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Subclasses of uniformly convex and starlike functions associated with Bessel functions

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Abstract: In recent years, applications of Bessel differential equations have been commonly used in univalent functions theory. The main object of the present paper is to give some characteristic properties for some subclasses of uniformly starlike and convex functions which are defined here by means of the normalized form of the generalized Bessel function to be univalent in the open unit disc. Furthermore, we also establish some results of these subclasses related to a particular integral operator. Some corresponding consequences of our main results are also considered.

Key words: Analytic functions, Bessel function, starlike functions, convex functions

1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$s(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in open unit disc $U = \{z : |z| < 1\}$ and normalized by $s(0) = 0$, and $s'(0) - 1 = 0$.

Let $\mathcal{T}$ [19] denote the subclass of $\mathcal{A}$ consisting of functions with negative coefficients in the form as

$$s(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0. \tag{1.2}$$

Let $\mathcal{ST}$ and $\mathcal{CV}$ denote the subclasses of $\mathcal{A}$ that are respectively, starlike and convex. In 1991, Goodman [12, 13] generalized the concept of starlike and convex functions. A function $s \in \mathcal{A}$ is uniformly convex if $s(z)$ maps every circular arc ζ contained in U with center $\zeta \in U$ onto a convex arc. The class of all uniformly convex functions is denoted by $\mathcal{UCV}$ and holds the following condition

$$s \in \mathcal{UCV} \Longleftrightarrow s \in \mathcal{A} \text{ and } Re\left\{\frac{\left(zs'(z)\right)'}{s'(z)}\right\} > \left|\frac{zs''(z)}{s'(z)}\right|, \quad z \in U. \tag{1.3}$$

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A function $s \in \mathcal{A}$ is uniformly starlike if $s(z)$ maps every circular arc ζ contained in U with center $\zeta \in U$ onto a starlike arc. Let $\mathcal{UST}$ represent the class of all uniformly starlike functions whose representation is given by

$$s \in \mathcal{UST} \iff s \in \mathcal{A} \text{ and } Re\left\{\frac{zs'(z)}{s(z)}\right\} > \left|\frac{zs'(z)}{s(z)} - 1\right|, \quad z \in U. \tag{1.4}$$

The representations of $\mathcal{UCV}$ and $\mathcal{UST}$ given in (1.3) and (1.4) were given in the references [14, 17, 20].

In 1997, Bharati et al. [7] introduced the subclasses of starlike and convex functions of order α $(\alpha \geq 0)$ and β $(0 \leq \beta < 1)$. Goodman [12, 13], Ma and Minda [14], and Ronning [16, 17] have also discussed some interesting facts of the class of uniformly convex and starlike functions of order α $(\alpha \geq 0)$ and β $(0 \leq \beta < 1)$.

In this study, we give the following new classes based on the paper by Cho et al. [8] and previous studies mentioned above. Throughout this study, unless otherwise stated, the parameters of alpha, beta, and eta are considered $(\alpha \geq 0)$, $(0 \leq \beta < 1)$, and $(0 \leq \eta \leq 1)$.

Definition 1.1 *A function s of the form (1.1) is said to be in the class $Q_q(\alpha, \beta, \eta)$ if it satisfies the following condition:*

$$Re\left\{\frac{zs'(z) + \eta z^2 s''(z)}{s(z)}\right\} \geq \alpha \left|\frac{zs'(z) + \eta z^2 s''(z)}{s(z)} - 1\right| + \beta. \tag{1.5}$$

Definition 1.2 *A function s of the form (1.1) is said to be in the class $Q_qCV(\alpha, \beta, \eta)$ if it satisfies the following condition:*

$$Re\left\{\frac{s'(z) + \eta z s''(z)}{s'(z)}\right\} \geq \alpha \left|\frac{s'(z) + \eta z s''(z)}{s'(z)} - 1\right| + \beta. \tag{1.6}$$

Definition 1.3 *A function s of the form (1.1) is said to be in the class $PS(\alpha, \eta)$ if it satisfies the following condition:*

$$Re\left\{\frac{zs'(z) + \eta z^2 s''(z)}{s(z)}\right\} + \alpha \geq \left|\frac{zs'(z) + \eta z^2 s''(z)}{s(z)} - \alpha\right|. \tag{1.7}$$

Definition 1.4 *A function s of the form (1.1) is said to be in the class $PCV(\alpha, \eta)$ if it satisfies the following condition:*

$$Re\left\{\frac{s'(z) + \eta z s''(z)}{s'(z)}\right\} + \alpha \geq \left|\frac{s'(z) + \eta z s''(z)}{s'(z)} - \alpha\right|. \tag{1.8}$$

We note that $PST(\alpha, \eta) = PS(\alpha, \eta) \cap T$ and $PCVT(\alpha, \eta) = PCV(\alpha, \eta) \cap T$.

For special value of parameter, we obtain some of the previously studied classes. Some of them are listed below.

i $\quad Q_q(\alpha, \beta, 0) = SP(\alpha, \beta)$ [8].

ii $\quad Q_qCV(\alpha, \beta, 1) = UCV(\alpha, \beta)$ [7].

iii $\quad PS(\alpha, 0) = q(\alpha)$ [8].

iv $PCV(\alpha,1) = CP(\alpha)$ [8].

Let us consider a second-order linear homogeneous differential equation,

$$z^2 w^{''}(z) + bzw^{'}(z) + \left[dz^2 - q^2(1-b)\right] w(z) = 0 \qquad (b,q,d \in \mathbb{C}). \tag{1.9}$$

A particular solution of (1.9) give a generalized Bessel functions of the first kind of order q, given in (1.10) defined by Baricz [4].

$$w(z) = w_{q,b,d} = \sum_{n=2}^{\infty} \frac{(-1)^n d^n}{(n)!\Gamma\left(q+n+\frac{b+1}{2}\right)} (\frac{z}{2})^{2n+q}. \tag{1.10}$$

The function $w_{q,b,d}$ is not univalent in U but the series given by (1.9) is convergent. Cho et al. [8] defined the following transformation

$$u_{q,b,d}(z) = 2^q \Gamma\left(q + \frac{b+1}{2}\right) z^{-\frac{q}{2}} w_{q,b,d}\left(\sqrt{z}\right) \qquad \sqrt{1} = 1.$$

Using the well-known Pochhammer symbol, the following Gamma function can be defined

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & n = 0 \\ a(a+1)(a+2)\cdots(a+n-1) & n \in \mathbb{N}. \end{cases}$$

We can express $u_{q,b,d}(z) = u_q(z)$ and $u_q(z)$ can be written as follows.

$$u_q(z) = u_{q,b,d}(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{d}{4}\right)^n}{\left(q+\frac{b+1}{2}\right)_n (n)!} z^n, \text{ where } q + \frac{b+1}{2} \in N = \{1,\ 2,\ 3, ...\}. \tag{1.11}$$

We write $m = q + \frac{b+1}{2}$ throughout this paper for convenience.

In geometric function theory, the study of generalized Bessel functions is an important topic. In this study, we refer to the studies by Baricz [3–6], Akgul [1, 2], Sakar and Aydogan [18], Cho et al. [8] , Mondal and Swaminathan [15], Deniz [11], Deniz et al. [10], and Choi and Agarwal [9]. Studies on Struve functions can be found in a recent investigation by Srivastava et al. [20].

In the present paper, we obtained sufficient conditions to be in $Q_q(\alpha,\beta,\eta)$ and $Q_qCV(\alpha,\beta,\eta)$. We also determined necessary and sufficient conditions to be in the $PS(\alpha,\eta)$ and $PCV(\alpha,\eta)$. Furthermore, we determined sufficient conditions for zu_q to be in $Q_q(\alpha,\beta,\eta)$ and $Q_qCV(\alpha,\beta,\eta)$ also for $z(2-u_q)$ to be in the function classes $PS(\alpha,\eta)$ and $PCV(\alpha,\eta)$. We consider an integral operator related to the function u_q. Also, some corollaries related to main theorems have been presented.

2. Main results

In this section, some theorems and corollaries related to our main results will be given.

Theorem 2.1 *A sufficient condition for a function s of the form (1.1) to be in the class $Q_q(\alpha,\beta,\eta)$ is that the following inequality (2.1) holds.*

$$\sum_{n=2}^{\infty} \left[n(\eta n - \eta + 1)(1+\alpha) - \alpha - \beta\right] |a_n| \leq 1 - \beta. \tag{2.1}$$

Proof It is sufficient to show that

$$\alpha\left|\frac{zs^{'}(z)+\eta z^{2}s^{''}(z)}{s(z)}-1\right|-Re\left\{\frac{zs^{'}(z)+\eta z^{2}s^{''}(z)}{s(z)}-1\right\}\leq 1-\beta.$$

Let us consider the following inequalities

$$\alpha\left|\frac{zs^{'}(z)+\eta z^{2}s^{''}(z)}{s(z)}-1\right|-Re\left\{\frac{zs^{'}(z)+\eta z^{2}s^{''}(z)}{s(z)}-1\right\}\leq (1+\alpha)\left|\frac{zs^{'}(z)+\eta z^{2}s^{''}(z)-s(z)}{s(z)}\right|$$

$$\leq \frac{(1+\alpha)\sum_{n=2}^{\infty}\left(\eta n^{2}-\eta n+n-1\right)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}.$$

This last expression is bounded above by $1-\beta$ if the following inequality holds

$$\sum_{n=2}^{\infty}\left[n(\eta n-\eta+1)(1+\alpha)-\alpha-\beta\right]|a_n|\leq 1-\beta.$$

□

It is remarkable that a necessary and sufficient condition for a function s of the form (1.2) to be in the class $Q_q(\alpha,\beta,\eta)$ is that the condition (2.1) is satisfied.

Theorem 2.2 *A sufficient condition for a function s of the form (1.1) to be in the class $Q_qCV(\alpha,\beta,\eta)$ is that the following inequality (2.2) holds.*

$$\sum_{n=2}^{\infty}n\left[\eta(n-1)(1+\alpha)+1-\beta\right]|a_n|\leq 1-\beta. \tag{2.2}$$

Proof It is sufficient to show that

$$\alpha\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}-1\right|-Re\left\{\frac{zs^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}-1\right\}\leq 1-\beta.$$

Let s consider the following inequalities

$$\alpha\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}-1\right|-Re\left\{\frac{zs^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}-1\right\}\leq (1+\alpha)\left|\frac{\eta zs^{''}(z)}{s^{'}(z)}\right|$$

$$\leq \frac{\eta(1+\alpha)\sum_{n=2}^{\infty}n(n-1)|a_n|}{1-\sum_{n=2}^{\infty}n|a_n|}.$$

This last expression is bounded above by $1-\beta$ if the following inequality holds

$$\sum_{n=2}^{\infty}n\left[\eta(n-1)(1+\alpha)+1-\beta\right]|a_n|\leq 1-\beta.$$

□

It is remarkable that a necessary and sufficient condition for a function s of the form (1.2) to be in the class $Q_qCV(\alpha,\beta,\eta)$ is that the condition (2.2) is satisfied.

Theorem 2.3 *A necessary and sufficient condition for a function s of the form (1.2) to be in the class $PS(\alpha,\eta)$ is that the following inequality (2.3) holds.*

$$\sum_{n=2}^{\infty}\left[\eta n(n-1)+n-\alpha\right]|a_n|\leq 1+\alpha. \tag{2.3}$$

Proof Let us consider the following inequalities

$$Re\left\{\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right\}+\alpha\geq\left|\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}-\alpha\right|,$$

which leads to

$$\left|\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right|-Re\left\{\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right\}\leq 2\alpha.$$

Consider

$$\begin{aligned}\left|\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right|-Re\left\{\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right\}&\leq 2\left|\frac{zs^{'}(z)+\eta z^2s^{''}(z)}{s(z)}\right|\\&=2\frac{\left|z-\sum_{n=2}^{\infty}\left(\eta n^2-\eta n+n\right)a_nz^n\right|}{\left|z-\sum_{n=2}^{\infty}a_nz^n\right|}\\&=2\frac{\left|1-\sum_{n=2}^{\infty}\left(\eta n^2-\eta n+n\right)a_nz^{n-1}\right|}{\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|}.\end{aligned}$$

This last expression is bounded above by 2α if the following inequalities hold

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(\eta n^2-\eta n+n\right)|a_n|-1\leq\alpha\left(1+\sum_{n=2}^{\infty}|a_n|\right)\\&\sum_{n=2}^{\infty}\left[\eta n(n-1)+n-\alpha\right]|a_n|\leq 1+\alpha.\end{aligned}$$

□

Theorem 2.4 *A necessary and sufficient condition for a function s of the form (1.2) to be in the class $PCV(\alpha,\eta)$ is that the following inequality (2.4) holds.*

$$\sum_{n=2}^{\infty}\left[\eta n(n-1)+n(1-\alpha)\right]|a_n|\leq 1+\alpha. \tag{2.4}$$

Proof Let us consider the following inequalities

$$Re\left\{\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right\}+\alpha\geq\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}-\alpha\right|,$$

which leads to

$$\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right|-Re\left\{\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right\}\leq 2\alpha.$$

Consider

$$\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right|-Re\left\{\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right\}\leq 2\left|\frac{s^{'}(z)+\eta zs^{''}(z)}{s^{'}(z)}\right|$$

$$=2\frac{\left|1-\sum\limits_{n=2}^{\infty}\left(\eta n^2-\eta n+n\right)a_nz^{n-1}\right|}{\left|1-\sum\limits_{n=2}^{\infty}na_nz^{n-1}\right|}.$$

This last expression is bounded above by 2α if the following inequality holds

$$\sum_{n=2}^{\infty}\left[\eta n(n-1)+n(1-\alpha)\right]|a_n|\leq 1+\alpha.$$

□

Theorem 2.5 *If $d<0$ and $m<0$ then $zu_q\in Q_q(\alpha,\beta,\eta)$ if*

$$\eta(1+\alpha)u_q^{''}(1)+(2\eta+1)(1+\alpha)u_q^{'}(1)+(1-\beta)\left[u_q(1)-1\right]\leq 1-\beta. \tag{2.5}$$

Proof Since

$$zu_q(z)=z+\sum_{n=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}z^n, \tag{2.6}$$

on account of Theorem 2.1, it is sufficient to show that

$$£(d,m,\alpha,\beta,\eta)=\sum_{n=2}^{\infty}\left[n(\eta n-\eta+1)(1+\alpha)-(\alpha+\beta)\right]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}\leq 1-\beta.$$

Doing some simple calculations, we can obtain the equalities given below:

$$£(d,m,\alpha,\beta,\eta)=\sum_{n=2}^{\infty}\left[n(\eta n-\eta+1)(1+\alpha)-(\alpha+\beta)\right]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$=\sum_{n=2}^{\infty}\left[(1+\alpha)(n-1)(\eta n-1)+1-\beta\right]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$=\sum_{n=2}^{\infty}\left[\eta(1+\alpha)\left(n^2-n\right)+(1+\alpha)(n-1)+1-\beta\right]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=2}^{\infty} \eta (1+\alpha)(n-1)(n-2) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} (2\eta+1)(1+\alpha)(n-1) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$+ \sum_{n=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=2}^{\infty} \eta (1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-3)!} + \sum_{n=2}^{\infty} (2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-2)!}$$

$$+ \sum_{n=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=0}^{\infty} \eta (1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n+2}}{(m)_{n+2}(n)!} + \sum_{n=0}^{\infty} (2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n)!} + \sum_{n=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n+1)!}$$

$$= \eta (1+\alpha) \frac{\left(-\frac{d}{4}\right)^{2}}{m(m+1)} u_{q+2}(1) + (2\eta+1)(1+\alpha) \frac{\left(-\frac{d}{4}\right)}{m} u_{q+1}(1) + (1-\beta)\left[u_q(1) - 1\right]$$

$$= \eta (1+\alpha) u_q''(1) + (2\eta+1)(1+\alpha) u_q'(1) + (1-\beta)\left[u_q(1)-1\right]. \tag{2.1}$$

Therefore, the last expression (2.7) is bounded above by $1-\beta$, if the condition (2.5) is satisfied. □

Corollary 2.6 *If $d<0$ and $m<0$ then, $z(2-u_q) \in Q_q(\alpha,\beta,\eta)$ if and only if (2.5) is satisfied.*

Proof Since

$$z(2-u_q) = z - \sum_{n=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} z^n,$$

by using the technique given in the proof of Theorem 2.5, we arrive immediately Corollary 2.6. □

Theorem 2.7 *If $d<0$ and $m<0$ then $zu_q \in Q_qCV(\alpha,\beta,\eta)$ if*

$$\eta(1+\alpha) u_q''(1) + \left[2\eta(1+\alpha) + (1-\beta)\right] u_q'(1) + (1-\beta)\left[u_q(1)-1\right] \le 1-\beta. \tag{2.8}$$

Proof Since

$$zu_q(z) = z + \sum_{n=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} z^n, \tag{2.9}$$

on account of Theorem 2.2, it is sufficient to show that

$$f(d,m,\alpha,\beta,\eta) = \sum_{n=2}^{\infty} n\left[\eta(n-1)(1+\alpha) + (1-\beta)\right] \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} \le 1-\beta.$$

Doing some simple calculations, we can obtain the equalities given below:

$$f(d,m,\alpha,\beta,\eta) = \sum_{n=2}^{\infty} \eta(1+\alpha)\left(n^2-n\right) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} n(1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=2}^{\infty} \eta(1+\alpha)\left[(n-1)(n-2)+2(n-1)\right] \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$+ \sum_{n=2}^{\infty} (1-\beta)\left[(n-1)+1\right] \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=2}^{\infty} \eta(1+\alpha)(n-1)(n-2) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} 2\eta(1+\alpha)(n-1) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$+ \sum_{n=2}^{\infty} (1-\beta)(n-1) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} + \sum_{n=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=3}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-3)!} + \sum_{n=2}^{\infty} 2\eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-2)!} +$$

$$\sum_{n=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$= \sum_{n=0}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n+2}}{(m)_{n+2}(n)!} + \sum_{n=0}^{\infty} 2\eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n)!} + \sum_{n=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n)!}$$

$$+ \sum_{n=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n+1)!}$$

$$= \sum_{n=0}^{\infty} \eta(1+\alpha) \frac{\left(-\frac{d}{4}\right)^{n+2}}{(m)_{n+2}(n)!} + \sum_{n=0}^{\infty} \left[2\eta(1+\alpha)+(1-\beta)\right] \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n)!}$$

$$+ \sum_{n=0}^{\infty} (1-\beta) \frac{\left(-\frac{d}{4}\right)^{n+1}}{(m)_{n+1}(n+1)!}$$

$$= \eta(1+\alpha) u_q^{''}(1) + \left[2\eta(1+\alpha)+(1-\beta)\right] u_q^{'}(1) + (1-\beta)\left[u_q(1)-1\right].$$

Therefore, the last expression is bounded above by $1-\beta$, if the condition (2.8) is satisfied. □

Corollary 2.8 *If $d<0$ and $m<0$ then, $z(2-u_q) \in Q_qCV(\alpha,\beta,\eta)$ if and only if the inequality (2.8) is satisfied.*

Proof Since

$$z(2-u_q) = z - \sum_{n=2}^{\infty} \frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!} z^n,$$

by using the same technique given in the proof of Theorem 2.7, we immediately reach Corollary 2.8. □

Theorem 2.9 *If* $d<0$ *and* $m<0$ *then* $z(2-u_q)\in PS(\alpha,\eta)$ *if*

$$\eta u_q^{''}(1)+(2\eta+1)u_q^{'}(1)+(1-\alpha)u_q(1)\leq 2. \tag{2.10}$$

Proof Since

$$z(2-u_q(z))=z-\sum_{n=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}z^n,$$

on account of Theorem 2.3, it is sufficient to show that

$$r(d,m,\alpha,\eta)=\sum_{n=2}^{\infty}[\eta n(n-1)+n-\alpha]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}\leq 1+\alpha.$$

Doing some simple calculations, we can obtain the equalities given below:

$$r(d,m,\alpha,\eta)=\sum_{n=2}^{\infty}[\eta n(n-1)+n-\alpha]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$\begin{aligned}&=\sum_{n=2}^{\infty}\eta(n-1)(n-2)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}+\sum_{n=2}^{\infty}(2\eta+1)(n-1)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}\\&+\sum_{n=2}^{\infty}(1-\alpha)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}\\&=\eta u_q^{''}(1)+(2\eta+1)u_q^{'}(1)+(1-\alpha)[u_q(1)-1].\end{aligned}$$

Therefore, the last expression is bounded above by $1+\alpha$ if the condition (2.10) is satisfied.

$$\eta u_q^{''}(1)+(2\eta+1)u_q^{'}(1)+(1-\alpha)u_q(1)\leq 2.$$

□

Theorem 2.10 *If* $d<0$ *and* $m<0$ *then* $z(2-u_q)\in PCV(\alpha,\eta)$ *if and only if*

$$\eta u_q^{''}(1)+(2\eta+1-\alpha)u_q^{'}(1)+(1-\alpha)u_q(1)\leq 2.$$

Proof Since

$$z(2-u_q(z))=z-\sum_{n=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}z^n,$$

on account of Theorem 2.4, it is sufficient to show that

$$h(d,m,\alpha,\eta)=\sum_{n=2}^{\infty}[\eta n(n-1)+n(1-\alpha)]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}\leq 1+\alpha.$$

Doing some simple calculations, we can obtain the equalities given below:

$$h(d,m,\alpha,\eta)=\sum_{n=2}^{\infty}\left[\eta n(n-1)+n(1-\alpha)\right]\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$=\sum_{n=2}^{\infty}\eta(n-1)(n-2)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}+\sum_{n=2}^{\infty}(2\eta+1-\alpha)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$+\sum_{n=2}^{\infty}(1-\alpha)\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}$$

$$=\eta u_q^{''}(1)+(2\eta+1-\alpha)u_q^{'}(1)+(1-\alpha)\left[u_q(1)-1\right].$$

Therefore, the last expression is bounded above by $1+\alpha$ if the following inequality holds

$$\eta u_q^{''}(1)+(2\eta+1-\alpha)u_q^{'}(1)+(1-\alpha)u_q(1)\leq 2.$$

□

In the next two theorems given below, we obtain results of similar types in connection with a particular integral operator $T(d,m,z)$ stated by

$$T(d,m,z)=\int_0^z\left[2-u_q(t)\right]dt. \tag{2.11}$$

Theorem 2.11 *If $d<0$ and $m<0$ then $T(d,m,z)\in Q_q(\alpha,\beta,\eta)$ if and only if the condition (2.5) is satisfied.*

Proof Since

$$T(d,m,z)=z-\sum_{n=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}z^n,$$

on account of Theorem 2.1, we need only to show that

$$\sum_{n=2}^{\infty}\left[n(\eta n-\eta+1)(1+\alpha)-\alpha-\beta\right]|a_n|\leq 1-\beta.$$

The rest of the proof of this theorem is similar to the proof of Theorem 2.5, so we omit the details. □

Theorem 2.12 *If $d<0$ and $m<0$ then $T(d,m,z)\in PS(\alpha,\eta)$ if and only if the condition (2.10) is satisfied.*

Proof Since

$$T(d,m,z)=z-\sum_{n=2}^{\infty}\frac{\left(-\frac{d}{4}\right)^{n-1}}{(m)_{n-1}(n-1)!}z^n,$$

on account of Theorem 2.3, we need only to show that

$$\sum_{n=2}^{\infty}\left[\eta n(n-1)+n-\alpha\right]|a_n|\leq 1+\alpha.$$

The rest of the proof of this theorem is similar to the proof of Theorem 2.9, so we omit the details. □

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