space of all sequences of scalars (x_n) such that $(x_n a_n^k) \in \ell$ with the topology generated by the seminorms

$$\|(x_n)\|_k = \left\|(x_n a_n^k)\right\|.$$

For any linear operator $T : X \longrightarrow Y$ between Fréchet spaces we consider the following operator seminorms

$$\|T\|_{p,q} = \sup \left\{ \|Tx\|_p : \|x\|_q \leq 1 \right\}, \quad p, q \in \mathbb{N}$$

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2010 *AMS Mathematics Subject Classification:* 46A45

which may take the value $+\infty$. In particular, for any one dimensional operator $T = u \otimes x$, we have

$$\|T\|_{p,q} = \|u\|_q^* \|x\|_p$$

The operator T is continuous if and only if for all k there is $N(k)$ such that

$$\|T\|_{k,N(k)} < \infty,$$

T is bounded if and only if there is $N \in \mathbb{N}$ such that for all $r \in \mathbb{N}$,

$$\|T\|_{r,N} < \infty.$$

We write $(X, Y) \in \mathcal{B}$ if every continuous linear operator on X to Y is bounded. Zahariuta [8] obtained that if the matrices A and B satisfy the conditions d_2 and d_1, repectively, then $(\lambda^{l_1}(A), \lambda^{l_1}(B)) \in \mathcal{B}$. This phenomenon was studied extensively by Vogt [7] not only for Köthe spaces but also for the general case of Fréchet spaces. In case of ℓ-Köthe spaces, there is no characterization of pairs (X, Y) with the property $\mathcal{B}$.

For Fréchet spaces X and Y, in [7], Vogt proved that $(X, Y) \in \mathcal{B}$ if and only if for every sequence $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_0 \in \mathbb{N}$ and $C > 0$ with

$$\|T\|_{r,N} \leq C \max_{1 \leq k \leq k_0} \|T\|_{k,N(k)} \tag{1.1}$$

for all $T \in \mathcal{L}(X, Y)$.

An operator $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ is called quasidiagonal if there exists $k : \mathbb{N} \longrightarrow \mathbb{N}$ and constants m_n such that

$$Te_n = m_n \tilde{e}_{k(n)}, \quad n \in \mathbb{N}$$

Following [4], a pair of Köthe spaces $(\lambda^\ell(B), \lambda^\ell(A))$ satisfies the condition $\mathcal{S}$ if,

$$\forall p \quad \exists q, k \quad \forall s, l \quad \exists r, C : \frac{b_m^s}{a_n^k} \leq C \max\left\{\frac{b_m^q}{a_n^p}, \frac{b_m^r}{a_n^l}\right\} \tag{1.2}$$

In [3] it was proved that the existence of an unbounded continuous linear operator from nuclear l_1-Köthe space to another implies the existence of a continuous unbounded quasi-diagonal operator. Moreover, if the both Köthe spaces are nuclear, in [5], Nurlu and Terzioğlu proved that the existence of an unbounded continuous linear operator on $\lambda^{l_1}(A)$ to $\lambda^{l_1}(B)$ implies, under some conditions, the existence of a common basic subspaces of $\lambda^{l_1}(A)$ and $\lambda^{l_1}(B)$. Djakov and Ramanujan generalized these results by omitting nuclearity condition [1].

Let $X = \lambda^\ell(A)$ and $Y = \lambda^\ell(B)$ be the ℓ- Köthe spaces. Here, we modify Proposition 1 in [1] for ℓ- Köthe spaces and using it we obtain a necessary and sufficient condition in terms of corresponding Köthe matrices when $(X, Y) \in \mathcal{B}$. Furthermore, we observe a common basic subspace between ℓ- Köthe spaces X and Y when $(X, Y) \notin \mathcal{B}$ and $(Y, X) \in \mathcal{S}$ following the same lines in [1].

2. Bounded and unbounded operators in ℓ-Köthe spaces

Let $\lambda^\ell(A), \lambda^\ell(B)$ be ℓ-Köthe spaces. As in [1] we obtain the following.

Proposition 2.1 *Let $\lambda^\ell(A)$ and $\lambda^\ell(B)$ be ℓ- Köthe spaces. If there exists a continuous linear unbounded operator $T:\lambda^\ell(A)\longrightarrow\lambda^\ell(B)$, then there exists a continuous unbounded quasidiagonal operator on $\lambda^\ell(A)$ to $\lambda^\ell(B)$.*

Proof Let $T:\lambda^\ell(A)\longrightarrow\lambda^\ell(B)$ be continuous and unbounded. We may assume without loss of generality that

$$\|Tx\|_k \le \frac{1}{2^k}\|x\|_k\,, \quad \forall x\in\lambda^\ell(A) \tag{2.1}$$

$$\sup_n \frac{\|Te_n\|_{k+1}}{\|e_n\|_k}=\infty, \quad k\in\mathbb{N}. \tag{2.2}$$

Indeed, one may obtain these by using appropriate multipliers and passing to a subsequence of seminorms, if necessary. More precisely, we proceed as in the Lemma in [6] and denote by U_k, V_k the closed unit balls defined by the k-th seminorms on $\lambda^\ell(A)$, $\lambda^\ell(B)$ respectively. Starting with an arbitrary V_1 we choose by continuity U_1 such that $T(U_1)\subset V_1$; but $T(U_1)$ is not a bounded subset of $\lambda^\ell(B)$. Thus, we choose V_2 containing V_1 such that $T(U_1)$ is not absorbed by V_2, that is $\forall\alpha>0$ $T(U_1)\not\subset\alpha V_2$. Again using the contiuity, we choose U_2 containing U_1 such that $T(U_2)\subset V_2$. Notice that $\frac{e_n}{\|e_n\|_k}\in U_k$, $\forall n$. Hence, using the continuity and unboundedness of T alternately, we find U_k and V_k such that (2.1) and (2.2) are satisfied. Let (k_j) be a sequence of integers such that each $k\in\mathbb{N}$ appears in it infinitely many times and choose an increasing subsequence (n_j) such that

$$\frac{\|Te_{n_j}\|_{k_j+1}}{\|e_{n_j}\|_{k_j}}\ge 2^j, \quad \forall j$$

Let us remind that $\|\tilde{e_v}\|_k=b_v^k$ and $\|e_n\|_k=a_n^k$ and let $Te_n=\sum_v \theta_{nv}\tilde{e_v}$. Note that

$$\begin{aligned}\sup_{|\alpha_v|\le1}\left|\sum_v \theta_{nv}\alpha_v\left(\sup_k\frac{b_v^k}{a_n^k}\right)\tilde{e_v}\right| &\le \sum_k\left(\frac{b_v^k}{a_n^k}\right)\left(\sup_{|\alpha_v|\le1}\left|\sum_v\theta_{nv}\alpha_v\tilde{e_v}\right|\right)\\ &\le \sum_k\frac{1}{a_n^k}\sup_{|\alpha_v|\le1}\left|\sum_v\theta_{nv}\alpha_v b_v^k\tilde{e_v}\right|\\ &\le \sum_k\frac{\|Te_n\|_k}{\|e_n\|_k}\le\sum_k\frac{1}{2^k}\le 1\end{aligned}$$

Therefore, we obtain that

$$\sup_{|\alpha_v|\le1}\left|\sum_v\theta_{n_jv}\alpha_v\left(\sup_k\frac{b_v^k}{a_{n_j}^k}\right)\tilde{e_v}\right|\le 1\le\frac{1}{2^j}\sup_{|\alpha_v|\le1}\left|\sum_v\theta_{n_jv}\alpha_v\frac{b_v^{k_j+1}}{a_{n_j}^{k_j}}\tilde{e_v}\right| \tag{2.3}$$

Thus, there is a v_j such that

$$t_j:=\sup_k\frac{b_{v_j}^k}{a_{n_j}^k}\le\frac{1}{2^j}\frac{b_{v_j}^{k_j+1}}{a_{n_j}^{k_j}}$$

Otherwise we obtain a contradiction to (2.3) by monotonicity of $\|.\|$.

Now, consider the quasidiagonal operator $D : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ defined by

$$De_{n_j} = t_j^{-1} \tilde{e_{v_j}}, \quad j \in \mathbb{N}$$

$$De_n = 0 \quad \text{if} \quad n \neq n_j$$

Let $x = \sum_j x_{n_j} e_{n_j} \in \lambda^\ell(A)$. Thus, $Dx = \sum_j x_{n_j} t_j^{-1} \tilde{e}_{v_j}$. Since $\left| x_{n_j} t_j^{-1} b_{v_j}^k \right| \leq \left| x_{n_j} a_{n_j}^k \right|$, by monotonicity we obtain that $\left\| \left(x_{n_j} t_j^{-1} b_{v_j}^k \right) \right\| \leq \left\| \left(x_{n_j} a_{n_j}^k \right) \right\|$, i.e.

$$\|Dx\|_k \leq \|x\|_k \quad \forall k$$

Hence, D is continuous.

Similarly, it is easy to see that D is unbounded since for a fixed k, there is a subsequence (j_m) such that $k_{j_m} = k$, $m \in \mathbb{N}$ and

$$\frac{\|De_{n_{j_m}}\|_{k+1}}{\|e_{n_{j_m}}\|_k} \geq 2^{j_m} \to \infty$$

as $m \to \infty$. This completes the proof. □

Proposition 2.1 enables us to prove the sufficiency part of the following theorem. Notice that sufficiency cannot be obtained directly for a general linear map.

Theorem 2.2 *Let $\lambda^\ell(A)$ and $\lambda^\ell(B)$ be ℓ-Köthe spaces. $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$ if and only if for every sequence $N(k) \uparrow \infty$ there exists $N \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with*

$$\frac{b_v^r}{a_i^N} \leq C \max_{1 \leq k \leq k_0} \frac{b_v^k}{a_i^{N(k)}}$$

for all $v \in \mathbb{N}, i \in \mathbb{N}$.

Proof Suppose $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$. Consider $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ with $T = e_i' \otimes e_v$ where $e_i{}'(x) = x_i$ for all $x \in \lambda^\ell(A)$.

Since T is the operator of rank one, we note that

$$\|T\|_{k,N(k)} = \|e_i{}'\|_{N(k)} \|e_v\|_k = \frac{b_v^k}{a_j^{N(k)}}$$

Similarly, $\|T\|_{r,N} = \frac{b_v^r}{a_i^N}$. The result follows from (1.1).

Conversely, we want to show that every continuous linear quasidiagonal operator is bounded. Let $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$ be a continuous quasidiagonal operator defined by $T(e_i) = t_i \tilde{e}_{z(i)}$. By continuity, $\exists N(k)$ such that

$$\sup_i \frac{\|Te_i\|_k}{\|e_i\|_{N(k)}} = \sup_i \frac{|t_i| \, b_{z(i)}^k}{a_i^{N(k)}} = C(k) < \infty.$$

Thus, for this $N(k)$, $\exists N \in \mathbb{N}$ such that $\forall r \in \mathbb{N}$ we have $k_o \in \mathbb{N}$ and $C > 0$ with

$$\frac{|t_i|\, b^r_{z(i)}}{a_i^N} \leq C \max_{1\leq k\leq k_0} \frac{|t_i|\, b^k_{z(i)}}{a_i^{N(k)}} \leq C \max_{1\leq k\leq k_0} C(k).$$

Hence, $\|T\|_{r,N} < \infty$, i.e. T is bounded. In view of Proposition 2.1, we obtain the result. □

$\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace if there is a quasidiagonal operator $T : \lambda^\ell(A) \to \lambda^\ell(B)$ such that the restriction of T to some infinite dimensional basic subspace of $\lambda^\ell(A)$ is an isomorphism. We observe the following extension of Proposition 3 in [1] to the ℓ-Köthe space case. The proof is the same as in [1].

Corollary 2.3 *If $(\lambda^\ell(B), \lambda^\ell(A)) \in \mathcal{S}$ and there exists a continuous unbounded operator $T : \lambda^\ell(A) \longrightarrow \lambda^\ell(B)$, then $\lambda^\ell(A)$ and $\lambda^\ell(B)$ have a common basic subspace.*

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