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Existence of solutions of BVPs for impulsive fractional Langevin equations involving Caputo fractional derivatives

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Abstract: The standard Caputo fractional derivative is generalized for the piecewise continuous functions. A more general boundary value problem for the impulsive Langevin fractional differential equation involving the Caputo fractional derivatives is studied. New existence results for solutions of concerned problems are established.

Key words: Impulsive fractional Langevin equation, boundary value problem, integral equation, Caputo fractional derivative

1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [9, 10, 15].

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [5]. For some new developments on the fractional Langevin equation in physics, see, for example, [1, 2, 4, 6, 11, 22]. Lizana et al. [11] studied a single-particle equation of motion starting with a microscopic description of a tracer particle in a one-dimensional many-particle system with a general two-body interaction potential and they have shown that the resulting dynamical equation belongs to the class of fractional Langevin equations using a harmonization technique. In [6], Gambo et al. discussed the Caputo modification of the Hadamard fractional derivative. Ahmad et al. [1, 3, 4] considered solutions of nonlinear Langevin equation involving two fractional orders. In [7, 14, 17–22, 24, 26, 28, 29], Tariboon et al. studied the existence and uniqueness of solutions of the nonlinear Langevin equation of Hadamard-Caputo-type fractional derivatives with nonlocal fractional integral conditions using a variety of fixed point theorems. Tariboon and Ntouyas [19] discussed the existence and uniqueness of solutions for Langevin impulsive q-difference equations with boundary conditions.

In recent years, some authors have studied solvability or existence and uniqueness of solutions of boundary value problems (BVPs for short) for impulsive Langevin fractional differential equations see [25, 27].

In [30], Zhao studied the existence and uniqueness of solutions to the impulsive boundary value problems

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(IBVP for short) for the following two classes of fractional differential equation with constant coefficients

$$\begin{cases} {}^cD^{\alpha}_{0,t}[{}^cD^{\beta}_{0,t}+\lambda]x(t)=f\left(t,x(t)\right), a.e., t\in J\setminus\{t_1,\cdots,t_m\}, \\ x(t_k^+)-x(t_k^-)=y_k, k\in\mathbb{N}_1^m, \\ ax(0)+bx(1)=c,\ {}^cD^{\beta}_{0,t}x(t_i)=d_i, i\in\mathbb{N}_0^m, \end{cases} \quad (1.1)$$

and

$$\begin{cases} {}^cD^{\alpha}_{0,t}[{}^cD^{\beta}_{0,t}+\lambda]x(t)=f\left(t,x(t)\right), a.e., t\in J\setminus\{t_1,\cdots,t_m\}, \\ x(t_k^+)-x(t_k^-)=y_k, k\in\mathbb{N}_1^m, \\ a{}^cD^{\beta}_{0,t}x(0)+b{}^cD^{\beta}_{0,t}x(t_m)=c,\ x(t_k)=d_k, \in\mathbb{N}_1^{m+1}, \end{cases} \quad (1.2)$$

where $J=[0,1]$, $0<\alpha,\beta<1$ with $\alpha+\beta<1$, $\lambda>0$, ${}^cD^{*}_{0,t}$ is the Caputo fractional derivative, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $f:J\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a>0,b,c,d_k\geq 0$ are constants, $\mathbb{N}_k^l=\{k,k+1,\cdots,l\}$ for the integers k and l.

In [23], the authors studied the existence results of solutions for the following impulsive fractional Langevin equations with two different fractional derivatives

$$\begin{cases} {}^cD^{\alpha}_{t}[{}^cD^{\beta}_{t}-\lambda]x(t)=f\left(t,x(t)\right), a.e., t\in J\setminus\{t_1,\cdots,t_m\}, \\ x(t_k^+)-x(t_k^-)=I_k, k\in\mathbb{N}_1^m, \\ x(0)=x(\eta_i)=x(1)=0, \eta_i\in(t_i,t_{i+1}), i\in\mathbb{N}_0^{m-1}, \end{cases} \quad (1.3)$$

where $0<\alpha,\beta<1$ with $\alpha+\beta<1$, ${}^cD^{\beta}_t$ is the Caputo fractional derivative, $J=[0,1]$, $0=t_0<\eta_0<t_1<\eta_1<t_2<\cdots<t_{m-1}<\eta_{m-1}<t_m<\eta_m=t_{m+1}=1$ $\lambda\in\mathbb{R}$, $f:J\times\mathbb{R}\to\mathbb{R}$ is a given function.

Motivated by [23, 30], in this paper, we consider the following more general boundary value problem for the impulsive Langevin fractional differential equation

$$\begin{cases} {}^cD^{\alpha}_{0+}[{}^cD^{\beta}_{0+}-\lambda]x(t)=P(t)f\left(t,x(t)\right), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^{m-1}, \\ \Delta x(t_i)=x(t_i^+)-x(t_i)=I(t_i,x(t_i)), i\in\mathbb{N}_1^{m-1}, \\ A_1x(0)-B_1{}^cD^{\beta}_{0+}x(0)=C_1,\ A_2x(1)+B_2{}^cD^{\beta}_{0+}x(1)=C_2, \\ x(\eta_i)=D_i,\ i\in\mathbb{N}_1^{m-1}, \end{cases} \quad (1.4)$$

where

(a) $\alpha,\beta\in(0,1)$, $\lambda\in\mathbb{R}$, ${}^cD^{*}_{0+}$ is the Caputo fractional derivative of order $*>0$ and with the starting point 0, see Definition 2.3,

for integers $a<b$ $k<l$, $\mathbb{N}_a^b=\{a,a+1,a+2,\cdots,b\}$, $A_i,B_i,C_i\in\mathbb{R}(i=1,2), D_i\in\mathbb{R}(i\in\mathbb{N}_0^{m-1})$ are constants, $0=t_0<t_1<t_2<\cdots<t_{m-1}<t_m=1$, $\eta_i\in(t_{i-1},t_i](i\in\mathbb{N}_1^{m-1})$ with $\eta_m<1$ are fixed points, m is a positive integer,

(b) $f:(0,1)\times\mathbb{R}\mapsto\mathbb{R}$ is a Carathéodory function, see Definition 2.4, $I:\{t_i:i\in\mathbb{N}_1^{m-1}\}\times\mathbb{R}\mapsto\mathbb{R}$ is a discrete Carathódory function, see Definition 2.,

(c) $P:(0,1)\to\mathbb{R}$ is continuous and there exists constant $\sigma>-1$ and

$$\tau\in(\max\{-\rho,-\varrho-\rho,-\rho-\sigma,-\varrho-\rho-\sigma\},0]$$

such that $|P(t)|\leq t^{\sigma}(1-t)^{\tau}$ for all $t\in(0,1)$.

A function $u:(0,1]\mapsto \mathbb{R}$ is called a solution of BVP(1.4) if

$$u|_{(t_i,t_{i+1}]}\in C^0(t_i,t_{i+1}], i\in \mathbb{N}_0^{m-1},\ \lim_{t\to t_i^+}u(t) \text{ are finite}, i\in \mathbb{N}_0^{m-1}$$

and all equations in (1.4) are satisfied.

The first purpose of this paper is to provide a method to convert boundary value problems for impulsive Langevin fractional differential equations involving two fractional derivatives to integral equations. Then we establish existence results for solutions of BVP(1.4) by using Schauder's fixed point theorem [12] under some suitable assumptions. It is noted that the lower point of the fractional differential equations involved is 0 which is different from those ones used in [13].

The remainder of the paper is organized as follows: In Section 2, the related definitions are introduced firstly. Then we seek continuous solutions of a class of linear Langevin fractional differential equations and we also seek piecewise continuous solutions of a class of linear Langevin fractional differential equations). In Section 3, the equivalent integral equations of BVP(1.4) are presented. Finally in Section 4, we establish sufficient conditions for the existence of solutions of BVP(1.4).

2. Preliminary results

In this section, we firstly present some necessary definitions from the fractional calculus theory which can be found in the literature [8, 15]. Then we get exact continuous solutions of a class of fractional Landevin equations. Thirdly, we get exact piecewise continuous solutions of a class of impulsive fractional Langevin equations.

Denote by $L^1(a,b)$ the set of all integrable functions on (a,b), $C^0(a,b]$ the set of all continuous functions on $(a,b]$. For $\varphi\in L^1(a,b)$, denote by $||\varphi||_1=\int_a^b|\varphi(s)|ds$. For $\varphi\in C^0[a,b]$, denote by $||\varphi||_0=\max_{t\in[a,b]}|\varphi(t)|$. Let the Gamma and beta functions $\Gamma(\alpha)$, $\mathbf{B}(p,q)$, and the Mitag-Leffler function $E_{\alpha,\delta}(x)$ be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\ \ \mathbf{B}(p,q)=\int_0^1x^{p-1}(1-x)^{q-1}dx,\ \alpha>0,p>0,q>0,$$

$$\mathbf{E}_{\alpha,\delta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\delta)},\ \ \mathbf{E}_{\alpha}(x)=\mathbf{E}_{\alpha,1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+1)},\alpha>0,\delta>0.$$

Definition 2.1 (page 69 in [8]) *Let $-\infty<a<b<+\infty$. The Riemann–Liouville fractional integrals $I_{a^+}^{\alpha}g$ and $I_{b^-}^{\alpha}g$ of order $\alpha\in\mathbf{C}(\mathbf{R}(\alpha)>0)$ are defined by*

$$I_{a^+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}g(s)ds,t>a,$$

$$I_{b^-}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}g(s)ds,t<b,$$

respectively. These integrals are called the left side and the right side fractional integrals.

Definition 2.2 (page 70 in [8]) *Let $-\infty<a<b<+\infty$. The Riemann–Liouville fractional derivatives $D_{a^+}^{\alpha}g$ and $D_{b^-}^{\alpha}g$ of order $\alpha\in\mathbf{C}(\mathbf{R}(\alpha)\geq 0)$ are defined by*

$$D_{a^+}^{\alpha}g(t)=\left(\frac{d}{dt}\right)^nI_{a^+}^{n-\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{g(s)}{(t-s)^{\alpha-n+1}}ds,t>a,$$

$$D_{b^-}^{\alpha}g(t)=\left(-\frac{d}{dt}\right)^nI_{b^-}^{n-\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b\frac{g(s)}{(s-t)^{\alpha-n+1}}ds,t<b,$$

where $n = [\mathbf{R}(\alpha)] + 1$*. In particular, when* $\alpha = n \in \mathbb{N}$*, then* $D^0_{a+}g(t) = D^0_{b-}g(t) = g(t)$ *and* $D^n_{a+}g(t) = g^{(n)}(t)$, $D^n_{b-}g(t) = (-1)^n g^{(n)}(t)$*, where* $g^{(n)}(t)$ *is the usual derivative of* $g(t)$ *of order* n*.*

Definition 2.3 (page 91 in [8]) *Let* $-\infty < a < b < +\infty$*. The Caputo fractional derivatives* ${}^cD^\alpha_{a+}g$ *and* ${}^cD^\alpha_{b-}g$ *of order* $\alpha \in \mathbf{C}(\mathbf{R}(\alpha) \geq 0)$ *are defined via the Riemann–Liouville fractional derivatives by*

$$ {}^cD^\alpha_{a+}g(t) = D^\alpha_{a+}\left[g(t) - \sum_{j=0}^{n-1}\frac{g^{(j)}(a)}{j!}(t-a)^j\right], t > a, $$

$$ {}^cD^\alpha_{b-}g(t) = D^\alpha_{b-}\left[g(t) - \sum_{j=0}^{n-1}\frac{g^{(j)}(b)}{j!}(b-t)^j\right], t < b, $$

respectively, where $n = [\mathbf{R}(\alpha)] + 1$ *for* $\alpha \notin \mathbb{N}$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}$*. These derivatives are called left side and right side Caputo fractional derivatives of order* α*.*

For a piecewise function $g : \cup(t_i, t_{i+1}] \to \mathbb{R}$ with $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, we give the following definition:

Definition 2.4 *The Caputo fractional derivative* ${}^cD^\alpha_{0+}g$ *of order* $\alpha \in \mathbf{C}(\mathbf{R}(\alpha) \geq 0)$ *are defined via the Riemann–Liouville fractional derivatives by*

$$ {}^cD^\alpha_{0+}g(t) = D^\alpha_{0+}g(t) - \sum_{\sigma=1}^{i}\sum_{\mu=0}^{n-1}\frac{\Delta g^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha} - \sum_{\mu=0}^{n-1}\frac{g^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, t \in (t_i, t_{i+1}], i \in \mathbb{N}^m_0, $$

where $n = [\mathbf{R}(\alpha)] + 1$ *for* $\alpha \notin \mathbb{N}$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}$*. This derivative is called left side Caputo fractional derivative of order* α*.*

Remark 2.5 *If* $x \in AC^n(t_i, t_{i+1}](i \in \mathbb{N}^m_0)$*, we have*

$$ {}^cD^\alpha_{0+}x(t) = \frac{\int_0^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds}{\Gamma(n-\alpha)} = \frac{\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha-1}x^{(n)}(s)ds + \int_{t_i}^t (t-s)^{n-\alpha-1}x^{(n)}(s)ds}{\Gamma(n-\alpha)} $$

$$ = \frac{\left[\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha}dx^{(n-1)}(s) + \int_{t_i}^t (t-s)^{n-\alpha}dx^{(n-1)}(s)\right]'_t}{\Gamma(n-\alpha+1)} $$

$$ = \frac{\left[\sum_{\sigma=0}^{i-1}\left((t-s)^{n-\alpha}x^{(n-1)}(s)\Big|_{t_\sigma}^{t_{\sigma+1}} + (n-\alpha)\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha-1}x^{(n-1)}(s)ds\right)\right]'_t}{\Gamma(n-\alpha+1)} $$

$$ + \frac{\left[(t-s)^{n-\alpha}x^{(n-1)}(s)\Big|_{t_i}^t + (n-\alpha)\int_{t_i}^t (t-s)^{n-\alpha-1}x^{(n-1)}(s)ds\right]'_t}{\Gamma(n-\alpha+1)} $$

$$ = \frac{\sum_{\sigma=0}^{i-1}\left((t-t_{\sigma+1})^{n-\alpha-1}x^{(n-1)}(t^-_{\sigma+1}) - (t-t_\sigma)^{n-\alpha-1}x^{(n-1)}(t^+_\sigma)\right) - (t-t_i)^{n-\alpha-1}x^{(n-1)}(t^+_i)}{\Gamma(n-\alpha)} $$

$$ + \frac{\left[\sum_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha-1}x^{(n-1)}(s)ds + \int_{t_i}^t (t-s)^{n-\alpha-1}x^{(n-1)}(s)ds\right]'}{\Gamma(n-\alpha)} $$

$$= \frac{\sum\limits_{\sigma=0}^{i-1}\left((t-t_{\sigma+1})^{n-\alpha-1}x^{(n-1)}(t_{\sigma+1}^-)-(t-t_\sigma)^{n-\alpha-1}x^{(n-1)}(t_\sigma^+)\right)-(t-t_i)^{n-\alpha-1}x^{(n-1)}(t_i^+)}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum\limits_{\sigma=0}^{i-1}\int_{t_\sigma}^{t_{\sigma+1}}(t-s)^{n-\alpha}dx^{(n-2)}(s)+\int_{t_i}^{t}(t-s)^{n-\alpha}dx^{(n-2)}(s)\right]''}{\Gamma(n-\alpha+1)}=\cdots\cdots$$

$$= D_{0^+}^\alpha x(t)-\frac{\sum\limits_{\sigma=1}^{i}(t-t_\sigma)^{n-\alpha-1}\Delta x^{(n-1)}(t_\sigma)}{\Gamma(n-\alpha)}-\frac{\sum\limits_{\sigma=1}^{i}(t-t_\sigma)^{n-\alpha-2}\Delta x^{(n-2)}(t_\sigma)}{\Gamma(n-\alpha-1)}-\cdots-\frac{\sum\limits_{\sigma=1}^{i}(t-t_\sigma)^{-\alpha}\Delta x(t_\sigma)}{\Gamma(1-\alpha)}$$

$$-\frac{t^{n-\alpha-1}x^{(n-1)}(0)}{\Gamma(n-\alpha)}-\frac{t^{n-\alpha-2}x^{(n-2)}(0)}{\Gamma(n-\alpha-1)}-\cdots\cdots-\frac{t^{-\alpha}x(0)}{\Gamma(1-\alpha)}$$

$$= D_{0^+}^\alpha x(t)-\sum_{\sigma=1}^{i}\sum_{\mu=0}^{n-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha}-\sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.$$

Definition 2.6 $h:(0,1)\times\mathbb{R}\mapsto\mathbb{R}$ *is called a Carathéodory function if*

(i) $t\mapsto h(t,x)$ *is integrable function on* $(0,1)$ *for every* $x\in\mathbb{R}$*,*

(ii) $x\mapsto h(t,x)$ *is continuous on* $\mathbb{R}$ *for each* $t\in(t_i,t_{i+1}](i\in\mathbb{N}_0^m)$*,*

(iii) *for each* $r>0$*, there exists* $M_r>0$ *such that* $|x|\le r$ *implies that*

$$|h(t,x)|\le M_r, t\in(t_i,t_{i+1}), i\in\mathbb{N}_0^m.$$

Definition 2.7 $I:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}\mapsto\mathbb{R}$ *is a discrete Carathéodory function if*

(i) $x\mapsto I(t_i,x)$ *is continuous on* $\mathbb{R}$ *for each* $i\in\mathbb{N}_1^m$*,*

(ii) *for each* $r>0$*, there exists* $M_{I,r}>0$ *such that* $|x|\le r$ *implies that*

$$|I(t_i,x)|\le M_{I,r}, i\in\mathbb{N}_1^m.$$

Definition 2.8 *Banach space: Let* n *be a positive integer,* $\alpha\in(n-1,n)$*,* $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$*. Denote*

$$PC_0(0,1]=\left\{x:(0,1]\mapsto\mathbb{R}:x|_{(t_k,t_{k+1}]}\in C^0(t_k,t_{k+1}],\ \lim_{t\to t_k^+}x(t)\ \text{are finite}, i\in\mathbb{N}_0^m\right\}.$$

Let us define

$$||x||=\max\left\{\sup_{t\in(t_k,t_{k+1}]}|x(t)|:k\in\mathbb{N}_0^m\right\}, x\in PC_0(0,1].$$

Then $PC_0(0,1]$ *is a Banach space.*

Now, we seek continuous solutions of linear Langevin fractional differential equations (LFDEs for short) with the Caputo fractional derivatives and the Riemann–Liouville fractional derivatives, respectively.

Let n,l be positive integers, $\lambda\in\mathbb{R}$, $\rho\in(n-1,n)$ and $\varrho\in(l-1,l)$. Consider

$$^cD_{0^+}^\rho[^cD_{0^+}^\varrho-\lambda]x(t)=P(t), a.e., t\in[0,1], \tag{2.1}$$

where $P:(0,1)\to\mathbb{R}$ is continuous and there exists constants $\sigma>-1$ and

$$\tau\in(\max\{-\rho+n-1,-\varrho-\rho+n-1,-\rho-\sigma+n-1,-\varrho-\rho-\sigma+n-1\},0]$$

such that $|P(t)|\le t^\sigma(1-t)^\tau$ for all $t\in(0,1)$.

Lemma 2.9 *x is a solution of (2.1) if and only if there exist constants $x_i, y_i \in \mathbb{R}(i \in \mathbb{N}_0^{l-1}, j \in \mathbb{N}_0^{n-1})$ such that*

$$x(t) = \sum_{i=0}^{l-1} x_i t^i \mathbf{E}_{\varrho,i+1}(\lambda t^\varrho) + \sum_{j=0}^{n-1} \Gamma(j+1) y_j t^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda t^\varrho)$$
$$+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du, t \in (0,1]. \tag{2.2}$$

Proof The proof follows from [16, 23] in Section 3 by using the Laplace transform [8] and is omitted. □

Now, we seek piecewise continuous solutions of linear impulsive Langevin fractional differential equations (ILFDEs for short) with the Caputo fractional derivatives.

Let n, l be positive integers, $\lambda \in \mathbb{R}$, $\rho \in (n-1, n)$, and $\varrho \in (l-1, l)$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$. Consider the piecewise continuous solution of the following equation

$${}^cD_{0+}^\rho [{}^cD_{0+}^\varrho - \lambda] x(t) = P(t), a.e., t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m. \tag{2.3}$$

where $P : (0,1) \to \mathbb{R}$ is continuous and there exists

$$\sigma > -1, \tau \in (\max\{-\rho + n - 1, -\varrho - \rho + n - 1, -\rho - \sigma + n - 1, -\varrho - \rho - \sigma + n - 1\}, 0]$$

such that $|P(t)| \leq t^\sigma (1-t)^\tau$ for all $t \in (0,1)$.

Lemma 2.10 *x is a piecewise continuous solution of (2.3) if and only if there exist $c_{\nu i}, d_{\nu j} \in \mathbb{R}(i \in \mathbb{N}_0^{l-1}, j \in \mathbb{N}_0^{n-1}$ such that*

$$x(t) = \sum_{\nu=0}^{k} \sum_{i=0}^{l-1} c_{\nu i} (t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda (t-t_\nu)^\varrho)$$
$$+ \sum_{\nu=0}^{k} \sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j} (t-t_\nu)^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda (t-t_\nu)^\varrho) \tag{2.4}$$
$$+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m.$$

Proof The proof is very long since the careful computation is needed. We divide it into the following two steps.

Step 1. Note that the starting point of the derivatives is 0 similar to [13]. We prove that x satisfies (2.4) if x is a piecewise continuous solution of (2.3).

By Lemma 2.1, we know that there exists $c_{0i}, d_{0j} \in \mathbb{R}(i \in \mathbb{N}_0^{l-1}, j \in \mathbb{N}_0^{n-1})$ such that

$$x(t) = \sum_{i=0}^{l-1} c_{0i} t^i \mathbf{E}_{\varrho,i+1}(\lambda t^\varrho) + \sum_{j=0}^{n-1} \Gamma(j+1) d_{0j} t^{\varrho+j} \mathbf{E}_{\varrho,\varrho+j+1}(\lambda t^\varrho)$$
$$+ \int_0^t (t-u)^{\varrho+\rho-1} \mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho) P(u) du, t \in (t_0, t_1].$$

We note that

$$d_{0j} = \frac{[{}^cD_{0+}^\varrho x - \lambda x]^{(j)}(0)}{j!} = \frac{[{}^cD_{0+}^\varrho x]^{(j)}(0)}{j!} - \lambda \frac{x^{(j)}(0)}{j!}, j \in \mathbb{N}_0^{n-1},$$
$$c_{0i} = x^{(i)}(0), \quad i \in \mathbb{N}_0^{l-1}. \tag{2.5}$$

Hence, (2.4) holds for $k=0$. Now suppose that (2.4) holds for $k=0,1,\cdots,\omega$, i.e. there exist constants $c_{\nu i}, d_{\nu j} \in \mathbb{R}(i \in \mathbb{N}_1^{l-1}, j \in \mathbb{N}_1^{n-1}, \nu \in \mathbb{N}_0^{\omega})$

$$\begin{aligned} x(t) &= \sum_{\nu=0}^{k}\sum_{i=0}^{l-1} c_{\nu i}(t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \sum_{\nu=0}^{k}\sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j}(t-t_\nu)^{\varrho+j}\mathbf{E}_{\varrho,\varrho+j+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \int_0^t (t-u)^{\varrho+\rho-1}\mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho)P(u)du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^{\omega}. \end{aligned} \tag{2.6}$$

We also note that

$$\begin{aligned} c_{\nu i} &= \Delta x^{(i)}(t_\nu), i \in \mathbb{N}_1^{l-1}, \nu \in \mathbb{N}_0^{\omega} \\ d_{\nu j} &= \frac{\Delta[{}^cD_{0^+}^{\varrho}x - \lambda x]^{(j)}(t_\nu)}{j!}, j \in \mathbb{N}_1^{n-1}, \nu \in \mathbb{N}_0^{\omega}. \end{aligned} \tag{2.7}$$

We will prove that (2.4) holds for $k=\omega+1$. Then by mathematical induction method, (2.4) holds for all $k \in \mathbb{N}_0^m$. Then this step is completed.

In order to get the exact expression of x on $(t_{\omega+1}, t_{\omega+2}]$, we suppose that there exists Φ such that

$$\begin{aligned} x(t) &= \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}(t-t_\nu)^i \mathbf{E}_{\varrho,i+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1) d_{\nu j}(t-t_\nu)^{\varrho+j}\mathbf{E}_{\varrho,\varrho+j+1}(\lambda(t-t_\nu)^\varrho) \\ &+ \int_0^t (t-u)^{\varrho+\rho-1}\mathbf{E}_{\varrho,\varrho+\rho}(\lambda(t-u)^\varrho)P(u)du + \Phi(t), t \in (t_{\omega+1}, t_{\omega+2}]. \end{aligned} \tag{2.8}$$

Using Definition 2.3′, we know for $t \in (t_{\omega+1}, t_{\omega+2}]$ by direct computation that

$${}^cD_{0^+}^{\rho}x(t) = D_{0^+}^{\rho}x(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1} \frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho} - \sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}.$$

Using Definition 2.2, (2.6), and (2.8), we get by direct computation that

$$\begin{aligned} D_{0^+}^{\rho}x(t) &= \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_\nu)^{\chi\varrho-\rho+i} \\ &+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_\nu)^{\chi\varrho-\rho+\varrho+j} \\ &+ \sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t (t-u)^{\chi\varrho+\varrho-1}P(u)du + D_{t_{\omega+1}^+}^{\rho}\Phi(t). \end{aligned}$$

It follows for $t \in (t_{\omega+1}, t_{\omega+2}]$ that

$$\begin{aligned} {}^cD^{\rho}_{0^+}x(t) &= \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_{\nu})^{\chi\varrho-\rho+i} \\ &+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1)d_{\nu j} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_{\nu})^{\chi\varrho-\rho+\varrho+j} \\ &+ \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)} \int_0^t (t-u)^{\chi\varrho+\varrho-1}P(u)du + D^{\rho}_{t^+_{\omega+1}}\Phi(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1} \frac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\rho+1)}(t-t_{\sigma})^{\mu-\rho} \\ &- \sum_{\mu=0}^{n-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\rho+1)} t^{\mu-\rho}, t \in (t_{\omega+1}, t_{\omega+2}]. \end{aligned} \tag{2.9}$$

Similarly we get for $t \in (t_{\omega+1}, t_{\omega+2}]$ that

$${}^cD^{\varrho}_{0^+}x(t) = D^{\varrho}_{0^+}x(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1} \frac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\varrho+1)}(t-t_{\sigma})^{\mu-\varrho} - \sum_{\mu=0}^{l-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)} t^{\mu-\varrho}.$$

Using Definition 2.2, (2.6), and (2.8), we get by direct computation that

$$\begin{aligned} D^{\varrho}_{0^+}x(t) &= \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_{\nu})^{\chi\varrho+i-\varrho} \\ &+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1)d_{\nu j} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j+1)}(t-t_{\nu})^{\chi\varrho+j} \\ &+ \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)}(t-u)^{\chi\varrho+\rho-1}P(u)du + D^{\varrho}_{t^+_{\omega+1}}\Phi(t). \end{aligned}$$

It follows that

$$\begin{aligned} {}^cD^{\varrho}_{0^+}x(t) &= \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_{\nu})^{\chi\varrho+i-\varrho} \\ &+ \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1} \Gamma(j+1)d_{\nu j} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+j+1)}(t-t_{\nu})^{\chi\varrho+j} + \int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}}{\Gamma(\chi\varrho+\varrho)}(t-u)^{\chi\varrho+\rho-1}P(u)du \\ &+ D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1} \frac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\varrho+1)}(t-t_{\sigma})^{\mu-\varrho} - \sum_{\mu=0}^{l-1} \frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)} t^{\mu-\varrho}, t \in (t_{\omega+1}, t_{\omega+2}]. \end{aligned} \tag{2.10}$$

Similarly for $t \in (t_\tau, t_{\tau+1}](\tau \in \mathbb{N}_0^\omega)$, we have

$$\begin{aligned}
&{}^cD_{0^+}^\varrho x(t) = \sum_{\nu=0}^{\tau}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}\\
&+\sum_{\nu=0}^{\tau}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}\\
&+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t (t-u)^{\chi\varrho+\rho-1}P(u)du\\
&-\sum_{\sigma=1}^{\tau}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho+1)}(t-t_\sigma)^{\mu-\varrho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)}t^{\mu-\varrho}, t\in(t_\tau,t_{\tau+1}](\tau\in\mathbb{N}_0^\omega).
\end{aligned} \tag{2.11}$$

On the other hand, we have for $t \in (t_{\omega+1}, t_{\omega+2}]$ that

$${}^cD_{0^+}^\rho\, {}^cD_{0^+}^\varrho x(t) = D_{0^+}^\rho\, {}^cD_{0^+}^\varrho x(t) - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD_{0^+}^\varrho x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho}-\sum_{\mu=0}^{n-1}\frac{[{}^cD_{0^+}^\varrho x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}.$$

Using (2.10), (2.11), and direct computation, we get that

$$\begin{aligned}
&D_{0^+}^\rho\, {}^cD_{0^+}^\varrho x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho}\\
&+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho}+P(t)+\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho)}\int_0^t(t-u)^{\chi\varrho-1}P(u)du\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho}+D_{t_{\omega+1}^+}^\rho D_{t_{\omega+1}^+}^\varrho\Phi(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
&{}^cD_{0^+}^\rho\, {}^cD_{0^+}^\varrho x(t) = \sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1} c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho}\\
&+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho}\\
&+P(t)+\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho)}\int_0^t(t-u)^{\chi\varrho-1}P(u)du+D_{t_{\omega+1}^+}^\rho D_{t_{\omega+1}^+}^\varrho\Phi(t)\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho}\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD_{0^+}^\varrho x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho}-\sum_{\mu=0}^{n-1}\frac{[{}^cD_{0^+}^\varrho x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}, t\in(t_{\omega+1},t_{\omega+2}].
\end{aligned} \tag{2.12}$$

Then for $t \in (t_{\omega+1}, t_{\omega+2}]$, from (2.9) and (2.12), we get

$$\begin{aligned}
&{}^cD^{\rho}_{0^+}\,{}^cD^{\varrho}_{0^+}x(t) - \lambda{}^cD^{\rho}_{0^+}x(t) = D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t) + P(t)\\
&+\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\tfrac{\lambda^{\chi}}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_{\nu})^{\chi\varrho+i-\varrho-\rho} + \sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\tfrac{\lambda^{\chi}}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_{\nu})^{\chi\varrho+j-\rho}\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\tfrac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\sigma})^{\mu-\varrho-\rho} - \sum_{\mu=0}^{l-1}\tfrac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho}\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x]^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\rho+1)}(t-t_{\sigma})^{\mu-\rho} - \sum_{\mu=0}^{n-1}\tfrac{[{}^cD^{\varrho}_{0^+}x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}\\
&-\lambda\Bigg[\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\tfrac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_{\nu})^{\chi\varrho-\rho+i}\\
&+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\tfrac{\lambda^{\chi}}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_{\nu})^{\chi\varrho-\rho+\varrho+j}\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\tfrac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\rho+1)}(t-t_{\sigma})^{\mu-\rho} - \sum_{\mu=0}^{n-1}\tfrac{x^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho}\Bigg].
\end{aligned}$$

By using (2.7), (substituting $c_{\nu i}, d_{\nu j}$), we get

$$\begin{aligned}
&{}^cD^{\rho}_{0^+}\,{}^cD^{\varrho}_{0^+}x(t) - \lambda{}^cD^{\rho}_{0^+}x(t) = D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t) + P(t)\\
&+\sum_{\nu=1}^{\omega}\sum_{i=0}^{l-1}\tfrac{\Delta x^{(i)}(t_{\nu})}{\Gamma(i-\varrho-\rho+1)}(t-t_{\nu})^{i-\varrho-\rho}\\
&+\sum_{\nu=1}^{\omega}\sum_{j=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(j)}(t_{\nu})}{\Gamma(j-\rho+1)}(t-t_{\nu})^{j-\rho} - \sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{l-1}\tfrac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\sigma})^{\mu-\varrho-\rho}\\
&-\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x]^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\rho+1)}(t-t_{\sigma})^{\mu-\rho} + \lambda\sum_{\sigma=1}^{\omega+1}\sum_{\mu=0}^{n-1}\tfrac{\Delta x^{(\mu)}(t_{\sigma})}{\Gamma(\mu-\rho+1)}(t-t_{\sigma})^{\mu-\rho}\\
&= D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t) + P(t) - \sum_{\mu=0}^{l-1}\tfrac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}\\
&-\sum_{\mu=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}.
\end{aligned}$$

So

$$\begin{aligned}
P(t) &= D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t) + P(t)\\
&-\sum_{\mu=0}^{l-1}\tfrac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}\\
&-\sum_{\mu=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}, t \in (t_{\omega+1}, t_{\omega+2}].
\end{aligned}$$

It follows that

$$\begin{aligned}
&D^{\rho}_{t^+_{\omega+1}}D^{\varrho}_{t^+_{\omega+1}}\Phi(t) - \lambda D^{\rho}_{t^+_{\omega+1}}\Phi(t) - \sum_{\mu=0}^{l-1}\tfrac{\Delta x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}\\
&-\sum_{\mu=0}^{n-1}\tfrac{\Delta[{}^cD^{\varrho}_{0^+}x-\lambda x]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho} = 0,\ t \in (t_{\omega+1}, t_{\omega+2}].
\end{aligned} \tag{2.13}$$

From (2.8), we have $\Delta x^{(\mu)}(t_{\omega+1})=\Phi^{(\mu)}(t_{\omega+1})$ and $\Delta[{}^cD_{0^+}^{\varrho}x-\lambda x]^{(\mu)}(t_{\omega+1})=[{}^cD_{t_{\omega+1}^+}^{\varrho}\Phi-\lambda\Phi]^{(\mu)}(t_{\omega+1})$. Then (2.13) becomes

$$\begin{aligned}&D_{t_{\omega+1}^+}^{\rho}D_{t_{\omega+1}^+}^{\varrho}\Phi(t)-\lambda D_{t_{\omega+1}^+}^{\rho}\Phi(t)-\sum_{\mu=0}^{l-1}\frac{\Phi^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}\\&-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD_{t_{\omega+1}^+}^{\varrho}\Phi-\lambda\Phi\right]^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}=0,\ t\in(t_{\omega+1},t_{\omega+2}].\end{aligned}\tag{2.14}$$

One sees from Definition 2.3 that

$$\begin{aligned}&{}^cD_{t_{\omega+1}^+}^{\rho}[{}^cD_{t_{\omega+1}^+}^{\varrho}x-\lambda x](t)=D_{t_{\omega+1}^+}^{\rho}[{}^cD_{t_{\omega+1}^+}^{\varrho}x-\lambda x](t)-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD_{t_{\omega+1}^+}^{\varrho}x-\lambda x\right]^{(\mu)}}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}\\&=D_{t_{\omega+1}^+}^{\rho}D_{t_{\omega+1}^+}^{\varrho}x-\lambda D_{t_{\omega+1}^+}^{\rho}x(t)-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(t_{\omega+1})}{\Gamma(\mu-\varrho-\rho+1)}(t-t_{\omega+1})^{\mu-\varrho-\rho}\\&-\sum_{\mu=0}^{n-1}\frac{\left[{}^cD_{t_{\omega+1}^+}^{\varrho}x-\lambda x\right]^{(\mu)}}{\Gamma(\mu-\rho+1)}(t-t_{\omega+1})^{\mu-\rho}.\end{aligned}$$

It follows (2.14) that

$${}^cD_{t_{\omega+1}^+}^{\rho}\,{}^cD_{t_{\omega+1}^+}^{\varrho}\Phi(t)-\lambda{}^cD_{t_{\omega+1}^+}^{\rho}\Phi(t)=0,\ t\in(t_{\omega+1},t_{\omega+2}].$$

It follows from Lemma 2.1 (with the starting point being replaced by $t_{\omega+1}$ and $P(t)$ being replaced by 0) that there exist constants $c_{\omega+1i},d_{\omega+1j}\in\mathbb{R}(i\in\mathbb{N}_0^{n-1},j\in\mathbb{N}_0^{l-1})$ such that

$$\begin{aligned}\Phi(t)&=\sum_{\mu=0}^{l-1}c_{\omega+1,i}(t-t_{\omega+1})^{\mu}\mathbf{E}_{\varrho,\mu+1}(\lambda(t-t_{\omega+1})^{\varrho})\\&+\sum_{\mu=0}^{n-1}\Gamma(j+1)d_{\omega+1j}(t-t_{\omega+1})^{\varrho+\mu}\mathbf{E}_{\varrho,\mu+\varrho}(\lambda(t-t_{\omega+1})^{\varrho}),t\in(t_{\omega+1},t_{\omega+2}].\end{aligned}$$

Substituting Φ into (2.8), we know that (2.4) holds for $k=\omega+1$. By mathematical induction method, we know that (2.4) holds for $k\in\mathbb{N}_0^m$.

Note that the starting point of the derivatives is 0 similar to [13]. We prove that x is a piecewise continuous solution of (2.3) if x satisfies (2.4).

Since x satisfies (2.4), by using Definition 2.3′ and direct computation similar to the proof of (2.9) in

Step 1, we get for $t\in(t_\omega,t_{\omega+1}](\omega\in\mathbb{N}_0^m)$ that

$$\begin{aligned}{}^cD_{0^+}^{\rho}x(t)&=D_{0^+}^{\rho}x(t)-\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{n-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha}-\sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}\\
&=\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+i+1)}(t-t_\nu)^{\chi\varrho-\rho+i}\\
&+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho-\rho+\varrho+j+1)}(t-t_\nu)^{\chi\varrho-\rho+\varrho+j}\\
&+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t(t-u)^{\chi\varrho+\varrho-1}P(u)du\\
&-\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{n-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\alpha+1)}(t-t_\sigma)^{\mu-\alpha}-\sum_{\mu=0}^{n-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha},\end{aligned}\tag{2.15}$$

and similar to the proof of (2.11), we get for $t\in(t_\tau,t_{\tau+1}](\tau\in\mathbb{N}_0^m)$ that

$$\begin{aligned}{}^cD_{0^+}^{\varrho}x(t)&=\sum_{\nu=0}^{\tau}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho}\\
&+\sum_{\nu=0}^{\tau}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j+1)}(t-t_\nu)^{\chi\varrho+j}\\
&+\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+\varrho)}\int_0^t(t-u)^{\chi\varrho+\rho-1}P(u)du\\
&-\sum_{\sigma=1}^{\tau}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho+1)}(t-t_\sigma)^{\mu-\varrho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho+1)}t^{\mu-\varrho},t\in(t_\tau,t_{\tau+1}],\tau\in\mathbb{N}_0^m,\end{aligned}$$

Then similar to the proof of (2.12), for $t\in(t_\omega,t_{\omega+1}]$ we have that

$$\begin{aligned}{}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t)&=\sum_{\nu=0}^{\omega}\sum_{i=0}^{l-1}c_{\nu i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+i-\varrho-\rho+1)}(t-t_\nu)^{\chi\varrho+i-\varrho-\rho}\\
&+\sum_{\nu=0}^{\omega}\sum_{j=0}^{n-1}\Gamma(j+1)d_{\nu j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho+j-\rho+1)}(t-t_\nu)^{\chi\varrho+j-\rho}\\
&+P(t)+\sum_{\chi=1}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\varrho)}\int_0^t(t-u)^{\chi\varrho-1}P(u)du\\
&-\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{l-1}\frac{\Delta x^{(\mu)}(t_\sigma)}{\Gamma(\mu-\varrho-\rho+1)}(t-t_\sigma)^{\mu-\varrho-\rho}-\sum_{\mu=0}^{l-1}\frac{x^{(\mu)}(0)}{\Gamma(\mu-\varrho-\rho+1)}t^{\mu-\varrho-\rho}\\
&-\sum_{\sigma=1}^{\omega}\sum_{\mu=0}^{n-1}\frac{\Delta[{}^cD_{0^+}^{\varrho}x]^{(\mu)}(t_\sigma)}{\Gamma(\mu-\rho+1)}(t-t_\sigma)^{\mu-\rho}-\sum_{\mu=0}^{n-1}\frac{[{}^cD_{0^+}^{\varrho}x]^{(\mu)}(0)}{\Gamma(\mu-\rho+1)}t^{\mu-\rho},t\in(t_{\omega+1},t_{\omega+2}].\end{aligned}\tag{2.16}$$

Then for $t\in(t_\omega,t_{\omega+1}]$ we get by using (2.15), (2.16), and direct computation that

$${}^cD_{0^+}^{\rho}\,{}^cD_{0^+}^{\varrho}x(t)-\lambda{}^cD_{0^+}^{\rho}x(t)=P(t),t\in(t_\omega,t_{\omega+1}],\omega\in\mathbb{N}_0^m.$$

Hence, x is a piecewise continuous solution of (2.3). The proof is completed. □

3. Equivalent integral equations of BVP(1.4)

In this section, we present equivalent integral equations of BVP (1.4) by using Lemma 2.2. For ease of expression, denote

$$(Ff)(t) = \int_0^t (t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du,$$

$$\Theta = B_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta) + [A_1-\lambda B_1]\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta),$$

$$\Xi = A_2(1-t_{m-1})^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_{m-1})^\beta) + B_2\mathbf{E}_{\beta,1}(\lambda(1-t_{m-1})^\beta),$$

$$\Phi = \Xi\prod_{k=2}^{m-1}\left[(\eta_k-t_{k-1})^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_{k-1})^\beta)\right].$$

Then by direct computation, we get

$${}^cD_{0+}^\beta(Ff)(t) = \int_0^t (t-u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du.$$

Denote

$$M_{\nu,k} = (\eta_k-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k-t_\nu)^\beta),\ k\in\mathbb{N}_2^{m-1},\ \nu\in\mathbb{N}_1^{k-1},$$

$$M_{\nu,m} = A_2(1-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_\nu)^\beta) + B_2\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta),\ \nu\in\mathbb{N}_1^{m-1},$$

$$\begin{aligned}M_k = {}& D_k - \tfrac{\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)[\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1]}{\Theta}\\ &-\tfrac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]\eta_k^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta} - \sum_{\nu=1}^{k-1}\mathbf{E}_{\beta,1}(\lambda(\eta_k-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\ &+\tfrac{[A_1-\lambda B_1]\eta_k^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1) + \tfrac{B_1\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1) - (Ff)(\eta_k), k\in\mathbb{N}_2^{m-1},\end{aligned}$$

$$\begin{aligned}M_m = {}& C_2 - \tfrac{[\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1][A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta} - \tfrac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\\ &-[A_2+\lambda B_2]\sum_{\nu=1}^{m-1}\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\ &+\left[\tfrac{B_1[A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta} + \tfrac{[A_1-\lambda B_1][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\right](Ff)(\eta_1)\\ &-A_2(Ff)(1) - B_2D_{0+}^\beta(Ff)(1).\end{aligned}$$

Let $d_\nu(\nu\in\mathbb{N}_1^{m-1})$ satisfy the following iterative equations:

$$M_{1,2}d_1 = M_2,\quad M_{1,3}d_1 + M_{2,3}d_2 = M_3,\quad M_{1,4}d_1 + M_{2,4}d_2 + M_{3,4}d_3 = M_4,$$

$$\cdots\cdots\cdots$$

$$M_{1,m}d_1 + M_{2,m}d_2 + \cdots + M_{m-1,m}d_{m-1} = M_m.$$

Suppose that (a)–(c) hold and $\Theta \neq 0, \Xi \neq 0$. Then BVP(1.4) is equivalent to the following integral equation

$$\begin{aligned} x(t) &= \tfrac{\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1}{\Theta}\mathbf{E}_{\beta,1}(\lambda t^\beta)+\tfrac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta} \\ &-\left[\tfrac{B_1\mathbf{E}_{\beta,1}(\lambda t^\beta)}{\Theta}+\tfrac{[A_1-\lambda B_1]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\right](Ff)(\eta_1) \\ &+\sum_{\nu=1}^{k}\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)I(t_\nu,x(t_\nu))+\sum_{\nu=1}^{k}d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta) \\ &+(Ff)(t), t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^{m-1}. \end{aligned} \tag{3.1}$$

Proof Suppose that x is a solution of BVP(1.4). From Lemma 2.2 (choose $l=n=1$, $\rho=\alpha, \varrho=\beta$, replacing $P(t)$ by $P(t)f(t,x(t))$), there exist $c_\nu, d_\nu \in \mathbb{R}(\nu\in\mathbb{N}_0^{m-1})$ such that

$$\begin{aligned} x(t) &= \sum_{\nu=0}^{k}c_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)+\sum_{\nu=0}^{k}d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta) \\ &+\int_0^t(t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du, t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^{m-1}. \end{aligned} \tag{3.2}$$

By Direct computation, for $t\in(t_i,t_{i+1}]$ we can get that

$$\begin{aligned} {}^cD_{0^+}^\beta x(t) &= \sum_{\nu=0}^{i}c_\nu\sum_{\chi=1}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi\beta+1-\beta)}(t-t_\nu)^{\chi\beta-\beta}+\sum_{\nu=0}^{i}d_\nu\sum_{\chi=0}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi\beta+1)}(t-t_\nu)^{\chi\beta} \\ &+\begin{cases} \int_0^t\sum_{\chi=0}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi\beta+\alpha)}(t-u)^{\chi\beta+\alpha-1}P(u)f(u,x(u))du, \alpha+\beta<1, \\ \tfrac{\int_0^t(t-s)^{-\beta}p(s)f(s,x(s))ds}{\Gamma(1-\beta)}+\int_0^t\sum_{\chi=1}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi\beta+1-\beta)}(t-u)^{\chi\beta-\beta}P(u)f(u,x(u))du, \alpha+\beta=1, \\ \int_0^t\sum_{\chi=0}^{\infty}\tfrac{\lambda^\chi}{\Gamma(\chi\beta+\alpha)}(t-u)^{\chi\beta+\alpha-1}P(u)f(u,x(u))du, \alpha+\beta>1. \end{cases} \end{aligned}$$

It follows that

$$\begin{aligned} {}^cD_{0^+}^\beta x(t) &= \lambda\sum_{\nu=0}^{i}c_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)+\sum_{\nu=0}^{i}d_\nu\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta) \\ &+\int_0^t(t-u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(t-u)^\beta)P(u)f(u,x(u))du. \end{aligned} \tag{3.3}$$

By $\Delta x(t_k)=I(t_k,x(t_k))$ and (3.2), we get

$$c_k=I(t_k,x(t_k)), k\in\mathbb{N}_1^{m-1}. \tag{3.4}$$

By $x(\eta_i) = D_i,\ i \in \mathbb{N}_1^{m-1}$ and (3.2), using (3.4), we get

$$\begin{aligned} & c_0\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta) + \sum_{\nu=0}^{k-1} d_\nu(\eta_k - t_\nu)^\beta \mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k - t_\nu)^\beta) \\ & = D_k - \sum_{\nu=1}^{k-1} I(t_\nu, x(t_\nu))\mathbf{E}_{\beta,1}(\lambda(\eta_k - t_\nu)^\beta) \\ & - \int_0^{\eta_k} (\eta_k - u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(\eta_k - u)^\beta)P(u)f(u, x(u))du, k \in \mathbb{N}_1^{m-1}. \end{aligned} \tag{3.5(k)}$$

By $A_1x(0) - B_1D_{0+}^\beta x(0) = C_1$ and $A_2x(1) + B_2D_{0+}^\beta x(1) = C_2$ and (3.2), (3.3), using (3.4), we get

$$[A_1 - \lambda B_1]c_0 - B_1d_0 = C_1, \tag{3.6}$$

and

$$\begin{aligned} & [A_2 + \lambda B_2]\mathbf{E}_{\beta,1}(\lambda)c_0 + \sum_{\nu=0}^{m-1} \left[A_2(1 - t_\nu)^\beta \mathbf{E}_{\beta,\beta+1}(\lambda(1 - t_\nu)^\beta) + B_2\mathbf{E}_{\beta,1}(\lambda(1 - t_\nu)^\beta)\right] d_\nu \\ & = C_2 - [A_2 + \lambda B_2] \sum_{\nu=1}^{m-1} \mathbf{E}_{\beta,1}(\lambda(1 - t_\nu)^\beta)I(t_\nu, x(t_\nu)) \\ & -A_2 \int_0^1 (1 - u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(1 - u)^\beta)P(u)f(u, x(u))du \\ & -B_2 \int_0^1 (1 - u)^{\alpha-1}\mathbf{E}_{\beta,\alpha}(\lambda(1 - u)^\beta)P(u)f(u, x(u))du. \end{aligned} \tag{3.7}$$

Now, we seek solutions $c_0, d_i(i \in \mathbb{N}_0^{m-1})$ from (3.5(k)) ($k \in \mathbb{N}_1^{m-1}$), (3.6), and (3.7). We remember $\Theta = B_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta) + [A_1 - \lambda B_1]\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)$. By (3.5(1)) and (3.6), using $\Theta \neq 0$, we get

$$\begin{aligned} c_0 &= \frac{\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1 + B_1D_1}{\Theta} - \frac{B_1}{\Theta}(Ff)(\eta_1), \\ d_0 &= \frac{[A_1 - \lambda B_1]D_1 - C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)}{\Theta} - \frac{A_1 - \lambda B_1}{\Theta}(Ff)(\eta_1). \end{aligned}$$

Then (3.5) becomes

$$\begin{aligned} & \sum_{\nu=1}^{k-1} (\eta_k - t_\nu)^\beta \mathbf{E}_{\beta,\beta+1}(\lambda(\eta_k - t_\nu)^\beta)d_\nu = D_k - \frac{\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)[\eta_1^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1 + B_1D_1]}{\Theta} \\ & - \frac{[(A_1 - \lambda B_1)D_1 - C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]\eta_k^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta} - \sum_{\nu=1}^{k-1} \mathbf{E}_{\beta,1}(\lambda(\eta_k - t_\nu)^\beta)I(t_\nu, x(t_\nu)) \\ & + \frac{[A_1 - \lambda B_1]\eta_k^\beta \mathbf{E}_{\beta,\beta+1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1) + \frac{B_1\mathbf{E}_{\beta,1}(\lambda\eta_k^\beta)}{\Theta}(Ff)(\eta_1) - (Ff)(\eta_k), k \in \mathbb{N}_2^{m-1}. \end{aligned} \tag{3.8(k)}$$

On the other hand, (3.7) becomes

$$\begin{aligned}
&\sum_{\nu=1}^{m-1}\left[A_2(1-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(1-t_\nu)^\beta)+B_2\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)\right]d_\nu\\
&=C_2-\tfrac{[\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1][A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta}\\
&-\tfrac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\\
&-[A_2+\lambda B_2]\sum_{\nu=1}^{m-1}\mathbf{E}_{\beta,1}(\lambda(1-t_\nu)^\beta)I(t_\nu,x(t_\nu))\\
&+\left[\tfrac{B_1[A_2+\lambda B_2]\mathbf{E}_{\beta,1}(\lambda)}{\Theta}+\tfrac{[A_1-\lambda B_1][A_2\mathbf{E}_{\beta,\beta+1}(\lambda)+B_2\mathbf{E}_{\beta,1}(\lambda)]}{\Theta}\right](Ff)(\eta_1)\\
&-A_2(Ff)(1)-B_2D_{0^+}^\beta(Ff)(1).
\end{aligned}\tag{3.9}$$

Since $\Xi\neq 0$, we can get unique solution $(d_1,d_2,\cdots,d_{m-1}$ from (3.8)(k) and (3.9). Substituting $c_i,d_i(i\in\mathbb{N}_0^{m-1})$ into (3.2), we get (3.1).

$$\begin{aligned}
x(t)&=\tfrac{\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1}{\Theta}\mathbf{E}_{\beta,1}(\lambda t^\beta)+\tfrac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\\
&-\left[\tfrac{B_1\mathbf{E}_{\beta,1}(\lambda t^\beta)}{\Theta}+\tfrac{[A_1-\lambda B_1]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\right](Ff)(\eta_1)\\
&+\sum_{\nu=1}^{k}\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)I(t_\nu,x(t_\nu))+\sum_{\nu=1}^{k}d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta)\\
&+(Ff)(t),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^{m-1}.
\end{aligned}\tag{3.10}$$

On the other hand, if x satisfies (3.1), we can prove that x is a solution of BVP(1.4). The proof is completed. □

4. Solvability of BVP(1.4)

In this section, we establish existence results for solutions of BVP(1.4). We list the following assumptions:

there exist nondecreasing functions $\varphi_f,\varphi_I:[0,\infty)\to[0,\infty)$ such that

$$|f(t,x)|\le\varphi_f(|x|),t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m,x\in\mathbb{R},$$

$$|I(t_i,x)|\le\varphi_{I1}(|x|),i\in\mathbb{N}_1^m,x\in\mathbb{R}.$$

there exist constants $M_f,M_I\ge 0$ such that

$$|f(t,x)|\le M_f,t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m,x\in\mathbb{R},$$

$$|I(t_i,x)|\le M_{I1},i\in\mathbb{N}_1^m,x\in\mathbb{R}.$$

Let us denote

$$Q_0 = \frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}$$

$$+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$

$$\Bigg[|C_2|+\sum_{k=2}^{m-1}|D_k|+\frac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+\frac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}$$

$$+\frac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Bigg],$$

$$Q_f = (m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$

$$\Bigg[\frac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}$$

$$+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}+\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+|A_2|+1\Bigg]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)$$

$$+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)$$

$$+\left[\frac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}+\frac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}\right]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)$$

$$+\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|),$$

$$Q_I = (m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\left[\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]\times$$

$$[m\mathbf{E}_{\beta,1}(|\lambda|)+m(|A_2|+|\lambda||B_2|)\mathbf{E}_{\beta,1}(|\lambda|)]+m\mathbf{E}_{\beta,1}(|\lambda|).$$

Suppose that (a)–(c), (H1) hold, $\Theta\neq 0, \Xi\neq 0$. Then BVP(1.4) has at least one solution if there exists $r_0>0$ such that

$$Q_0+Q_f\varphi_f(r_0)+Q_I(r_0)\leq r_0. \tag{4.1}$$

Proof Suppose that $M_{\nu k}, M_\nu(k\in\mathbb{N}_1^{m-1}, \nu\in\mathbb{N}_1^{k-1})$ are defined in Section 3. Define the operator T on $PC_0[0,1]$ for $x\in PC_0[0,1]$ by

$$\begin{aligned}(Tx)(t) &= \frac{\eta_1^\beta\mathbf{E}_{\beta,\beta+1}(\lambda\eta_1^\beta)C_1+B_1D_1}{\Theta}\mathbf{E}_{\beta,1}(\lambda t^\beta)+\frac{[(A_1-\lambda B_1)D_1-C_1\mathbf{E}_{\beta,1}(\lambda\eta_1^\beta)]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\\ &\quad-\left[\frac{B_1\mathbf{E}_{\beta,1}(\lambda t^\beta)}{\Theta}+\frac{[A_1-\lambda B_1]t^\beta\mathbf{E}_{\beta,\beta+1}(\lambda t^\beta)}{\Theta}\right](Ff)(\eta_1)\\ &\quad+\sum_{\nu=1}^k\mathbf{E}_{\beta,1}(\lambda(t-t_\nu)^\beta)I(t_\nu,x(t_\nu))+\sum_{\nu=1}^k d_\nu(t-t_\nu)^\beta\mathbf{E}_{\beta,\beta+1}(\lambda(t-t_\nu)^\beta)\\ &\quad+(Ff)(t), t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^{m-1},\end{aligned} \tag{4.2}$$

where $d_i(i\in\mathbb{N}_1^{m-1})$ satisfy the following iterative equations:

$$\sum_{\nu=1}^{k-1}M_{\nu,k}d_\nu=M_k, k\in\mathbb{N}_2^{m-1},\ \ \sum_{\nu=1}^{m-1}M_{\nu,m}d_\nu=M_m. \tag{4.3}$$

By a standard method, we can prove that $T : PC_0[0,1] \to PC_0[0,1]$ is well defined and x is a solution of BVP(1.5) if and only if x is a fixed point of T in $PC_0[0,1]$ by Theorem 3.1. One sees that (4.3) is transformed to

$$\begin{pmatrix} d_1 \\ d_2 \\ \cdots \\ d_{m-1} \end{pmatrix} = \begin{pmatrix} M_{1,2} & 0 & 0 & \cdots & 0 \\ M_{1,3} & M_{2,3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{1,m} & M_{2,m} & M_{3,m} & \cdots & M_{m-1,m} \end{pmatrix}^{-1} \begin{pmatrix} M_2 \\ M_3 \\ \cdots \\ M_m \end{pmatrix}.$$

One sees from the definition of $M_{\nu,k}$ that

$$|M_{\nu,k}| \le \mathbf{E}_{\beta,\beta+1}(|\lambda|),\ k \in \mathbb{N}_2^{m-1},\ \nu \in \mathbb{N}_1^{k-1},$$

$$|M_{\nu,m}| \le |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|),\ \nu \in \mathbb{N}_1^{m-1}.$$

Then

$$|M_{\nu,k}| \le \mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|),\ \ k \in \mathbb{N}_2^{m}, \nu \in \mathbb{N}_1^{k-1}.$$

Denote

$$\begin{pmatrix} M_{1,2} & 0 & 0 & \cdots & 0 \\ M_{1,3} & M_{2,3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{1,m} & M_{2,m} & M_{3,m} & \cdots & M_{m-1,m} \end{pmatrix} =: \begin{pmatrix} N_{1,1} & N_{1,2} & N_{1,3} & \cdots & N_{1,m-1} \\ N_{2,1} & M_{2,2} & N_{2,3} & \cdots & N_{2,m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ N_{m-1,1} & N_{m-1,2} & N_{m-1,3} & \cdots & N_{m-1,m-1} \end{pmatrix}.$$

Then the algebraic complement $N_{i,j}^*$ of $N_{i,j}$ satisfies

$$\begin{aligned} |N_{i,j}^*| &\le (m-1)! \max\left\{|N_{i,j}| : i,j \in \mathbb{N}_1^{m-1}\right\} \\ &\le (m-1)! \left[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)\right]. \end{aligned} \tag{4.4}$$

Let $\Omega_0 = \{x \in PC_0[0,1] : ||x|| \le r_0\}$. For $x \in \Omega_0$, we get by (H1) that

$$|f(t,x(t))| \le \varphi_f(||x||) \le \varphi_f(r_0), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^{m-1},$$

$$|I(t_i, x(t_i))| \le \varphi_I(||x||) \le \varphi_I(r_0), i \in \mathbb{N}_1^{m-1}.$$

Then for $t \in (t_i, t_{i+1}]$, we get

$$\begin{aligned} |(Ff)(t)| &\le \textstyle\int_0^t (t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(\lambda(t-u)^\beta)|P(u)||f(u,x(u))|du \\ &\le \textstyle\int_0^t (t-u)^{\beta+\alpha-1}\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)u^\sigma(1-s)^\tau\varphi_f(r_0)du \\ &\le \textstyle\int_0^t (t-u)^{\beta+\alpha+\tau-1}\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)u^\sigma du\varphi_f(r_0) \\ &\le \mathbf{B}(\beta+\alpha+\tau, \sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0). \end{aligned}$$

Furthermore,

$$|D_{0^+}^{\beta}(Ff)(t)| \leq \mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\varphi_f(r_0).$$

Then for $k \in \mathbb{N}_2^{m-1}$, we have

$$\begin{aligned}
&|M_k| \leq |D_k| + \tfrac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}\\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + m\mathbf{E}_{\beta,1}(|\lambda|)\varphi_I(r_0)\\
&+\tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}|(Ff)(\eta_1)| + \tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}|(Ff)(\eta_1)| + |(Ff)(\eta_k)|, k \in \mathbb{N}_2^{m-1},\\
&|M_m| \leq |C_2| + \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}\\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\\
&+m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)\varphi_I(r_0)\\
&+\left[\tfrac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\right]|(Ff)(\eta_1)|\\
&+|A_2||(Ff)(1)| + |B_2||D_{0^+}^{\beta}(Ff)(1)|.
\end{aligned}$$

It follows for all $k \in \mathbb{N}_2^m$ that

$$\begin{aligned}
&|M_k| \leq |C_2| + \sum_{k=2}^{m-1}|D_k| + \tfrac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|}\\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|}\\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\\
&+[m\mathbf{E}_{\beta,1}(|\lambda|) + m[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)]\,\varphi_I(r_0)\\
&+\Big[\Big(\tfrac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Big)\\
&+\tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + |A_2| + 1\Big]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0)\\
&+|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\varphi_f(r_0).
\end{aligned} \tag{4.5}$$

Hence, (4.4) and (4.5) imply that

$$\begin{aligned}
&|d_\nu| = \left|\sum_{\tau=1}^{m-1} N^*_{\tau,\nu} M_{\tau+1}\right| \le \sum_{\tau=1}^{m-1} |N^*_{\tau,\nu}||M_{\tau+1}| \\
&\le m!\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times \\
&\Bigg[|C_2| + \sum_{k=2}^{m-1}|D_k| + \tfrac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|} \\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} \\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|} \\
&+[m\mathbf{E}_{\beta,1}(|\lambda|) + m[|A_2| + |\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)]\,\varphi_I(r_0) \\
&+\Bigg[\left(\tfrac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\right) \\
&+\tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + |A_2| + 1\Bigg]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0) \\
&+|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\varphi_f(r_0)\Bigg].
\end{aligned}$$

Thus,

$$\begin{aligned}
&|(Tx)(t)| \le \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} \\
&+\left[\tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}\right]|(Ff)(\eta_1)| \\
&+\sum_{\nu=1}^{k}\mathbf{E}_{\beta,1}(|\lambda|)|I(t_\nu,x(t_\nu))| + \sum_{\nu=1}^{k}|d_\nu|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |(Ff)(t)| \\
&\le \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} \\
&+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times \\
&\Bigg[|C_2| + \sum_{k=2}^{m-1}|D_k| + \tfrac{\mathbf{E}_{\beta,1}(|\lambda|)[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|]}{|\Theta|} \\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{[\mathbf{E}_{\beta,\beta+1}(|\lambda|)|C_1|+|B_1||D_1|][|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} \\
&+\tfrac{[(|A_1|+|\lambda||B_1|)|D_1|+|C_1|\mathbf{E}_{\beta,1}(|\lambda|)][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\Bigg] \\
&+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times \\
&[m\mathbf{E}_{\beta,1}(|\lambda|) + m[|A_2| + |\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)]\,\varphi_I(r_0) \\
&+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times \\
&\Bigg[\left(\tfrac{|B_1|[|A_2|+|\lambda||B_2|]\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|][|A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|)+|B_2|\mathbf{E}_{\beta,1}(|\lambda|)]}{|\Theta|}\right) \\
&+\tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|} + \tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + |A_2| + 1\Bigg]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0) \\
&+(m+1)!\mathbf{E}_{\beta,\beta+1}(|\lambda|)\,[\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |A_2|\mathbf{E}_{\beta,\beta+1}(|\lambda|) + |B_2\mathbf{E}_{\beta,1}(|\lambda|)] \times \\
&|B_2|\mathbf{B}(\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\alpha}(|\lambda|)\varphi_f(r_0) \\
&+\left[\tfrac{|B_1|\mathbf{E}_{\beta,1}(|\lambda|)}{|\Theta|} + \tfrac{[|A_1|+|\lambda||B_1|]\mathbf{E}_{\beta,\beta+1}(|\lambda|)}{|\Theta|}\right]\mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0) \\
&+m\mathbf{E}_{\beta,1}(|\lambda|)\varphi_I(r_0) + \mathbf{B}(\beta+\alpha+\tau,\sigma+1)\mathbf{E}_{\beta,\beta+\alpha}(|\lambda|)\varphi_f(r_0) \\
&= Q_0 + Q_f\varphi_f(r_0) + Q_I(r_0) \le r_0.
\end{aligned}$$

Hence, $T\Omega_0 \subset \Omega_0$. Then Schauder's fixed point theorem implies that T has at least one solution in Ω_0, which is a solution of BVP(1.4). The proof is completed. □

Corollary 4.1 *Suppose that (a)–(c), (H2) hold. Then BVP(1.4) has at least one solution.*

Proof Choose $\varphi_f(x) = M_f$ and $\varphi_I(x) = M_I$. It is easy to see that (4.1) has positive solution. By Theorem 4.1, we get this result. □

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References

[1] Ahmad B, Eloe PW. A nonlocal boundary value problem for a nonlinear fractional differential equation with two indices. Communications on Applied Nonlinear Analysis 2010; 17: 69-80.

[2] Ahmad B, Nieto JJ. Solvability of nonlinear Langevin equation involving two fractional orders with Dirichlet boundary conditions. International Journal of Differential Equations 2010; 2010.

[3] Ahmad B, Nieto JJ, Alsaedi A. A nonlocal three-point inclusion problem of Langevin equation with two different fractional orders. Advance in Difference Equations 2012; 54.

[4] Ahmad B, Nieto JJ, Alsaedi A, El-Shahed M. A study of nonlinear Langevin equation involving two fractional orders in different intervals. Nonlinear Analysis Real World Applications 2012; 13: 599-606.

[5] Coffey W, Kalmykov Y, Waldron J. The Langevin Equation, 2nd ed. Singapore: World Scientific, 2004.

[6] Gambo Y, Jarad F, Baleanu D, Abdeljawad T. On Caputo modification of the Hadamard fractional derivative. Advance in Difference Equations 2014; 10.

[7] Gao Z, Yu X, Wang J. Nonlocal problems for Langevin-type differential equations with two fractional-order derivatives. Boundary Value Problems 2016; 52.

[8] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations, in North-Holland Mathematics Studies, vol: 204, Amsterdam, the Netherlands: Elsevier Science B. V., 2006.

[9] Kilbas AA, Marichev OI, Samko SG. Fractional Integral and Derivatives (Theory and Applications). Switzerland: Gordon and Breach, 1993.

[10] Kilbas AA, Trujillo JJ. Differential equations of fractional order: methods resultsd and problems-I. Applicable Analysis 2010; 78(1): 153-192.

[11] Lizana L, Ambjornsson T, Taloni A, Barkai E, Lomholt MA. Foundation of fractional Langevin equation: harmonization of a many-body problem. Physical Review E 2010; 81.

[12] Mawhin J. Toplogical degree methods in nonlinear boundary value problems. In: NSFCBMS Regional Conference Series in Math; Providence, RI, USA: American Math Soc, 1979.

[13] Nickolai K. Initial value problems of fractional order with fractional impulsive conditions. Results in Mathematics, 2013; 63: 1289-1310.

[14] Ntouyas SK, Tariboon J. Fractional integral problems for Hadamard-Caputo fractional Langevin differential inclusions. Journal of Applied Mathematical Computations 2016; 51(1-2): 11-22.

[15] Podlubny I. Fractional Differential Equations. Mathmatics in Science and Engineering. Vol. 198. San Diego, CA, USA: Academic Press, 1999.

[16] Prabhakar TR. A singular integral equation with generalized Mittag-Leffler function in the kernel. Yokohama Mathematical Journal 1971; 19(7): 7-15.

[17] Sudsutad W, Ntouyas SK, Tariboon J. Systems of fractional Langevin equations of Riemann–Liouville and Hadamard types. Advance in Difference Equations 2015; 235.

[18] Sudsutad W, Ntouyas SK, Tariboon J. Systems of fractional Langevin equations of Riemann–Liouville and Hadamard types. Advance in Difference Equations 2015; 1: 1-24.

[19] Tariboon J, Ntouyas SK. Nonlinear second-order impulsive q-difference Langevin equation with boundary conditions. Boundary Value Problems 2015; 85.

[20] Sudsutad W, Tariboon J. Nonlinear fractional integro-differential Langevin equation involving two fractional orders with three-point multi-term fractional integral boundary conditions. Journal of Applied Mathematical Computations 2013; 43: 507-522.

[21] Thaiprayoon C, Ntouyas SK, Tariboon J. On the nonlocal Katugampola fractional integral conditions for fractional Langevin equation. Advance in Difference Equations 2015; 1: 1-16.

[22] Tariboon J, Ntouyas SK, Thaiprayoon C. Nonlinear Langevin equation of Hadamard-Caputo type fractional derivatives with nonlocal fractional integral conditions. Advances in Mathematical Physics 2014; 2014.

[23] Wang J, Feckan M, Zhou Y. Presentation of solutions of impulsive fractional Langevin equations and existence results. The European Physical Journal Special Topics 2013; 222(8): 1857-1874.

[24] Wang H, Lin X. Existence of solutions for impulsive fractional Langevin functional differential equations with variable parameter. Revista de la Real Academia de Ciencias Exactas, Fasicas y Naturales. Serie A. Matematicas 2016; 110(1): 79-96.

[25] Wang J, Zhang Y. On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives. Applied Mathematics Letters, 2015; 39: 85-90.

[26] Wang G, Zhang L, Song G. Boundary value problem of a nonlinear Langevin equation with two different fractional orders and impulses. Fixed Point Theory and Applications 2012; 1: 1-17.

[27] Yukunthorna W, Ahmad B, Ntouyas SK, Tariboon J. On Caputo-Hadamard type fractional impulsive hybrid systems with nonlinear fractional integral conditions. Nonlinear Analysis Hybrid Systems 2016; 19: 77-92.

[28] Yu T, Deng K, Luo M. Existence and uniqueness of solutions of initial value problems for nonlinear Langevin equation involving two fractional orders. Communications in Nonlinear Science and Numerical Simulation 2014; 19(6): 1661-1668.

[29] Yukunthorn W, Ntouyas SK, Tariboon J. Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann–Liouville fractional integral conditions. Advance in Difference Equations 2014; (35).

[30] Zhao K. Impulsive boundary value problems for two classes of fractional differential equation with two different Caputo fractional derivatives. Mediterranean Journal of Mathematics 2016; 13(3): 1033-1050.