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Some properties for a class of analytic functions defined by a higher-order differential inequality

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Abstract: Let $\mathcal{B}_p(\alpha,\beta,\lambda;j)$ be the class consisting of functions $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k$, $p\in\mathbb{N}$ which satisfy

$$\mathrm{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\}>\lambda,\ \ (z\in\mathbb{U}=\{z:\ |z|<1\}),$$

for some λ $(\lambda<p!\{\alpha+(p-j)\beta+(p-j)(p-j-1)(\beta-\alpha)/2\}/(p-j)!)$ and $j=0,1,...,p$, where $p+1-j+2\alpha/(\beta-\alpha)>0$ or $\alpha=\beta=1$. The extreme points of $\mathcal{B}_p(\alpha,\beta,\lambda;j)$ are determined and various sharp inequalities related to $\mathcal{B}_p(\alpha,\beta,\lambda;j)$ are obtained. These include univalence criteria, coefficient bounds, growth and distortion estimates and bounds for certain linear operators. Furthermore, inclusion properties are investigated and estimates on λ are found so that functions of $\mathcal{B}_p(\alpha,\beta,\lambda;j)$ are p-valent starlike in $\mathbb{U}$. For instance, $\mathrm{Re}\{zf''(z)\}>(5-12\ln 2)/(44-48\ln 2)\approx-0.309$ is sufficient condition for any normalized analytic function f to be starlike in $\mathbb{U}$. The results improve and include a number of known results as their special cases.

Key words: Starlike functions, p-valent functions, Jack's lemma, univalent functions, extreme points, convex functions, distortion and growth theorem, coefficient bounds, differential inequality

1. Introduction and preliminaries

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k,\quad p\in\mathbb{N}=\{1,2,3,...\},$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f\in\mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^*(\lambda)$ of p-valent starlike functions of order λ in $\mathbb{U}$, if it satisfies

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\lambda,\ (0\le\lambda<p;\ z\in\mathbb{U}).$$

The corresponding class $\mathcal{CV}_p(\lambda)$ of p-valent convex functions in $\mathbb{U}$ consists of functions f defined by $zf'(z)\in\mathcal{S}_p^*(\lambda)$. Furthermore, let $\mathcal{S}$ be the subclass of functions in $\mathcal{A}_1$ that are univalent in $\mathbb{U}$. The subclasses of $\mathcal{S}$

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consisting of starlike and convex functions in $\mathbb{U}$ are defined respectively by $\mathcal{S}_1^*(0) := \mathcal{S}^*$ and $\mathcal{CV}_1(0) := \mathcal{CV}$. Let $\mathcal{R}$ denote the class of functions $f \in \mathcal{A}_1$ which satisfy

$$\mathrm{Re}\{f'(z) + zf''(z)\} > 0, \quad (z \in \mathbb{U}).$$

Chichra [3] proved that $\mathcal{R} \subset \mathcal{S}$. Indeed, Singh and Singh [13] showed that $\mathcal{R} \subset \mathcal{S}^*$. Owa et al. [11] studied the subclass $\mathcal{A}_p(\alpha, \beta, \lambda; j)$ of $\mathcal{A}_p$ consisting of functions f defined by

$$\mathrm{Re}\left\{\alpha \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} > \lambda, \quad (z \in \mathbb{U}),$$

for some α $(\alpha > 0)$, β $(\beta > 0)$ and λ $(0 \leq \lambda < p!\{\alpha + (p-j)\beta\}/(p-j)!$, where $j = 0, 1, ..., p-1$. The class is studied for the starlikeness when $j = p = \alpha = 1$ by Gao and Zhou [5], while for $j = p = \alpha = \beta = 1$ was studied earlier by Silverman [12]. Wang et al. [15] found radius of univalency for the special case when $j = 0$ and $p = 1$. Several authors, including [8, 9, 14, 16], studied classes of analytic functions associated with $\mathcal{R}$. We can deduce that [11, Theorem 2.1] which presents characterization for functions of $\mathcal{A}_p(\alpha, \beta, \lambda; j)$ holds true with the same proof, whenever the parameter conditions extended to be $p + 1 - j + \alpha/\beta > 0$, $\lambda < p!\{\alpha + (p-j)\beta\}/(p-j)!$ and $j = 0, 1, ..., p$.

In this paper, we consider the subclass $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left\{\alpha \frac{f^{(j)}(z)}{z^{p-j}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}} + \left(\frac{\beta - \alpha}{2}\right) \frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\} > \lambda, \quad (z \in \mathbb{U}), \tag{1.1}$$

for some λ $(\lambda < p!\{\alpha + (p-j)\beta + (p-j)(p-j-1)(\beta-\alpha)/2\}/(p-j)!)$ and $j = 0, 1, ..., p$, where $p + 1 - j + 2\alpha/(\beta - \alpha) > 0$ or $\alpha = \beta = 1$. The extreme points and the sharp coefficient bounds for $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ are determined. These key results yield important information, like distortion and growth estimates, bounds for auxiliary linear operators acting on $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ that are helpful in investigating starlikeness conditions. Using an application of Jack's lemma, inclusion properties associated with the classes $\mathcal{A}_p(\alpha, \beta, \lambda; j)$ and $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ are investigated by suitably choosing the parameters. Estimates on λ are found that would ensure functions of certain subclasses of $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ are p-valent starlike in $\mathbb{U}$. The results improve and include a number of known results as their special cases.

From the definition of the class $\mathcal{B}_p(\alpha, \beta, \lambda; j)$, we see the following

Lemma 1.1 (i) $f(z) \in \mathcal{B}_p(j, j+2, \lambda; j) \iff zf'(z) \in \mathcal{B}_p(1, 1, \lambda; j)$.
(ii) $f(z) \in \mathcal{B}_p(1, 2, \lambda, 1) \iff zf'(z) \in \mathcal{B}_p(0, 1, \lambda, 0)$.

Proof Let $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$. Then

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j}. \tag{1.2}$$

Using the representation (1.2) and a mild computation, we obtain

$$\begin{aligned} zf'(z) \in \mathcal{B}_p(1, 1, \lambda; j) \quad &\iff \quad \mathrm{Re}\left\{\frac{p!p(p-j+1)}{(p-j)!} + \sum_{k=p+1}^{\infty} \frac{k!k(k-j+1)}{(k-j)!} a_k z^{k-p}\right\} > \lambda \\ &\iff \quad f(z) \in \mathcal{B}_p(j, j+2, \lambda; j). \end{aligned}$$

This proves (i). Now,

$$\begin{aligned} f(z) \in \mathcal{B}_p(1, 2, \lambda; 1) &\iff \operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}} + 2\frac{f''(z)}{z^{p-2}} + \frac{f'''(z)}{2z^{p-3}}\right\} > \lambda \\ &\iff zf'(z) \in \mathcal{B}_p(0, 1, \lambda; 0). \end{aligned}$$

Hence, the proof of (ii) follows. □

2. The extreme points of $\mathcal{B}_p(\alpha, \beta, \lambda; j)$

Theorem 2.1 *A function $f \in \mathcal{A}_p$ is in the class of $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ if and only if*

$$f(z) = z^p + 4(\sigma - \lambda) \int_{|x|=1} \left(\sum_{k=p+1}^{\infty} \frac{(k-j)! x^{k-p} z^k}{k!(k-j+1)\{(\beta - \alpha)(k-j) + 2\alpha\}} \right) d\mu(x), \tag{2.1}$$

where $\mu(x)$ is the probability measure on $X = \{x \in \mathbb{C} : \ |x| = 1\}$ and

$$\sigma = p!\{\alpha + (p-j)\beta + (p-j)(p-j-1)(\beta - \alpha)/2\}/(p-j)!. \tag{2.2}$$

Proof We define the function

$$F(z) = \frac{\alpha(f^{(j)}(z)/z^{p-j}) + \beta(f^{(j+1)}(z)/z^{p-j-1}) + (\beta - \alpha)(f^{(j+2)}(z)/2z^{p-j-2}) - \lambda}{\sigma - \lambda},$$

where $\sigma = p!\{\alpha + (p-j)\beta + (p-j)(p-j-1)(\beta - \alpha)/2\}/(p-j)!$. Then $F(z)$ is a Carathéodory function, since $F(0) = 1$ and Re $F(z) > 0$. Therefore, by the aid of Herglotz expressions of Carathéodory functions, we can write $F(z)$ in the following form:

$$F(z) = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x). \tag{2.3}$$

Since (2.3) is equivalent to

$$\alpha f^{(j)}(z) + \beta z f^{(j+1)}(z) + \frac{1}{2}(\beta - \alpha) z^2 f^{(j+2)}(z) = \lambda z^{p-j} + (\sigma - \lambda) z^{p-j} \int_{|x|=1} \left(1 + \sum_{k=1}^{\infty} 2x^k z^k \right) d\mu(x),$$

we have, for $\beta \neq \alpha$, that

$$\frac{2\alpha}{\beta - \alpha} \left(z f^{(j)}(z) \right)' + \left(z^2 f^{(j+1)}(z) \right)' = \frac{2}{\beta - \alpha} \left\{ \sigma z^{p-j} + 2(\sigma - \lambda) \int_{|x|=1} \left(\sum_{k=1}^{\infty} x^k z^{k+p-j} \right) d\mu(x), \right\}. \tag{2.4}$$

Integrating both sides of (2.4) from 0 to z gives

$$\frac{2\alpha}{\beta - \alpha} z f^{(j)}(z) + z^2 f^{(j+1)}(z) = \frac{2}{\beta - \alpha} \left\{ \frac{\sigma z^{p-j+1}}{p-j+1} + 2(\sigma - \lambda) \int_{|x|=1} \left(\sum_{k=1}^{\infty} \frac{x^k z^{k+p-j+1}}{k+p-j+1} \right) d\mu(x), \right\}. \tag{2.5}$$

Multiplying both sides of (2.5) by $z^{\frac{2\alpha}{\beta-\alpha}-2}$ yields

$$\left(z^{\frac{2\alpha}{\beta-\alpha}}\left[f^{(j)}(z)-\frac{p!}{(p-j)!}z^{p-j}\right]\right)'=\frac{4(\sigma-\lambda)}{\beta-\alpha}\int_{|x|=1}\sum_{k=1}^{\infty}\frac{x^k z^{\frac{2\alpha}{\beta-\alpha}+k+p-j-1}}{k+p-j+1}d\mu(x). \tag{2.6}$$

If $p+1-j+2\alpha/(\beta-\alpha)>0$, then by integrating (2.6) from 0 to z, we reach:

$$f^{(j)}(z)=\frac{p!z^{p-j}}{(p-j)!}+\int_{|x|=1}\sum_{k=1}^{\infty}\frac{4(\sigma-\lambda)x^k z^{k+p-j}}{(k+p-j+1)\{(\beta-\alpha)(k+p-j)+2\alpha\}}d\mu(x). \tag{2.7}$$

An integration of both sides in (2.7) gives

$$\begin{aligned} f^{(j-1)}(z)-f^{(j-1)}(0)&=\frac{p!}{(p-j+1)!}z^{p-j+1}\\ +\quad 4(\sigma-\lambda)&\int_{|x|=1}\left(\sum_{k=1}^{\infty}\frac{x^k z^{k+p-j+1}}{(k+p-j+1)^2\{(\beta-\alpha)(k+p-j)+2\alpha\}}\right)d\mu(x). \end{aligned}$$

Therefore, we have

$$f^{(j-1)}(z)=\frac{p!z^{p-j+1}}{(p-j+1)!}+\int_{|x|=1}\sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)x^{k-p}z^{k-j+1}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}d\mu(x). \tag{2.8}$$

Applying the same method for (2.8), we get

$$\begin{aligned} f^{(j-2)}(z)&=\frac{p!}{(p-j+2)!}z^{p-j+2} \qquad\qquad (2.9)\\ +\quad 4(\sigma-\lambda)&\int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{x^{k-p}z^{k-j+2}}{(k-j+2)(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}\right)d\mu(x). \end{aligned}$$

Since $f^{(p)}(0)=p!$ and $f^{(j)}(0)=0$, for $j=0,1,..,p-1$, then integrating $(j-2)$ times to both sides of (2.9) gives (2.1). If $\beta=\alpha=1$, then

$$f(z)=z^p+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\int_{|x|=1}\left(\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!(k-j+1)}x^{k-p}z^k\right)d\mu(x)$$

is given in [11, Theorem 2.1] for the class $\mathcal{A}_p(1,1,\lambda;j)$ which is equivalent to $\mathcal{B}_p(1,1,\lambda;j)$, where $j=0,1,...,p$. This completes the proof of Theorem 2.1. □

Remark 2.2 *Theorem 2.1 can be reduced to special known cases as follows*

(a) For $\alpha=\frac{1}{2}-\gamma$ *and* $\beta=\frac{1}{2}+\gamma$, $p=1$, $j=0$ *in (2.1), we have*

$$f(z)=z+4(1-\lambda)\int_{|x|=1}\left(\sum_{k=2}^{\infty}\frac{x^{k-1}z^k}{(k+1)(2\gamma k+1-2\gamma)}\right)d\mu(x)$$

is a characterization for the functions of the class $\mathcal{W}_\lambda(3\gamma+\frac{1}{2},\gamma)$ *given in [2, Theorem 2.1], whenever* $\varphi=0$ *and* $\gamma\le 3/2$.

(b) If $\alpha = \beta = p = j = 1$ *in (2.1), then*

$$f(z) = \int_{|x|=1} \left[\int_0^z \left(\frac{(2\lambda-1)t + 2(\beta-1)\bar{x}\log(1-xt)}{t} \right) dt \right] d\mu(x)$$

which is a result obtained in [12, Theorem 1].

(c) If $\alpha = \beta = 1$ *and* $j = 0$ *in (2.1), we obtain a result given in [15, Theorem 2.1].*

(d) If $\alpha = 0$, $\beta = 1$, $p = 1$ *and* $j = 0$ *in (2.1), then*

$$f(z) = z + 4(1-\lambda) \int_{|x|=1} \left(\sum_{k=2}^{\infty} \frac{x^{k-1}z^k}{k(k+1)} \right) d\mu(x)$$

leads to a result obtained from [5, Theorem 1].

Corollary 2.3 *The extreme points of the class* $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ *are*

$$f_x(z) = z^p + 4(\sigma - \lambda) \sum_{k=p+1}^{\infty} \frac{(k-j)!\, x^{k-p} z^k}{k!(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}, \quad (|x| = 1). \tag{2.10}$$

Proof Using the notation $f_x(z)$, (2.1) can be written as

$$f_\mu(z) = \int_{|x|=1} f_x(z)\, d\mu(x).$$

By Theorem 2.1, the map $\mu \to f_\mu$ is one-to-one, so the assertion follows (see [6]). □

Setting $p = 1$ and $j = 0$ in Theorem 2.1, we get the following

Corollary 2.4 *A function* $f \in \mathcal{A}_1$ *is in the class of* $\mathcal{B}_1(\alpha, \beta, \lambda; 0)$ *if and only if*

$$f(z) = z + 4(\alpha + \beta - \lambda) \int_{|x|=1} \left(\sum_{k=2}^{\infty} \frac{x^{k-1}z^k}{(k+1)\{(\beta-\alpha)k + 2\alpha\}} \right) d\mu(x), \tag{2.11}$$

where $\mu(x)$ *is the probability measure on* $X = \{x \in \mathbb{C} : |x| = 1\}$ *and* $\lambda < \alpha + \beta$.

If $p = 1$ and $j = 1$ in Theorem 2.1, then

Corollary 2.5 *A function* $f \in \mathcal{A}_1$ *is in the class of* $\mathcal{B}_1(\alpha, \beta, \lambda; 1)$ *if and only if*

$$f(z) = z + 4(\alpha - \lambda) \int_{|x|=1} \left(\sum_{k=2}^{\infty} \frac{x^{k-1}z^k}{k^2\{(\beta-\alpha)(k-1) + 2\alpha\}} \right) d\mu(x), \tag{2.12}$$

where $\mu(x)$ *is the probability measure on* $X = \{x \in \mathbb{C} : |x| = 1\}$ *and* $\lambda < \alpha$.

Corollary 2.6 *If $f \in \mathcal{B}_p(\alpha, \beta, \lambda; j)$, then*

$$|a_k| \leq \frac{4(\sigma - \lambda) \cdot (k-j)!}{k!(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}, \quad (k \geq p+1),$$

where $\sigma = p!\{\alpha + (p-j)\beta + (p-j)(p-j-1)(\beta-\alpha)/2\}/(p-j)!$. The result is sharp, where equality attained for functions defined by (2.10).

Remark 2.7 *For $\alpha = \beta = p = j = 1$, Corollary 2.6 yields the coefficient bounds obtained in [12, Corollary 1].*

Corollary 2.8 *If $f \in \mathcal{B}_p(\alpha, \beta, \lambda; j)$ and $z \in \mathbb{U}$, then*

$$\max\left\{0, \frac{p!|z|^{p-j}}{(p-j)!} - \sum_{k=1}^{\infty} \frac{4(\sigma-\lambda)|z|^{k+p-j}}{(k+p-j+1)\{(\beta-\alpha)(k+p-j)+2\alpha\}}\right\}$$

$$\leq \ | f^{(j)}(z) | \leq \frac{p!|z|^{p-j}}{(p-j)!} + \sum_{k=1}^{\infty} \frac{4(\sigma-\lambda)|z|^{k+p-j}}{(k+p-j+1)\{(\beta-\alpha)(k+p-j)+2\alpha\}}.$$

Setting $p = 1, \beta = 3, \alpha = 1$ in Corollary 2.8 and using the fact that $\sum_{k=1}^{\infty}(1/k^2) = \pi^2/6$, we obtain respectively the following two remarks for $j = 0$ and $j = 1$.

Remark 2.9 *If*

$$\mathrm{Re}\left\{\frac{f(z)}{z} + 3f'(z) + zf''(z)\right\} > \lambda, \quad (\lambda < 4, \ z \in \mathbb{U}),$$

then

$$|f(z)| < 1 + (4-\lambda)\left(\frac{\pi^2}{3} - \frac{5}{2}\right), \quad (z \in \mathbb{U}).$$

Remark 2.10 *If*

$$\mathrm{Re}\left\{f'(z) + 3zf''(z) + z^2 f'''(z)\right\} > \lambda, \quad (\lambda < 1, \ z \in \mathbb{U}),$$

then

$$|f'(z)| < 1 + (1-\lambda)\left(\frac{\pi^2}{3} - 2\right), \quad (z \in \mathbb{U}).$$

3. Bounds for auxiliary linear operators acting on $\mathcal{B}_p(\alpha, \beta, \lambda; j)$

In this section, sharp bounds for the linear operators $f^{(j)}(z)/z^{p-j}$, $f^{(j-1)}(z)/z^{p-j+1}$ and $f^{(j+1)}(z)/z^{p-j-1}$ are obtained. These bounds play main role in finding several univalence criteria and investigating p-valent starlike functions included in $\mathcal{B}_p(\alpha, \beta, \lambda; j)$.

Theorem 3.1 *If $f(z) \in \mathcal{B}_p(\alpha, \beta, \lambda; j)$ and $|z| \leq r$, where $j = 0, ..., p$, then*

$$\begin{aligned}\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} &\geq \frac{p!}{(p-j)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty} \frac{(-r)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}} \\ &> \frac{p!}{(p-j)!} + \sum_{k=p+1}^{\infty} \frac{4(\sigma-\lambda)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}} \qquad (3.1)\end{aligned}$$

and

$$\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\} \leq \frac{p!}{(p-j)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}$$
$$< \frac{p!}{(p-j)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{1}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.$$

The inequalities are both sharp.

Proof It suffices to consider the extreme point

$$g(z) = z^p + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}z^k. \tag{3.2}$$

Case I: if $p+1-j+2\alpha/(\beta-\alpha)>0$, then from (2.7), we obtain

$$\frac{g^{(j)}(z)}{z^{p-j}} = \frac{p!}{(p-j)!} + \sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)z^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}. \tag{3.3}$$
$$= \frac{p!}{(p-j)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1 s^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\frac{z}{1-stz}\ ds\ dt$$

Therefore,

$$\mathrm{Re}\left\{\frac{g^{(j)}(z)}{z^{p-j}}\right\} = \frac{p!}{(p-j)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1 s^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\mathrm{Re}\left\{\frac{z}{1-stz}\right\}dsdt. \tag{3.4}$$

Since, $k(z)=z/(1-\delta z)$, $(0\leq\delta\leq 1)$ is convex in $\mathbb{U}$, $k(\overline{z})=\overline{k(z)}$ and $k(z)$ maps real axis to real axis, we have

$$\frac{-r}{1+\delta r} \leq \mathrm{Re}\left\{\frac{z}{1-\delta z}\right\} \leq \frac{r}{1-\delta r}, \quad (|z|\leq r). \tag{3.5}$$

Applying (3.5) into (3.4) and expanding the integrand into a power series gives

$$\mathrm{Re}\left\{\frac{g^{(j)}(z)}{z^{p-j}}\right\} = \frac{p!}{(p-j)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1\sum_{k=p+1}^{\infty}s^{k-j}t^{\frac{2\alpha}{\beta-\alpha}+k-j-1}(-r)^{k-p}dsdt. \tag{3.6}$$

Performing the indicated integration yields

$$\mathrm{Re}\left\{\frac{g^{(j)}(z)}{z^{p-j}}\right\} \geq \frac{p!}{(p-j)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(-r)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}$$
$$> \frac{p!}{(p-j)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.$$

On the other hand, using (3.5), we have

$$\begin{aligned}\operatorname{Re}\left\{\frac{g^{(j)}(z)}{z^{p-j}}\right\} &\leq \frac{p!}{(p-j)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}\\ &< \frac{p!}{(p-j)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{1}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.\end{aligned}$$

The sharpness can be seen from (3.2).

Case II: if $\alpha=\beta=1$, we consider the extreme point

$$h(z)=z^p+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(k-j)!}{k!(k-j+1)}z^k \tag{3.7}$$

Applying similar technique of $g(z)$ for the function $h(z)$, we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{h^{(j)}(z)}{z^{p-j}}\right\} &\geq \frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(-r)^{k-p}}{k-j+1}\\ &> \frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(-1)^{k-p}}{k-j+1},\end{aligned}$$

while

$$\begin{aligned}\operatorname{Re}\left\{\frac{h^{(j)}(z)}{z^{p-j}}\right\} &\leq \frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{k-j+1}\\ &< \frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{1}{k-j+1}.\end{aligned}$$

The sharpness can be seen from (3.7). This completes the proof of Theorem 3.1. □

Remark 3.2 *If $\alpha=\beta=1$ in Theorem 3.1, we obtain a result found in [10, Lemma 5].*

Corollary 3.3 *If $f(z)\in\mathcal{B}_1(1,1,\lambda;1)$, then*

$$\operatorname{Re}\{f'(z)\}>1+2(1-\lambda)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}=(1-\lambda)(2\ln 2-1)+\lambda.$$

This result was obtained by several authors [1, 4, 5, 12].

Corollary 3.4 *$\mathcal{B}_1(\alpha,\beta,\lambda;1)\subset\mathcal{S}$ for $\lambda\geq\lambda_0$, where*

$$\lambda_0=\alpha+\frac{1}{4}\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)[(\beta-\alpha)k+2\alpha]}\right)^{-1} \tag{3.8}$$

The result cannot be extended to $\lambda<\lambda_0$.

Proof Let $f \in \mathcal{B}_1(\alpha,\beta,\lambda;1)$. From (3.1), we know that if

$$1 + 4(\alpha-\lambda)\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)\{(\beta-\alpha)k+2\alpha\}} \geq 0,$$

then $f \in \mathcal{S}$, that is, if $\lambda \geq \lambda_0$, we have $\mathcal{B}_1(\alpha,\beta,\lambda;1) \subset \mathcal{S}$. The result cannot be extended to $\lambda < \lambda_0$ because $g'(-1) = 0$ at $\lambda = \lambda_0$. Thus $g'(-r) = 0$ for some $r = r(\lambda) < 1$ when $\lambda < \lambda_0$. □

Example 3.5 *(i)* $\mathcal{B}_1(1,3,\lambda;1) \subset \mathcal{S}$, *for* $\lambda \geq (6-\pi^2)/(12-\pi^2) \approx -1.8164$.
(ii) $\mathcal{B}_1(0,1,\lambda;1) \subset \mathcal{S}$, *for* $\lambda \geq 1/(4-8\ln 2) \approx -0.6472$. *The bounds of* λ *in (i) and (ii) are best possible for univalence.*

Theorem 3.6 *If* $f(z) \in \mathcal{B}_p(\alpha,\beta,\lambda;j)$ *and* $|z| \leq r$, *where* $j = 1,...,p$, *then*

$$\begin{aligned}\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} &\geq \frac{p!}{(p-j+1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(-r)^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}} \\ &> \frac{p!}{(p-j+1)!} + \sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)(-1)^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\mathrm{Re}\left\{\frac{f^{(j-1)}(z)}{z^{p-j+1}}\right\} &\leq \frac{p!}{(p-j+1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}} \\ &< \frac{p!}{(p-j+1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{1}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}.\end{aligned}$$

The inequalities are both sharp.

Proof We need only to consider the extreme point $g(z)$ defined by (3.2).
Case I: if $p+1-j+2\alpha/(\beta-\alpha) > 0$, then from (2.7), we obtain

$$\begin{aligned}\frac{g^{(j-1)}(z)}{z^{p-j+1}} &= \frac{p!}{(p-j+1)!} + \sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)z^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}. \\ &= \frac{p!}{(p-j+1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1\int_0^1 (us)^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\frac{z}{1-stuz}\,ds\,dt\,du.\end{aligned} \tag{3.10}$$

Therefore,

$$\mathrm{Re}\left\{\frac{g^{(j-1)}(z)}{z^{p-j+1}}\right\} = \frac{p!}{(p-j+1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1\int_0^1 (us)^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\mathrm{Re}\left\{\frac{z}{1-stuz}\right\}ds\,dt\,du. \tag{3.11}$$

Applying (3.5) into (3.11) and expanding the integrand into a power series gives

$$\mathrm{Re}\left\{\frac{g^{(j-1)}(z)}{z^{p-j+1}}\right\} = \frac{p!}{(p-j+1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\int_0^1\int_0^1\int_0^1 \sum_{k=p+1}^{\infty}(us)^{k-j}t^{\frac{2\alpha}{\beta-\alpha}+k-j-1}(-r)^{k-p}\,ds\,dt\,du. \tag{3.12}$$

Perform the indicated integration yield

$$\begin{aligned}\mathrm{Re}\left\{\frac{g^{(j-1)}(z)}{z^{p-j+1}}\right\} &\geq \frac{p!}{(p-j+1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(-r)^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}\\ &> \frac{p!}{(p-j+1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(-1)^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}.\end{aligned}$$

On the other hand, using (3.5), we have

$$\begin{aligned}\mathrm{Re}\left\{\frac{g^{(j-1)}(z)}{z^{p-j+1}}\right\} &\leq \frac{p!}{(p-j+1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}\\ &< \frac{p!}{(p-j+1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{1}{(k-j+1)^2\{(\beta-\alpha)(k-j)+2\alpha\}}.\end{aligned}$$

The sharpness can be seen from (3.2).
Case II: if $\alpha=\beta=1$, we consider the extreme function $h(z)$ defined by (3.7). Applying similar technique of $g(z)$ for the function $h(z)$, we obtain

$$\begin{aligned}\mathrm{Re}\left\{\frac{h^{(j-1)}(z)}{z^{p-j+1}}\right\} &\geq \frac{p!}{(p-j+1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(-r)^{k-p}}{(k-j+1)^2}\\ &> \frac{p!}{(p-j+1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(-1)^{k-p}}{(k-j+1)^2}, \qquad (3.13)\end{aligned}$$

while

$$\begin{aligned}\mathrm{Re}\left\{\frac{h^{(j-1)}(z)}{z^{p-j+1}}\right\} &\leq \frac{p!}{(p-j+1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{r^{k-p}}{(k-j+1)^2}\\ &< \frac{p!}{(p-j+1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{1}{(k-j+1)^2}.\end{aligned}$$

The sharpness can be seen from (3.7). This completes the proof of Theorem 3.6. □

Theorem 3.7 *If $f(z)\in\mathcal{B}_p(\alpha,\beta,\lambda;j)$ and $|z|\leq r$, where $j=0,...p$, then*

$$\begin{aligned}\mathrm{Re}\left\{\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} &\geq \frac{p!}{(p-j-1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)(-r)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}\\ &> \frac{p!}{(p-j-1)!}+\sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)(k-j)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}} \qquad (3.14)\end{aligned}$$

and

$$\operatorname{Re}\left\{\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} \le \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)r^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}$$
$$< \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{k-j}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.$$

The inequalities are both sharp.

Proof We need only to consider the extreme point $g(z)$ defined by (3.2).
Case I: if $p+1-j+2\alpha/(\beta-\alpha)>0$, then from (2.7), we obtain

$$\frac{g^{(j+1)}(z)}{z^{p-j-1}} = \frac{p!}{(p-j-1)!} + \sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)(k-j)z^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}. \tag{3.15}$$
$$= \frac{p!}{(p-j-1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\frac{d}{du}\left(\int_0^1\int_0^1 (us)^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\frac{z}{1-stuz}\,ds\,dt\right)|_{u=1}.$$

Therefore,

$$\operatorname{Re}\left\{\frac{g^{(j+1)}(z)}{z^{p-j-1}}\right\} = \frac{p!}{(p-j-1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\frac{d}{du}\left(\int_0^1\int_0^1 (us)^{p+1-j}t^{\frac{2\alpha}{\beta-\alpha}+p-j}\operatorname{Re}\left\{\frac{z}{1-stuz}\right\}dsdt\right)|_{u=1}. \tag{3.16}$$

Applying (3.5) into (3.16) and expanding the integrand into a power series gives

$$\operatorname{Re}\left\{\frac{g^{(j+1)}(z)}{z^{p-j-1}}\right\} = \frac{p!}{(p-j-1)!} + \frac{4(\sigma-\lambda)}{\beta-\alpha}\frac{d}{du}\left(\int_0^1\int_0^1 \sum_{k=p+1}^{\infty}(us)^{k-j}t^{\frac{2\alpha}{\beta-\alpha}+k-j-1}(-r)^{k-p}dsdt\right)|_{u=1}. \tag{3.17}$$

Perform the indicated integration and differentiation yield

$$\operatorname{Re}\left\{\frac{g^{(j+1)}(z)}{z^{p-j-1}}\right\} \ge \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)(-r)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}$$
$$> \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.$$

On the other hand, using (3.5), we have

$$\operatorname{Re}\left\{\frac{g^{(j+1)}(z)}{z^{p-j-1}}\right\} \le \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(k-j)r^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}$$
$$< \frac{p!}{(p-j-1)!} + 4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{k-j}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.$$

The sharpness can be seen from (3.2).
Case II: if $\alpha=\beta=1$, we consider the extreme function $h(z)$ defined by (3.7). Applying similar technique of

$g(z)$ for the function $h(z)$, we obtain

$$\begin{aligned}\operatorname{Re}\left\{\frac{h^{(j+1)}(z)}{z^{p-j-1}}\right\} &\geq \frac{p!}{(p-j-1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(k-j)(-r)^{k-p}}{k-j+1}\\ &> \frac{p!}{(p-j-1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(k-j)(-1)^{k-p}}{k-j+1},\end{aligned}$$

while

$$\begin{aligned}\operatorname{Re}\left\{\frac{h^{(j+1)}(z)}{z^{p-j-1}}\right\} &\leq \frac{p!}{(p-j-1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\frac{\lambda}{\alpha}\right)\sum_{k=p+1}^{\infty}\frac{(k-j)r^{k-p}}{k-j+1}\\ &< \frac{p!}{(p-j-1)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\frac{\lambda}{\alpha}\right)\sum_{k=p+1}^{\infty}\frac{k-j}{k-j+1}.\end{aligned}$$

The sharpness can be seen from (3.7). This completes the proof of Theorem 3.7. □

For $j=\alpha=0$ and $p=\beta=1$ in Theorem 3.7, we have

Corollary 3.8 *If* $\operatorname{Re}\{f'(z)+(1/2)zf''(z)\}>\lambda$, $(0\leq\lambda<1)$, *then*

$$\operatorname{Re}\{f'(z)\}>1+2(1-\lambda)(1-2\log 2).$$

This result was obtained in [4], for the class $R(\lambda,1/2)$.

If $p=1$ and $j=0$ in Theorem 3.7, then

Corollary 3.9 $\mathcal{B}_1(\alpha,\beta,\lambda;0)\subset\mathcal{S}$ *for* $\lambda\geq\lambda_1$, *where*

$$\lambda_1=\alpha+\beta+\frac{1}{4}\left(\sum_{k=2}^{\infty}\frac{(-1)^{k-1}k}{(k+1)\{(\beta-\alpha)k+2\alpha\}}\right)^{-1}. \tag{3.18}$$

The result cannot be extended to $\lambda<\lambda_1$.

Proof Let $f\in\mathcal{B}_1(\alpha,\beta,\lambda;0)$. From (3.14), we know that if

$$1+4(\alpha+\beta-\lambda)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}k}{(k+1)\{(\beta-\alpha)k+2\alpha\}}\geq 0,$$

then $f\in\mathcal{S}$, that is, if $\lambda\geq\lambda_1$, we have $\mathcal{B}_1(\alpha,\beta,\lambda;0)\subset\mathcal{S}$. The result cannot be extended to $\lambda<\lambda_1$ because $g'(-1)=0$ at $\lambda=\lambda_1$. Thus, $g'(-r)=0$ for some $r=r(\lambda)<1$ when $\lambda<\lambda_1$. □

Example 3.10 *(i)* $\mathcal{B}_1(-1,1,\lambda;0)\subset\mathcal{S}$, *for* $\lambda\geq 2/(1-4\ln 2)\approx -1.1283$.
(ii) $\mathcal{B}_1(0,1,\lambda;0)\subset\mathcal{S}$, *for* $\lambda\geq(3-4\ln 2)/(2-4\ln 2)\approx -0.295$. *The bounds of* λ *in (i) and (ii) are the best possible ones for univalence.*

4. Inclusion properties

We need the following lemma by Jack [7] in investigating inclusion properties related to $\mathcal{B}_p(\alpha,\beta,\lambda;j)$ and $\mathcal{A}_p(\alpha,\beta,\lambda;j)$. An application of these inclusions will appear in the next section.

Lemma 4.1 *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. If there are $z_0\in\mathbb{U}$ such that*

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|,$$

then $z_0w'(z_0)=kw(z_0)$, $(k\geq 1)$.

Theorem 4.2 *If $f(z)\in\mathcal{A}_p(\alpha,\beta,\lambda;j+1)$, for $j=0,1,...,p-1$, then*

$$f(z)\in\mathcal{A}_p\left(\alpha-\beta,\beta,\frac{\lambda}{p-j}-\frac{1}{2};j\right),$$

for $\sigma-(p-j)/2-1/4\leq\lambda<\sigma$, where $\sigma=p![\alpha+\beta(p-j-1)]/(p-j-1)!$.

Proof For $f(z)\in\mathcal{A}_p(\alpha,\beta,\lambda;j+1)$, we define $w(z)$ by

$$(\alpha-\beta)\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}=\frac{w(z)}{1-w(z)}+\frac{\sigma}{p-j},\quad(w(z)\neq 1).\tag{4.1}$$

Then, we have

$$(\alpha-\beta)f^{(j)}(z)+\beta zf^{(j+1)}(z)=\frac{z^{p-j}w(z)}{1-w(z)}+\frac{\sigma}{p-j}z^{p-j}.\tag{4.2}$$

By differentiating both sides of (4.2), we obtain

$$\alpha f^{(j+1)}(z)+\beta zf^{(j+2)}(z)=\sigma z^{p-j-1}+\frac{(p-j)z^{p-j-1}w(z)}{1-w(z)}+\frac{z^{p-j}w'(z)}{(1-w(z))^2}.$$

It follows that

$$\alpha\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\beta\frac{f^{(j+2)}(z)}{z^{p-j-2}}=\sigma+\frac{(p-j)w(z)}{1-w(z)}+\frac{zw'(z)}{(1-w(z))^2}.$$

Therefore, $f(z)\in\mathcal{A}_p(\alpha,\beta,\lambda;j+1)$ satisfies

$$\operatorname{Re}\left\{\alpha\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\beta\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\}=\sigma+(p-j)\operatorname{Re}\left\{\frac{w(z)}{1-w(z)}\right\}+\operatorname{Re}\left\{\frac{zw'(z)}{(1-w(z))^2}\right\}>\lambda$$

for $z\in\mathbb{U}$. Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$, if there are $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1,$$

then, by Lemma 4.1, we can write

$$w(z_0)=e^{i\theta},\quad z_0w'(z_0)=kw(z_0)=ke^{i\theta},\quad(k\geq 1).$$

For such a point $z_0 \in \mathbb{U}$, we obtain

$$\begin{aligned}\mathrm{Re}\left\{\alpha\frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}}+\beta\frac{f^{(j+2)}(z_0)}{z_0^{p-j-2}}\right\} &= \sigma+(p-j)\mathrm{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\}+\mathrm{Re}\left\{\frac{ke^{i\theta}}{(1-e^{i\theta}))^2}\right\}\\ =\sigma-\frac{p-j}{2}-\frac{k}{2(1-\cos\theta)} &\leq \sigma-\frac{p-j}{2}-\frac{1}{2(1-\cos\theta)}\leq\sigma-\frac{p-j}{2}-\frac{1}{4}\leq\lambda\end{aligned}$$

which contradicts our assumption. Hence, there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$, which means that $|w(z)| < 1$ for all $z \in \mathbb{U}$. Note that

$$\mathrm{Re}\left\{\frac{w(z)}{1-w(z)}\right\} > -\frac{1}{2}, \quad (z \in \mathbb{U})$$

for $|w(z)| < 1$. Therefore, by (4.1), we obtain

$$\mathrm{Re}\left\{(\alpha-\beta)\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} > -\frac{1}{2}+\frac{\sigma}{p-j} > -\frac{1}{2}+\frac{\lambda}{p-j}$$

which shows that

$$f(z) \in \mathcal{A}_p\left(\alpha-\beta,\beta,\frac{\lambda}{p-j}-\frac{1}{2};j\right).$$

□

Remark 4.3 *Theorem 4.2 corrects some errors appeared in [11, Theorem 3.2].*

Setting $p = 1$ and $j = 0$ in Theorem 4.2, we get

Corollary 4.4 $\mathcal{A}_1(\alpha,\beta,\lambda;1) \subseteq \mathcal{A}_1(\alpha-\beta,\beta,\lambda-1/2;0)$, *for* $\alpha - 3/4 \leq \lambda < \alpha$ *and* $\alpha/\beta > -1$.

Choosing $\alpha = 0$ and $\beta = 1$ in the previous corollary yields

Example 4.5 *If* $f \in \mathcal{A}_1$ *satisfies* $\mathrm{Re}\{f''(z)\} > \lambda$, $(-3/4 \leq \lambda < 0)$, *then* $\mathrm{Re}\{-f(z)/z + f'(z)\} > \lambda - 1/2$.

Theorem 4.6 *If* $f(z) \in \mathcal{B}_p(\alpha,\beta,\lambda;j)$, *for* $j = 0,1,...,p$, *then*

$$f(z) \in \mathcal{A}_p\left(\alpha,\frac{\beta-\alpha}{2},\frac{\lambda}{p-j+1}-\frac{1}{2};j\right),$$

for $\sigma - (p-j+1)/2 - 1/4 \leq \lambda < \sigma$, *where* σ *is defined by (2.2).*

Proof For $f(z) \in \mathcal{B}_p(\alpha,\beta,\lambda;j)$, we define $w(z)$ by

$$\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+1)}(z)}{z^{p-j-1}}=\frac{w(z)}{1-w(z)}+\frac{\sigma}{p-j+1}, \quad (w(z)\neq 1). \tag{4.3}$$

Then, we have

$$\alpha z f^{(j)}(z)+\left(\frac{\beta-\alpha}{2}\right)z^2 f^{(j+1)}(z)=\frac{z^{p-j+1}w(z)}{1-w(z)}+\frac{\sigma}{p-j+1}z^{p-j+1}. \tag{4.4}$$

By differentiating both sides of (4.4), we obtain

$$\alpha f^{(j)}(z)+\beta z f^{(j+1)}(z)+\left(\frac{\beta-\alpha}{2}\right)z^2 f^{(j+2)}(z)=(p-j+1)z^{p-j}\left(\frac{\sigma}{p-j+1}+\frac{w(z)}{1-w(z)}\right)+\frac{z^{p-j+1}w'(z)}{(1-w(z))^2}.$$

It follows that

$$\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}=(p-j+1)\left(\frac{\sigma}{p-j+1}+\frac{w(z)}{1-w(z)}\right)+\frac{zw'(z)}{(1-w(z))^2}.$$

Therefore, $f(z)\in\mathcal{B}_p(\alpha,\beta,\lambda;j)$ satisfies

$$\mathrm{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\}=\sigma+(p-j+1)\mathrm{Re}\left\{\frac{w(z)}{1-w(z)}\right\}+\mathrm{Re}\left\{\frac{zw'(z)}{(1-w(z))^2}\right\}>\lambda$$

for $z\in\mathbb{U}$. Since $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$, if there are $z_0\in\mathbb{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1,$$

then, by Lemma 4.1, we can write

$$w(z_0)=e^{i\theta},\quad z_0w'(z_0)=kw(z_0)=ke^{i\theta},\quad (k\ge 1).$$

For such a point $z_0\in\mathbb{U}$, we obtain

$$\begin{aligned}\mathrm{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\} &= \sigma+(p-j+1)\mathrm{Re}\left\{\frac{e^{i\theta}}{1-e^{i\theta}}\right\}+\mathrm{Re}\left\{\frac{ke^{i\theta}}{(1-e^{i\theta})^2}\right\}\\ =\sigma-\frac{p-j+1}{2}-\frac{k}{2(1-\cos\theta)} &\le \sigma-\frac{p-j+1}{2}-\frac{1}{2(1-\cos\theta)}\le\sigma-\frac{p-j+1}{2}-\frac{1}{4}\le\lambda\end{aligned}$$

which contradicts our assumption. Hence, there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=1$, which means that $|w(z)|<1$ for all $z\in\mathbb{U}$. Note that

$$\mathrm{Re}\left\{\frac{w(z)}{1-w(z)}\right\}>-\frac{1}{2},\quad (z\in\mathbb{U})$$

for $|w(z)|<1$. Therefore, by (4.3), we obtain

$$\mathrm{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\}>-\frac{1}{2}+\frac{\sigma}{p-j+1}>-\frac{1}{2}+\frac{\lambda}{p-j+1}$$

which shows that

$$f(z)\in\mathcal{A}_p\left(\alpha,\frac{\beta-\alpha}{2},\frac{\lambda}{p-j+1}-\frac{1}{2};j\right).$$

□

Taking $p=1$ and $j=0$ in Theorem 4.6, we obtain that

Corollary 4.7 *If* $f(z) \in \mathcal{B}_1(\alpha, \beta, \lambda; 0)$, *then*

$$f(z) \in \mathcal{A}_1\left(\alpha, \frac{\beta-\alpha}{2}, \frac{\lambda-1}{2}; 0\right),$$

for $\alpha + \beta - 5/4 \leq \lambda < \alpha + \beta$.

Theorem 4.8 *If* $f(z) \in \mathcal{A}_p(\alpha, \beta, \lambda; j+1)$, *for* $j = 0, 1, ..., p-1$, *then*

$$f(z) \in \mathcal{B}_p\left(\alpha - \beta, \alpha + \beta, \frac{p-j+1}{p-j}\lambda - \frac{1}{2}; j\right),$$

where $\delta - (p-j)/2 - 1/4 \leq \lambda < \delta$ *and* $\delta = p!\{\alpha + (p-j-1)\beta\}/(p-j-1)!$.

Proof From Theorem 4.2, we have that if

$$\operatorname{Re}\left\{\alpha \frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta \frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\} > \lambda, \tag{4.5}$$

then

$$\operatorname{Re}\left\{(\alpha - \beta)\frac{f^{(j)}(z)}{z^{p-j-1}} + \beta \frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\} > \frac{\lambda}{p-j} - \frac{1}{2}. \tag{4.6}$$

Combining inequality (4.5) along with inequality (4.6), we obtain

$$\operatorname{Re}\left\{(\alpha - \beta)\frac{f^{(j)}(z)}{z^{p-j-1}} + (\alpha+\beta)\frac{f^{(j+1)}(z)}{z^{p-j-1}} + \beta \frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\} > \frac{p-j+1}{p-j}\lambda - \frac{1}{2}.$$

That is

$$f(z) \in \mathcal{B}_p\left(\alpha - \beta, \alpha + \beta, \frac{p-j+1}{p-j}\lambda - \frac{1}{2}; j\right).$$

□

Taking $p = 1$ and $j = 0$ in Theorem 4.8, we obtain that

Corollary 4.9 *If* $f(z) \in \mathcal{A}_1(\alpha, \beta, \lambda; 1)$, *then*

$$f(z) \in \mathcal{B}_1\left(\alpha - \beta, \alpha + \beta, 2\lambda - \frac{1}{2}; 0\right),$$

where $\alpha - 3/4 \leq \lambda < \alpha$.

5. Criteria for p-valent starlike functions in $\mathcal{B}_p(\alpha, \beta, \lambda; j)$

To investigate sufficient conditions for functions in $\mathcal{B}_p(\alpha, \beta, \lambda; j)$ to be in $\mathcal{S}_p^*(j)$, we need the following lemma.

Lemma 5.1 *[10] If* $f \in \mathcal{A}_p$ *satisfies*

$$\operatorname{Re}\left\{\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}\right\} > 0, \quad (j = 1, ..., p; \ z \in \mathbb{U}),$$

then $f \in \mathcal{S}_p^*(j-1)$.

Theorem 5.2 *$\mathcal{B}_p(\alpha,\beta,\lambda;j)\subset\mathcal{S}_p^*(j)$ for $j=0,1,...,p-1$ and $\lambda\geq\lambda^*$, where λ^* is the solution of the equation*

$$\frac{p!}{(p-j)!}\left(\frac{(\beta+3\alpha)(p-j)+4\alpha}{4}\right)=\lambda^*+(\sigma-\lambda^*)\sum_{k=p+1}^{\infty}\frac{[(\beta-\alpha)(p-j)-4\alpha-2(\beta+\alpha)(k-j)](-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}},\quad(5.1)$$

such that $\alpha\leq 0$ and $-\alpha\geq\beta>(p-1-j)\alpha/(p+1-j)$.

Proof For $f\in\mathcal{B}_p(\alpha,\beta,\lambda;j)$, we have

$$\mathrm{Re}\left\{\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\}>\lambda,\quad(z\in\mathbb{U}).$$

Obviously,

$$\begin{aligned}(\beta-\alpha)\left(\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right) &= -(\beta+\alpha)\frac{f^{(j+1)}(z)}{z^{p-j-1}}-2\alpha\frac{f^{(j)}(z)}{z^{p-j}}\\ &+ 2\left(\alpha\frac{f^{(j)}(z)}{z^{p-j}}+\beta\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\left(\frac{\beta-\alpha}{2}\right)\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right).\end{aligned}\quad(5.2)$$

Therefore, by (5.2), (3.1), and (3.14), we have

$$\begin{aligned}&(\beta-\alpha)\mathrm{Re}\left\{\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\frac{f^{(j+2)}(z)}{z^{p-j-2}}\right\}-2\lambda=-(\beta+\alpha)\mathrm{Re}\left\{\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\}-2\alpha\mathrm{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}\right\}\\ \geq\ & -(\beta+\alpha)\left(\frac{p!}{(p-j-1)!}+\sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)(k-j)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}\right)\\ -\ & 2\alpha\left(\frac{p!}{(p-j)!}+\sum_{k=p+1}^{\infty}\frac{4(\sigma-\lambda)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}\right)\\ =\ & [-(\beta+\alpha)(p-j)-2\alpha]\frac{p!}{(p-j)!}-4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{[(\beta+\alpha)(k-j)+2\alpha](-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}.\end{aligned}\quad(5.3)$$

Now, let

$$\frac{zf^{(j+1)}(z)}{f^{(j)}(z)}=(p-j)\frac{1+w(z)}{1-w(z)}.$$

Then

$$\frac{f^{(j+1)}(z)}{z^{p-j-1}}+\frac{f^{(j+2)}(z)}{z^{p-j-2}}=(p-j)\frac{f^{(j)}(z)}{z^{p-j}}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2(p-j)+\frac{2zw'(z)}{(1-w(z))^2}\right].$$

To deduce the result, we claim that $|w(z)|<1$. If not, then using Lemma 4.1, there exists $z_0\in\mathbb{U}$ with $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, such that $z_0w'(z_0)=kw(z_0)=ke^{i\theta}$, where $k\geq 1$ and $0<\theta<2\pi$. It follows that

$$\begin{aligned}\mathrm{Re}\left\{\frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}}+\frac{f^{(j+2)}(z_0)}{z_0^{p-j-2}}\right\} &= (p-j)\mathrm{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2(p-j)+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right]\right\}\\ &= (p-j)\mathrm{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}\left[\frac{-\cos^2(\theta/2)}{\sin^2(\theta/2)}(p-j)-\frac{k}{2\sin^2(\theta/2)}\right]\right\}\\ &= -\frac{p-j}{2\sin^2(\theta/2)}\mathrm{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}\left[2\cos^2(\theta/2)(p-j)+k\right]\right\}\\ &\le -\frac{(p-j)k}{2\sin^2(\theta/2)}\mathrm{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}\right\}\\ &\le -\frac{p-j}{2}\mathrm{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}\right\}.\end{aligned}$$

Hence, by (3.14), we obtain

$$\mathrm{Re}\left\{\frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}}+\frac{f^{(j+2)}(z_0)}{z_0^{p-j-2}}\right\}\le-\frac{1}{2}\left[\frac{p!}{(p-j-1)!}+4(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{(p-j)(-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}}\right]. \tag{5.4}$$

If λ satisfies the inequality

$$\frac{p!}{(p-j)!}\left(\frac{(\beta+3\alpha)(p-j)+4\alpha}{4}\right)\le\lambda+(\sigma-\lambda)\sum_{k=p+1}^{\infty}\frac{[(\beta-\alpha)(p-j)-4\alpha-2(\beta+\alpha)(k-j)](-1)^{k-p}}{(k-j+1)\{(\beta-\alpha)(k-j)+2\alpha\}} \tag{5.5}$$

then we have a contradiction to (5.4) at $z=z_0$. The smallest λ that satisfies inequality (5.5) is λ^*, which is the solution of (5.1). Thus, if $\lambda\ge\lambda^*$, we see $|w(z)|<1$ and hence by Lemma 5.1, we obtain $f\in\mathcal{S}_p^*(j)$. This completes the proof of Theorem 5.2. □

Setting $p=1,\ j=0,\ \alpha=\alpha^*-\beta^*$, and $\beta=\alpha^*+\beta^*$ in Theorem 5.2 would ensure by Corollary 4.9 that

Corollary 5.3 *$\mathcal{A}_1(\alpha^*,\beta^*,\lambda;1)\subset\mathcal{S}^*$, for $\alpha^*>\lambda\ge(2\lambda^*+1)/4$, such that λ^* is the solution of the equation*

$$\frac{4\alpha^*-3\beta^*}{2}=\lambda^*+(2\alpha^*-\lambda^*)\sum_{k=2}^{\infty}\frac{(-1)^{k-1}[3\beta^*-2\alpha^*(k+1)]}{(k+1)(\beta^*k+\alpha^*)}.$$

where $\beta^*>\alpha^*>-\beta^*$ and $\beta^*-\alpha^*\ge\alpha^*+\beta^*>\alpha^*-\beta^*$.

If ($\alpha^*=0,\beta^*=1$) or ($\alpha^*=-1,\beta^*=2$) in Corollary 5.3, then we have the following interesting example

Example 5.4 *$f\in\mathcal{S}^*$ if it satisfies one of the following conditions:*

(i) $\mathrm{Re}\{zf''(z)\}>\dfrac{5-12\ln 2}{44-48\ln 2}\approx-0.309,\quad(z\in\mathbb{U}),$

(ii) $\mathrm{Re}\{-f'(z)+2zf''(z)\}>\dfrac{-60+12\ln 2+9\pi}{64-16\ln 2-12\pi}\approx-1.539,\quad(z\in\mathbb{U}).$

Theorem 5.5 *$\mathcal{B}_p(1,1,\lambda;j)\subset\mathcal{S}_p^*(j-1)$, for $j=1,...,p$ and $\lambda\geq\lambda^*$, where λ^* is the solution of the equation*

$$\frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda^*\right)\sum_{k=p+1}^{\infty}\frac{(p-j+1)(-1)^{k-p}}{(k-j+1)^2}=-2\lambda^*. \tag{5.6}$$

Proof If $f\in\mathcal{B}_p(1,1,\lambda;j)$, then

$$\operatorname{Re}\left\{\frac{f^{(j)}(z)}{z^{p-j}}+\frac{f^{(j+1)}(z)}{z^{p-j-1}}\right\}>\lambda.$$

Now, let

$$\frac{zf^{(j)}(z)}{f^{(j-1)}(z)}=(p-j+1)\frac{1+w(z)}{1-w(z)},\quad w(z)\neq 1.$$

Then

$$\frac{f^{(j)}(z)}{z^{p-j}}+\frac{f^{(j+1)}(z)}{z^{p-j-1}}=(p-j+1)\frac{f^{(j-1)}(z)}{z^{p-j-1}}\left[\left(\frac{1+w(z)}{1-w(z)}\right)^2(p-j+1)+\frac{2zw'(z)}{(1-w(z))^2}\right].$$

To deduce the result, we claim that $|w(z)|<1$. If not, then using Lemma 4.1, there exists $z_0\in\mathbb{U}$ with $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$, such that $z_0w'(z_0)=kw(z_0)=ke^{i\theta}$, where $k\geq 1$ and $0<\theta<2\pi$. It follows that

$$\begin{aligned}
\operatorname{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}+\frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}}\right\} &= (p-j+1)\operatorname{Re}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}}\left[\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2(p-j+1)+\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right]\right\}\\
&= (p-j+1)\operatorname{Re}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}}\left[\frac{-\cos^2(\theta/2)}{\sin^2(\theta/2)}(p-j+1)-\frac{k}{2\sin^2(\theta/2)}\right]\right\}\\
&= -\frac{p-j+1}{2\sin^2(\theta/2)}\operatorname{Re}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}}\left[2\cos^2(\theta/2)(p-j+1)+k\right]\right\}\\
&\leq -\frac{(p-j+1)k}{2\sin^2(\theta/2)}\operatorname{Re}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}}\right\}\\
&\leq -\frac{p-j+1}{2}\operatorname{Re}\left\{\frac{f^{(j-1)}(z_0)}{z_0^{p-j+1}}\right\}.
\end{aligned}\tag{5.7}$$

Hence, by (5.7) and (3.13), we obtain

$$\operatorname{Re}\left\{\frac{f^{(j)}(z_0)}{z_0^{p-j}}+\frac{f^{(j+1)}(z_0)}{z_0^{p-j-1}}\right\}\leq-\frac{1}{2}\left[\frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(p-j+1)(-1)^{k-p}}{(k-j+1)^2}\right]. \tag{5.8}$$

If λ satisfies the inequality

$$\frac{p!}{(p-j)!}+2\left(\frac{p!(p-j+1)}{(p-j)!}-\lambda\right)\sum_{k=p+1}^{\infty}\frac{(p-j+1)(-1)^{k-p}}{(k-j+1)^2}\quad\geq\quad-2\lambda \tag{5.9}$$

then we have a contradiction to (5.8) at $z = z_0$. The smallest λ that satisfies inequality (5.9) is λ^*, which is the solution of (5.6). Thus, if $\lambda \geq \lambda^*$, we see $|w(z)| < 1$ and hence by Lemma 5.1, $f \in \mathcal{S}_p^*(j-1)$. □

For $p = j = 1$ in Theorem 5.5, we have

Corollary 5.6 *For $f \in \mathcal{A}_1$, if*

$$\mathrm{Re}\{f'(z) + zf''(z)\} > \frac{6-\pi^2}{24-\pi^2},$$

then $f \in \mathcal{S}^*$.

This result was obtained by many authors, see [10, 12]. By the aid of the Alexander's relation $(f(z) \in \mathcal{CV} \Longleftrightarrow zf'(z) \in \mathcal{S}^*)$, we have the following consequence result from Theorem 5.5 and Lemma 1.1.

Corollary 5.7 *$\mathcal{B}_p(j, j+2, \lambda; j) \subset \mathcal{CV}_p(j-1)$, for $j = 1, ..., p$ and $\lambda \geq \lambda^*$, where λ^* is defined by (5.6). In particular, if*

$$\mathrm{Re}\{f'(z) + 3zf''(z) + z^2 f'''(z)\} > \frac{6-\pi^2}{24-\pi^2},$$

then $f \in \mathcal{CV}$.

We end this paper by posing the following problem:

Problem 5.8 *Find the least possible value of $\lambda(\alpha, \beta)$ such that $\mathcal{B}_p(\alpha, \beta, \lambda; j) \subset \mathcal{S}_p^*(j)$.*

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