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On a class of nonself-adjoint multidimensional periodic Schrödinger operators

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Abstract: We investigate the Schrödinger operator $L(q)$ in $L_2\left(\mathbb{R}^d\right)$ $(d \geq 1)$ with the complex-valued potential q that is periodic with respect to a lattice Ω. Besides, it is assumed that the Fourier coefficients q_γ of q with respect to the orthogonal system $\{e^{i\langle\gamma x\rangle} : \gamma \in \Gamma\}$ vanish if γ belongs to a half-space, where Γ is the lattice dual to Ω. We prove that the Bloch eigenvalues are $\mid \gamma + t \mid^2$ for $\gamma \in \Gamma$, where t is a quasimomentum and find explicit formulas for the Bloch functions. Moreover, we investigate the multiplicity of the Bloch eigenvalue and consider necessary and sufficient conditions on the potential which provide some root functions to be eigenfunctions. Besides, in case $d = 1$ we investigate in detail the root functions of the periodic and antiperiodic boundary value problems.

Key words: Periodic Schrödinger operator, Bloch eigenvalues, Bloch function

1. Introduction and preliminary facts

We consider the Schrödinger operator $L(q)$ generated in $L_2(\mathbb{R}^d)$ by the expression

$$-\Delta\Psi + q\Psi, \tag{1.1}$$

where the potential q is periodic relative to a lattice Ω and belongs to the class S of the complex-valued functions defined as follows. Let

$$\Gamma := \left\{\gamma \in \mathbb{R}^d : \langle\gamma, \omega\rangle \in 2\pi\mathbb{Z}, \ \forall\omega \in \Omega\right\}$$

be the lattice dual to Ω, where $\mathbb{Z}$ is the set of all integers and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^d$. Let $\{v_1, v_2, ..., v_d\}$ be any generator of the reciprocal lattice Γ, that is,

$$\Gamma = \{n_1v_1 + n_2v_2 + ... + n_dv_d : \ n_1 \in \mathbb{Z}, \ n_2 \in \mathbb{Z}, ..., n_d \in \mathbb{Z}\}.$$

Divide the lattice Γ into three parts $\Gamma(k)$, $\Gamma(k+)$, and $\Gamma(k-)$, where $\Gamma(k)$ is the sublatice of Γ generated by $\{v_1, v_2, ..., v_d\} \setminus \{v_k\}$ and

$$\Gamma(k\pm) = \{u \pm nv_k : u \in \Gamma(k), n \in \mathbb{N}\}, \ \mathbb{N} = \{1, 2, ...\}. \tag{1.2}$$

Denote the set of potentials q whose Fourier decompositions have the form

$$q(x) = \sum_{\gamma\in\Gamma(k\pm)} q_\gamma e^{i\langle\gamma,x\rangle} \tag{1.3}$$

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by $S(k\pm)$. Here the Fourier coefficients q_γ satisfy the following inequality

$$\sum_{\gamma\in\Gamma(k\pm)} | q_\gamma |= M < \infty, \tag{1.4}$$

where $q_\gamma = (q, e^{i\langle\gamma,x\rangle})$, $(\cdot,\cdot)$ is the inner product in $L_2(F)$ and $F := \mathbb{R}^d/\Omega$ is the fundamental domain (primitive cell) of the lattice Ω. In cases $d = 2$ and $d = 3$ the condition (1.4) is replaced by $q \in L_2(F)$. Without loss of generality we assume that the measure $\mu(F)$ of F is 1. Define S by

$$S = \cup_{k=1}^d (S(k+) \cup S(k-)). \tag{1.5}$$

The operator $L(q)$ is nonself-adjoint for each nonzero $q \in S$. However, $\{L(q) : q \in S\}$ contains a large class $\{L(q) : q \in S,\ q_\gamma \in \mathbb{R}, \forall\gamma \in \Gamma\}$ of PT symmetric operators which are important in the PT symmetric quantum theory (see e.g., [1]).

Let $L_t(q)$ be the operator generated in $L_2(F)$ by (1.1) and the quasiperiodic conditions

$$\Psi(x + \omega) = e^{i\langle t,\omega\rangle}\Psi(x),\ \forall\omega \in \Omega, \tag{1.6}$$

where $t \in F^\star := \mathbb{R}^d/\Gamma$. It is well known that the spectrum of $L_t(q)$ consists of the eigenvalues $\Lambda_1(t),\ \Lambda_2(t),....$ which are called Bloch eigenvalues of $L(q)$. The eigenfunction $\Psi_{N,t}(x)$ of $L_t(q)$ corresponding to the eigenvalue $\Lambda_N(t)$ is known as the Bloch function of $L(q)$:

$$L_t(q)\Psi_{N,t}(x) = \Lambda_N(t)\Psi_{N,t}(x). \tag{1.7}$$

In the case $q = 0$, the eigenvalues and normalized eigenfunctions of $L_t(q)$ are $| \gamma+t |^2$ and $e^{i\langle\gamma+t,x\rangle}$ for $\gamma \in \Gamma$:

$$L_t(0)e^{i\langle\gamma+t,x\rangle} =| \gamma + t |^2\ e^{i\langle\gamma+t,x\rangle}. \tag{1.8}$$

The one-dimensional case was considered in detail. First of all, Gasymov [5] proved the following remarkable results for the operator $L(q) := -\frac{d^2}{dx^2} + q$ with the potential q of the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx} \tag{1.9}$$

when $\sum | q_n |< \infty$.

Result 1: The spectrum $\sigma(L(q))$ of $L(q)$ is $[0,\infty)$. The spectral singularities on the spectrum are the numbers of the form $(\frac{n}{2})^2$.

Result 2: The equation

$$-y^{''}(x) + q(x)y(x) = \mu^2 y(x) \tag{1.10}$$

has the Floquet solution of the form

$$f(x,\mu) = e^{i\mu x}(1 + \sum_{n=1}^{\infty}\frac{1}{n+2\mu}\sum_{\alpha=n}^{\infty} v_{n,\alpha}e^{i\alpha x}),$$

where the following series converge

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha(\alpha-n)\mid v_{n,\alpha}\mid,\ \sum_{n=1}^{\infty}n\mid v_{n,\alpha}\mid.$$

Result 3: A spectral expansion was constructed by the Floquet solutions.

Result 4: It was shown that the Wronskian of the Floquet solutions

$$f_n(x):=\lim_{\mu\to\frac{n}{2}}(n-2\mu)f(x,-\mu)$$

and $f(x,\frac{n}{2})$ is equal to zero and $f_n(x)=s_nf(x,\frac{n}{2})$. It was proved that one can effectively reconstruct $\{q_n\}$ from $\{s_n\}$.

Some generalizations of the results of [5] were done by Gasymov's students (see [7, 11] and references therein).

Guillemin and Uribe [8, 9] investigated the boundary value problem generated on $[0,2\pi]$ by (1.10) and the periodic boundary conditions when $q\in Q_2^+$, where Q_2^+ is the set of $q\in L_2[0,2\pi]$ of the form (1.9). It was proved that the eigenvalues of this boundary value problem are n^2 for $n\in\mathbb{Z}$ and the root functions (eigenfunctions and associated functions) were studied. Moreover, the inverse method applied to Hill's equation was developed in the class of Hardy potentials and its applications to the N-soliton solutions of KdV were considered. For $L(q)$ with the potential $q\in Q_2^+$ the inverse spectral problem was investigated by Pastur and Tkachenko [12] and the alternative proofs of the equality $\sigma(L(q))=[0,\infty)$ were provided by Shin [14], Carlson [2], and Christiansen [3]. In the case $q(x)=Ae^{2\pi irx}$, where $A\in\mathbb{C}$ and $r\in\mathbb{Z}$, the periodic and antiperiodic boundary value problems were investigated in detail by Kerimov [10]. Several new and interesting observations from the point of view of physicists were made by Curtright and Mezincescu [4].

In the paper [15], we proved that if

$$q\in L_1[0,1],\ q(x+1)=q(x),\ q_n=0,\ \forall n=0,-1,-2,..., \tag{1.11}$$

then $\sigma(L(q))=[0,\infty)$ and $\sigma(L_t(q))=\{(2\pi n+t)^2:n\in\mathbb{Z}\}$ for all $t\in\mathbb{C}$, where $q_n=(q,e^{i2\pi nx})$ and $L_t(q)$ is the operator generated in $L_2[0,1]$ by the boundary value problem

$$-y^{''}(x)+q(x)y(x)=\lambda y(x),\ y(1)=e^{it}y(0),\ y^{'}(1)=e^{it}y^{'}(0). \tag{1.12}$$

Moreover, we proved that if $t\neq 0,\pi$, then $(2\pi n+t)^2$ is a simple eigenvalue of $L_t(q)$ and found explicit formulas for the corresponding eigenfunctions. Finally, we considered the inverse problem for the general case (1.11).

As far as I am concerned for the multidimensional Schrödinger operator with a potential from the set S defined in (1.5) there exist only two papers: [13] and [6]. In [13], Sarnak investigated the quasiperiodic potentials of the form

$$V(x)=\sum_{j=1}^{n}a_je^{2\pi i\langle\xi_j,x\rangle},$$

that is, V is taken to be a finite sum of exponents with generic frequencies, and he obtained various interesting results. Moreover, both continuous and discrete cases were considered at the same time. Here we recall only the following results of [13] which are connected with the investigations of this paper. He proved that if $\xi_1, \xi_2, ..., \xi_n$ are the elements of $\mathbb{R}^d$ lying inside a cone of angle less than π, then the spectrum of the Schrödinger operator with the potential V defined by the last equality is $[0, \infty)$ and he found an elegant formula for the solutions. It is clear that, the intersection of the set of the latter potentials V with the set of the periodic potentials coincides with the set $\mathbb{P}$ of the trigonometric polynomials from the set S. Therefore, in the case of periodic potentials of type (1.3) the recalled results of [13] are concerned with the set $\mathbb{P}$.

In [6], Gasimov investigated the three-dimensional operator $L(q)$ with periodic potential of the form

$$q(x) = \sum_{\gamma \in Z^+} q_\gamma e^{i\langle \gamma, x\rangle}, \quad \sum_{\gamma \in Z^+} \mid q_\gamma \mid < \infty, \tag{1.13}$$

where $Z^+ = \left\{(m_1, m_2, m_3) \in \mathbb{Z}^3 : m_1 + m_2 + m_3 \geq 1, \ m_j \geq 0, \ \forall j = 1, 2, 3\right\}$ is a part of $\mathbb{Z}^3$ lying in the first octant of $\mathbb{R}^3$. He found a formula for the resolvent kernel of $\left(L(q) - k^2\right)^{-1}$ with $\operatorname{Im} k \neq 0$, which indicates that $\left(L - k^2\right)^{-1}$ is bounded for such k. He also showed that the spectrum of L is $[0, \infty)$, and gave the Plancherel theorem, where the Fourier transformation was given in terms of certain solutions to $-\Delta u + qu = k^2 u$.

In this paper, by combining the methods in [15–19], we investigate the periodic multidimensional Schrödinger operator $L(q)$ of arbitrary dimension d and arbitrary lattice Ω when the potential q belongs to the set S which is larger than (1.13) and $\mathbb{P}$. The main results of this paper are formulated in Section 2, where we prove that if $q \in S$ then for all $t \in F^\star$ the eigenvalues of $L_t(q)$ consist of the numbers $|\gamma + t|^2$, for $\gamma \in \Gamma$, that is, the Bloch eigenvalues of $L(q)$ for $q \in S$ coincide with the Bloch eigenvalues of the free operator $L(0)$. It implies that the isoenergetic surfaces of $L(q)$ and $L(0)$ are the same. Moreover, we find explicit formulas for the Bloch functions. At the end of Section 2, we prove that in the two and three-dimensional cases the main results continue to hold if (1.4) is replaced by $q \in L_2(F)$. In Section 3, using the results of Section 2 and the approaches of the papers [8, 9], we investigate the multiplicity of the Bloch eigenvalues and consider necessary and sufficient conditions on the potential which provide some root functions to be eigenfunctions. Besides, we investigate in detail the root functions of the boundary value problem (1.12) for $t = 0, \pi$ when the potential q satisfies (1.11).

2. Main results

First let us formulate the main results. For this we introduce the following notations and recall some well known facts. For $b \in \Gamma$ the hyperplanes $\left\{x \in \mathbb{R}^d : |x| = |x + b|\right\}$ are called the diffraction hyperplanes. The number $\mid \gamma + t \mid^2$ is a simple eigenvalue of $L_t(0)$ if and only if $\gamma + t$ does not belong to any diffraction hyperplane, that is,

$$\mid \gamma + t \mid \neq \mid \gamma + b + t \mid, \ \forall b \in \Gamma, \ b \neq 0. \tag{2.1}$$

A number λ is a multiple eigenvalue of multiplicity m of $L_t(0)$ if and only if there exist m different vectors $b_1, b_2, ..., b_m$ of the lattice Γ such that

$$\lambda = \mid b_1 + t \mid^2 = \mid b_2 + t \mid^2 = ... = \mid b_m + t \mid^2 . \tag{2.2}$$

By the definitions of v_k and $\Gamma(k)$ (see (2)), for each b_j there exists $p_j \in \mathbb{Z}$ such that

$$b_j \in \Gamma(k, p_j) := \{p_j v_k + a : a \in \Gamma(k)\} . \tag{2.3}$$

Let us enumerate the vectors b_j so that $p_1 \geq p_2 \geq \ldots$. Then there exists s such that

$$p_1 = p_2 = \ldots = p_s > p_{s+1} \geq p_{s+2} \geq \ldots. \tag{2.4}$$

Finally recall that the isoenergetic surfaces of the operators $L(q)$ and $L(0)$ corresponding to the energy ρ^2 are the sets

$$I_\rho(q) = \{t \in F^* : \exists N, \Lambda_N(t) = \rho^2\} \quad \& \quad I_\rho(0) = \{t \in F^* : \exists \gamma \in \Gamma, \mid \gamma + t \mid = \rho\},$$

respectively. The isoenergetic surface $I_\rho(0)$ is the translation of the sphere $\{\mid x \mid = \rho\}$ by the vectors $\gamma \in \Gamma$ to the fundamental domain F^* of the reciprocal lattice Γ.

Theorem 2.1 ***(Main Results).*** *(a) If $q \in S$, then for any $t \in F^*$, the set of eigenvalues of $L_t(q)$ is $\{\mid \gamma + t \mid^2 : \gamma \in \Gamma\}$, that is,*

$$\sigma(L_t(q)) = \sigma(L_t(0)), \ \forall t \in F^* \ \ \& \ \ \sigma(L(q)) = \bigcup_{t \in F^*} \sigma(L_t(q)) = [0, \infty).$$

(b) For any $q \in S$ and $\rho \in [0, \infty)$ the isoenergetic surface $I_\rho(q)$ of $L(q)$ coincides with the isoenergetic surface $I_\rho(0)$ of the free operator $L(0)$.

(c) Let $\mid \gamma + t \mid^2$ be a simple eigenvalue of $L_t(0)$. If $q \in S(k+)$, then there exists only one eigenfunction $\Psi_{\gamma+t}(x)$ of $L_t(q)$ corresponding to $\mid \gamma + t \mid^2$. It can be normalized by

$$(\Psi_{\gamma+t}, e^{i\langle \gamma+t, x\rangle}) = 1 \tag{2.5}$$

and it satisfies (2.6) and (2.34) (see below and Theorem 2.7).

(d) Let λ be an eigenvalue of $L_t(0)$ of multiplicity m. If $q \in S(k+)$, then the functions $\Psi_{b_j+t}(x)$ for $j = 1, 2, \ldots, s$ defined in (2.8) are linearly independent eigenfunctions of $L_t(q)$ corresponding to λ, where b_j and s are defined in (2.2)–(2.4).

(e) The statements (c) and (d) continue to hold if $S(k+)$ and $\Gamma(k+)$ are replaced by $S(k-)$ and $\Gamma(k-)$, respectively. In the cases $d = 2$ and $d = 3$ the statements (a)–(d) continue to hold if the condition (1.4) is replaced by $q \in L_2(F)$.

(f) If $q \in S(k+)$, then for all l and j, the operator $L_t(q)$ has a root function $\varphi_{l,j}$ of the form (3.3). These functions for $l = 1$ are the eigenfunctions of the operator $L_t(q)$. For $l = 2$ they are eigenfunctions if and only if (3.9) holds.

(g) Suppose $d = 1$ and (1.11) holds. If the geometric multiplicity of the eigenvalue $(2\pi n)^2$ of the operator $L_0(q)$ is two, then (3.14) and (3.15) are linearly independent eigenfunctions of $L_0(q)$. If the geometric multiplicity of $(2\pi n)^2$ is one, then (3.15) and (3.14) are respectively the eigenfunction and associated function of $L_0(q)$. The geometric multiplicity of $(2\pi n)^2$ for $n \neq 0$ is two if and only if (3.21) holds.

Proof The proof of (a) follows from Theorems 2.4 and 2.5, where the relations $\sigma(L_t(q)) \subset \sigma(L_t(0))$ and $\sigma(L_t(0)) \subset \sigma(L_t(q))$ are proved respectively. (b) follows from (a). In Theorem 2.5 we prove that if (2.2)–(2.4) hold then the function $\Psi_{b_j+t}(x)$ defined in (2.8) is an eigenfunction. Since it is clear that the functions $\Psi_{b_j+t}(x)$ for $j = 1, 2, \ldots, s$ are linearly independent, we get the proof of (d), by proving Theorem 2.5. Theorems

2.5 and 2.7 imply (c), because only one eigenfunction may provide (2.6) and (2.34). We prove the theorems for $q\in S(k+)$. The proof of the case $q\in S(k-)$ is the same. By (1.5), they imply the proofs of the statements for $q\in S$. The last statement of (e) is proved in Theorem 2.8. The statements (f) and (g) are proved in Section 3 □

Let $\mid\gamma+t\mid^2$ be a simple eigenvalue of $L_t(0)$, which means that (2.1) holds. Introduce the function $\Psi_{\gamma+t}(x)$ defined by

$$\Psi_{\gamma+t}(x)=e^{i\langle\gamma+t,x\rangle}+A(\gamma)e^{i\langle\gamma+t,x\rangle}+\left(A(\gamma)\right)^2e^{i\langle\gamma+t,x\rangle}+...,\tag{2.6}$$

where $A(\gamma)$ is the linear transformation taking $e^{i\langle b+t,x\rangle}$ to

$$A(\gamma)e^{i\langle b+t,x\rangle}=\sum_{\gamma_1\in\Gamma(k+)}\frac{q_{\gamma_1}e^{i\langle b+\gamma_1+t,x\rangle}}{\mid\gamma+t\mid^2-\mid b+\gamma_1+t\mid^2},\tag{2.7}$$

where $b\in\left((\gamma+\Gamma(k+))\cup\{\gamma\}\right)$. In the proof of Theorem 2.5 we prove that the series (2.6) converges to some element of $L_2(F)$. Now we only note that if (2.1) holds then the denominators in (2.7) are not zero and $(A(\gamma))^n$ are defined for all $n=1,2,\ldots$.

Similarly, in cases (2.2)–(2.4), using the definition of $\Gamma(k+)$, one can readily see that the transformations $\left(A(b_j)\right)^n$ for $j=1,2,...,s$ and $n=1,2,...,$ are defined on $e^{i\langle b+t,x\rangle}$ for $b\in\left((b_j+\Gamma(k+))\cup\{b_j\}\right)$. Introduce the function $\Psi_{b_j+t}(x)$ defined by

$$\Psi_{b_j+t}(x)=e^{i\langle b_j+t,x\rangle}+\left(A(b_j)\right)e^{i\langle b_j+t,x\rangle}+\left(A(b_j)\right)^2e^{i\langle b_j+t,x\rangle}+....\tag{2.8}$$

Remark 2.2 *One can readily see that the transformation $A(b)$ can be defined by*

$$A(b)f=(\mid b+t\mid^2I+\Delta)^{-1}qf\tag{2.9}$$

in some subspaces. It is clear that if (2.1) holds then $(\mid\gamma+t\mid^2I+\Delta)^{-1}$ is well-defined in the subspace $E(\gamma)$ generated by the orthonormal system $\left\{e^{i\langle b+t,x\rangle}:b\in\Gamma\backslash\gamma\right\}$. On the other hand, it follows from (1.2) and the definition of $S(k+)$ that if $q\in S(k+)$ then $q^ne^{i\langle\gamma+t,x\rangle}\in E(\gamma)$ for $n=1,2,\ldots$. Therefore, $\left(A(\gamma)\right)^ne^{i\langle\gamma+t,x\rangle}$ exists for $n=1,2,\ldots$. Similarly, in cases (2.2)–(2.4), $(\mid b_j+t\mid^2I+\Delta)^{-1}$ is defined in the subspace $E(b_1,b_2,...,b_m)$ generated by the orthonormal system $\left\{e^{i\langle b+t,x\rangle}:b\in(\Gamma\backslash\{b_1,b_2,...,b_m\})\right\}$. On the other hand, by (1.2) if $q\in S(k+)$ and $j=1,2,...,s$ then $q^ne^{i\langle b_j+t,x\rangle}\in E(b_1,b_2,...,b_m)$ for $n=1,2,\ldots$. Therefore, $\left(A(b_j)\right)^ne^{i\langle b_j+t,x\rangle}$ exists for $n=1,2,....,$ and $j=1,2,...,s$.

To consider the Bloch eigenvalues $\Lambda_N(t)$ and Bloch functions $\Psi_{N,t}$ we use the following iteration of the formula

$$(\Lambda_N(t)-\mid\gamma+t\mid^2)(\Psi_{N,t},e^{i\langle\gamma+t,x\rangle})=(q\Psi_{N,t},e^{i\langle\gamma+t,x\rangle}),\tag{2.10}$$

which is obtained from (1.7) by multiplying by $e^{i\langle\gamma+t,x\rangle}$ and using (1.8). If

$$\Lambda_N(t)\neq\mid\gamma+t\mid^2,\ \forall\gamma\in\Gamma,\tag{2.11}$$

then (2.10) can be iterated as follows. Using the expansion (1.3) of $q\in S(k+)$ in (2.10), we get

$$(\Lambda_N(t)-\mid\gamma+t\mid^2)(\Psi_{N,t},e^{i\langle\gamma+t,x\rangle})=\sum_{\gamma_1\in\Gamma(k+)}q_{\gamma_1}(\Psi_{N,t},e^{i\langle\gamma-\gamma_1+t,x\rangle}).\tag{2.12}$$

On the other hand, replacing γ by $\gamma-\gamma_1$ in (2.10) and taking (2.11) into account we obtain

$$(\Psi_{N,t}, e^{i\langle\gamma-\gamma_1+t,x\rangle}) = \frac{(q\Psi_{N,t}, e^{i\langle\gamma-\gamma_1+t,x\rangle})}{\Lambda_N(t) - \mid \gamma-\gamma_1+t \mid^2}. \tag{2.13}$$

Now, using (2.13) in (2.12) we get

$$(\Lambda_N(t) - \mid \gamma+t \mid^2)(\Psi_{N,t}, e^{i\langle\gamma+t,x\rangle}) = \sum_{\gamma_1} \frac{q_{\gamma_1}(q\Psi_{N,t}, e^{i\langle\gamma-\gamma_1+t,x\rangle})}{\Lambda_N(t) - \mid \gamma-\gamma_1+t \mid^2}. \tag{2.14}$$

Repeating this process m times we obtain

$$(\Lambda_N(t) - \mid \gamma+t \mid^2)(\Psi_{N,t}, e^{i\langle\gamma+t,x\rangle}) = \sum_{\gamma_1,\gamma_2,\ldots,\gamma_m} \frac{q_{\gamma_1}q_{\gamma_2}\cdots q_{\gamma_m}(q\Psi_{N,t}, e^{i\langle\gamma+t-\gamma(m),x\rangle})}{\prod\limits_{s=1,2,\ldots,m}\left[\Lambda_N(t) - \left|\gamma+t-\gamma(s)\right|^2\right]}, \tag{2.15}$$

where $\gamma(j) =: \gamma_1+\gamma_2+\cdots+\gamma_j$ for $j=1,2,\ldots$ and the summations in (2.14) and (2.15) are taken under the conditions $\gamma_1 \in \Gamma(k+)$ and $\gamma_1,\gamma_2,\ldots,\gamma_m \in \Gamma(k+)$, respectively.

To estimate the right-hand side of (2.15) we use the following simple proposition.

Proposition 2.3 *For each $k \in \{1,2,\ldots d\}$ there exists a positive constant $c(k)$ such that if $\gamma_j \in \Gamma(k+)$ for $j=1,2,\ldots,$ then*

$$|\gamma_1+\gamma_2+\cdots+\gamma_s| \geq c(k)s, \tag{2.16}$$

for all $s \in \mathbb{N}$. The proposition continues to hold if $\Gamma(k+)$ is replaced by $\Gamma(k-)$.

Proof We prove the proposition for $\Gamma(k+)$. The proof for $\Gamma(k-)$ is the same. Let $P(k)$ be the hyperplane spanned by $v_1,v_2,\ldots,v_{k-1},v_{k+1},\ldots,v_d$. Since $v_k \notin P(k)$, it has the orthogonal decomposition

$$v_k = u_k + h_k,\ u_k \in P(k),\ h_k \perp P(k) = 0,\ h_k \neq 0. \tag{2.17}$$

On the other hand, by (1.2), if $\gamma_j \in \Gamma(k+)$ then $\gamma_j = a_j + n_j v_k$, where $a_j \in \Gamma(k) \subset P(k)$ and $n_j \in \mathbb{N}$. Therefore, there exist $u \in \Gamma(k) \subset P(k)$ and $w \in P(k)$ such that

$$\sum_{j=1}^{s}\gamma_j = u + \left(\sum_{j=1}^{s} n_j\right) v_k = w + \left(\sum_{j=1}^{s} n_j\right) h_k, \tag{2.18}$$

where $\langle u, h_k\rangle = 0$ (see (2.17)), $\langle w, h_k\rangle = 0$ and $n_1+n_2+\cdots+n_s \geq s$. Thus, using (2.18) and Pythagorean theorem we see that (2.16) holds for $c(k) = |h_k|$ □

Now we are ready to prove the following.

Theorem 2.4 *If $q \in S$ and $t \in F^*$, then for every eigenvalue $\Lambda_N(t)$ of $L_t(q)$ there exists $\gamma \in \Gamma$ such that $\Lambda_N(t) = |\gamma+t|^2$, that is, $\sigma(L_t(q)) \subset \sigma(L_t(0))$ for all $t \in F^*$.*

Proof By (1.5) it is enough to prove the theorem for $q \in (S(k+) \cup S(k-))$. We prove it for $q \in S(k+)$. The proof of the case $q \in S(k-)$ is the same. Suppose, to the contrary, that there exists N such that (2.11) holds. Then there exists a positive number c such that

$$\left|\Lambda_N(t) - |\gamma+t|^2\right| > c,\ \forall\gamma \in \Gamma. \tag{2.19}$$

Now using (1.4), (2.16), and (2.19) let us estimate the right-hand side of (2.15). It immediately follows from (1.4) that

$$\sum_{\gamma_1,\gamma_2,\ldots,\gamma_m} |q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_m}| \le M^m. \tag{2.20}$$

On the other hand, by (2.16) we have

$$|\gamma+t\pm\gamma(s)|^2 \ge (c(k)s-|\gamma+t|)^2 \tag{2.21}$$

and

$$\left|\Lambda_N(t)-|\gamma+t-\gamma(s)|^2\right| \ge (c(k)s-|\gamma+t|)^2-|\Lambda_N(t)|. \tag{2.22}$$

Now using (2.19), (2.20), and (2.22), one can readily see that the right side of (2.15) approaches 0 as $m\to\infty$. Therefore, letting m tend to ∞ in (2.15) and then using (2.11), we obtain

$$(\Lambda_N(t)-\mid\gamma+t\mid^2)(\Psi_{N,t},e^{i\langle\gamma+t,x\rangle})=0 \quad \& \quad (\Psi_{N,t},e^{i\langle\gamma+t,x\rangle})=0, \tag{2.23}$$

for all $\gamma\in\Gamma$. The last equality of (2.23) is a contradiction, since $\left\{e^{i\langle\gamma+t,x\rangle}:\gamma\in\Gamma\right\}$ is an orthonormal basis in $L_2(F)$ and $\|\Psi_{N,t}\|\ne 0$. The theorem is proved. □

Now we prove that $\sigma(L_t(0))\subset\sigma(L_t(q))$ and consider the Bloch functions.

Theorem 2.5 *Let $q\in S$ and $t\in F^*$. Then for all $\gamma\in\Gamma$, the numbers $\mid\gamma+t\mid^2$ are the eigenvalues of $L_t(q)$. In cases (2.1) and (2.2)–(2.4), the functions defined in (2.6) and (2.8) for $j=1,2,...,s$ are the eigenfunctions corresponding to $\mid\gamma+t\mid^2$ and λ, respectively.*

Proof First let us consider the case (2.1). It readily follows from (2.7) that

$$(A(\gamma))^n\, e^{i\langle\gamma+t,x\rangle} = \sum_{\gamma_1,\gamma_2,\ldots,\gamma_m} \frac{q_{\gamma_1}q_{\gamma_2}\ldots q_{\gamma_n}e^{i\langle\gamma+t+\gamma(n),x\rangle}}{\prod\limits_{s=1,2,\ldots,n}\left(\mid\gamma+t\mid^2-|\gamma+t+\gamma(s)|^2\right)}. \tag{2.24}$$

Using (2.20) and (2.21) in (2.24) one can readily see that there exists a constant c such that

$$\left|(A(\gamma))^n\, e^{i\langle\gamma+t,x\rangle}\right| < \frac{c}{2^n},\ \forall n=1,2,\ldots \tag{2.25}$$

Therefore, the series in the right hand side of (2.6) converges to some element, denoted by $\Psi_{\gamma+t}$, of $L_2(F)$ and $\Psi_{\gamma+t}$ satisfies (1.6). Moreover, $\Delta\Psi_{\gamma+t}\in L_2(F)$ and

$$A(\gamma)\Psi_{\gamma+t}=\Psi_{\gamma+t}-e^{i\langle\gamma+t,x\rangle}). \tag{2.26}$$

It, with (2.9) and (1.8), implies that

$$q\Psi_{\gamma+t}=(\Delta+\mid\gamma+t\mid^2 I)(\Psi_{\gamma+t}-e^{i\langle\gamma+t,x\rangle})=\Delta\Psi_{\gamma+t}+\mid\gamma+t\mid^2\Psi_{\gamma+t}, \tag{2.27}$$

that is, $\Psi_{\gamma+t}$ is an eigenfunction of $L_t(q)$ corresponding to $\mid\gamma+t\mid^2$.

Now we consider the cases (2.2)–(2.4). Using (2.4) and (1.2) and arguing as in the proof of (2.25) we see that one can replace γ by b_j for $j = 1, 2, ..., s$ in (2.25). Therefore, repeating the proofs of (2.26) and (2.27) one can easily verify that the functions $\Psi_{b_j+t}(x)$ defined by (2.8) for $j = 1, 2, ..., s$ are the eigenfunctions corresponding to the eigenvalue $\lambda =\mid b_j + t \mid^2$ □

Now we consider the Fourier decompositions of the Bloch functions. Let $\Psi_{\gamma+t}(x)$ be an arbitrary eigenfunction of $L_t(q)$ corresponding to the the simple eigenvalue $\mid \gamma + t \mid^2$ of $L_t(0)$, that is, (2.1) holds. Since $\left\{e^{i\langle \gamma+\delta+t,x\rangle} : \delta \in \Gamma\right\}$ is an orthonormal basis we have

$$\Psi_{\gamma+t}(x) = (\Psi_{\gamma+t}, e^{i\langle \gamma+t,x\rangle})e^{i\langle \gamma+t,x\rangle} + \sum_{\delta\in\Gamma\backslash\{0\}} (\Psi_{\gamma+t}, e^{i\langle \gamma+\delta+t,x\rangle})e^{i\langle \gamma+\delta+t,x\rangle}. \tag{2.28}$$

To find $(\Psi_{\gamma+t}, e^{i\langle \gamma+\delta+t,x\rangle})$ for $\delta \neq 0$, we use the formula

$$(\Psi_{\gamma+t}, e^{i\langle \gamma+\delta+t,x\rangle}) = \frac{1}{d(\gamma,\delta)} \sum_{\gamma_1\in\Gamma(k+)} q_{\gamma_1}(\Psi_{\gamma+t}, e^{i\langle \gamma+\delta-\gamma_1+t,x\rangle}) \tag{2.29}$$

obtained from (2.12) by replacing $\Lambda_N(t), \Psi_{N,t}$ and γ by $\mid \gamma + t \mid^2$, $\Psi_{\gamma+t}$ and $\gamma + \delta$, respectively, where

$$d(\gamma, \delta) =\mid \gamma + t \mid^2 - \mid \gamma + \delta + t \mid^2 \neq 0, \ \forall\delta \neq 0. \tag{2.30}$$

Now iterating (2.29) we obtain the Fourier decomposition of $\Psi_{\gamma+t}$ in the next theorem. Note that in the iteration we take the following into account.

Remark 2.6 *By definition of* $\Gamma(k)$, *for* $\delta \in \Gamma\backslash\{0\}$ *there exist* $a \in \Gamma(k)$ *and* $p \in \mathbb{Z}$, *such that*

$$\delta = (a + pv_k) \in \Gamma(k, p), \tag{2.31}$$

where $\Gamma(k, p)$ *is defined in* (2.3). *Let us consider the cases:* $p \leq 0$ *and* $p > 0$.

Case 1*: Let* $p \leq 0$. *If* $\gamma_1 \in \Gamma(k+), \gamma_2 \in \Gamma(k+), ...,$ *then* $\gamma_1 = a_1 + p_1v_k$, $\gamma_2 = a_2 + p_2v_k, ...,$ *where* $a_1 \in \Gamma(k)$, $a_2 \in \Gamma(k), ...$ *and* $p_1 > 0$, $p_2 > 0, ...$ *Therefore, by (44) we have*

$$\delta - \gamma(j) = u_j + s_jv_k, s_j < 0, \forall j = 1, 2, ...,$$

where $u_j \in \Gamma(k)$ *and* $\gamma(j)$ *is defined in* (2.15). *It with* (2.30) *implies that*

$$\delta - \gamma(j) \neq 0 \quad \& \ d(\gamma, \delta - \gamma(j)) \neq 0, \ \forall j = 1, 2, ... \tag{2.32}$$

Case 2*: Let* $p > 0$ *and* $\gamma_1 \in \Gamma(k+), \gamma_2 \in \Gamma(k+), ...$ *Then arguing as in* ***Case 1*** *we obtain that*

$$(\delta - \gamma(p)) \in \Gamma(k, s) \quad \& \quad s \leq 0. \tag{2.33}$$

Theorem 2.7 *If* $\Psi_{\gamma+t}(x)$ *is an eigenfunction of* $L_t(q)$ *corresponding to the simple eigenvalue* $\mid \gamma + t \mid^2$ *of* $L_t(0)$, *then it can be normalized by* (2.5) *and it satisfies*

$$\Psi_{\gamma+t}(x) = e^{i\langle \gamma+t,x\rangle} + \sum_{\delta\in\Gamma(k+)} c(\gamma, \delta)e^{i\langle \gamma+\delta+t,x\rangle} =$$

$$e^{i\langle \gamma+t,x\rangle} + \sum_{p\in\mathbb{N}} \sum_{\delta\in\Gamma(k,p)} c(\gamma, \delta)e^{i\langle \gamma+\delta+t,x\rangle}, \tag{2.34}$$

where $c(\gamma,\delta)$ *for* $\delta\in\Gamma(k,p)$ *and* $p\in\mathbb{N}$ *is defined by*

$$c(\gamma,\delta)=\frac{1}{d(\gamma,\delta)}\left(q_{\delta}+\sum_{j=1}^{p-1}\sum_{\gamma_1,\gamma_2,\ldots,\gamma_j}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_j}q_{\delta-\gamma(j)}}{d(\gamma,\delta-\gamma_1)d(\gamma,\delta-\gamma(2))...d(\gamma,\delta-\gamma(j))}\right) \tag{2.35}$$

and the summations are taken under conditions

$$\{\gamma_1,\gamma_2,...,\gamma_j,\delta-\gamma(j)\}\subset\Gamma(k+),\ \forall j=1,2,...,p-1. \tag{2.36}$$

Moreover, the right-hand sides of (2.6) *and* (2.34) *are the same.*

Proof First let us consider **Case** 1 of Remark 2.6. Then $\delta\in\Gamma\backslash\{0\}$ has the form (2.31) and $p\leq 0$, that is, $\delta\in\Gamma(k,p)$ and $p\leq 0$. In this case, iterating (2.29) m times and taking (2.32) into account, we obtain

$$(\Psi_{\gamma+t},e^{i\langle\gamma+\delta+t,x\rangle})=\sum_{\gamma_1,\gamma_2,\ldots,\gamma_m}\left(\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_{m+1}}(\Psi_{\gamma+t},e^{i\langle\gamma+\delta-\gamma(m+1)+t,x\rangle})}{d(\gamma,\delta)d(\gamma,\delta-\gamma_1)...d(\gamma,\delta-\gamma(m))}\right). \tag{2.37}$$

Moreover, repeating the arguments used in the proof of the statement that the right hand side of (2.15) approaches zero (see the proof of Theorem 2.4) we obtain that the right hand side of (2.37) approaches zero as $m\to\infty$, and hence

$$(\Psi_{\gamma+t},e^{i\langle\gamma+\delta+t,x\rangle})=0,\ \forall\delta\in\Gamma(k,p),\ p\leq 0,\ \delta\neq 0. \tag{2.38}$$

Now we consider **Case** 2 of Remark 2.6. Then δ has the form (2.31) and $p>0$. In this case, we iterate (2.29) as follows. Isolate the terms in the right-hand side of (2.29) containing the multiplicand $(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle})$ which occurs in the case $\gamma_1=\delta$ and use (2.29) for the other terms to get

$$(\Psi_{\gamma+t},e^{i\langle\gamma+\delta+t,x\rangle})=\frac{q_{\delta}}{d(\gamma,\delta)}(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle})+\sum_{\gamma_1,\gamma_2}\frac{q_{\gamma_1}q_{\gamma_2}(\Psi_{\gamma+t},e^{i\langle\gamma+\delta-\gamma_1-\gamma_2+t,x\rangle})}{d(\gamma,\delta)d(\gamma,\delta-\gamma_1)}.$$

Isolating again the terms containing the multiplicand $(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle})$ which occurs in the case $\gamma_2=\delta-\gamma_1$ and using again (2.29) for the other terms and repeating this process $p-1$ times we obtain

$$(\Psi_{\gamma+t},e^{i\langle\gamma+\delta+t,x\rangle})=c(\gamma,\delta)(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle})+ \sum_{\gamma_1,\gamma_2,\ldots,\gamma_p}\frac{q_{\gamma_1}q_{\gamma_2}...q_{\gamma_p}(\Psi_{\gamma+t},e^{i\langle\gamma+\delta-\gamma(p)+t,x\rangle})}{d(\gamma,\delta)d(\gamma,\delta-\gamma_1)...d(\gamma,\delta-\gamma(p-1))}, \tag{2.39}$$

where $c(\gamma,\delta)$ is defined in (2.35) and (2.36), $\delta-\gamma(p)\neq 0$ and $\delta-\gamma(p)\in\Gamma(k,s)$, $s\leq 0$ by (2.33). Therefore, it follows from (2.38) that the second term of the right-hand side of (2.39) is zero and hence we have

$$(\Psi_{\gamma+t},e^{i\langle\gamma+\delta+t,x\rangle})=c(\gamma,\delta)(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle}). \tag{2.40}$$

Since the system $\left\{e^{i\langle\gamma+t,x\rangle}:\gamma\in\Gamma\right\}$ is an orthonormal basis in $L_2(F)$ and $\|\Psi_{\gamma+t}\|\neq 0$, formulas (2.28), (2.38), and (2.40) imply that $(\Psi_{\gamma+t},e^{i\langle\gamma+t,x\rangle})\neq 0$. Hence, there exists an eigenfunction, denoted again by $\Psi_{\gamma+t}$, satisfying (2.5). Now using (2.28), (2.38), (2.40), and the obvious relation $\Gamma(k+)=\cup_{p\in\mathbb{N}}\Gamma(k,p)$ we get the proof of (2.34). Thus, we have proved that any eigenfunction normalized by (2.5) has the form (2.34). It implies that there is only one eigenfunction normalized by (2.5); hence, the right-hand sides of (2.6) and (2.34) are the same. The theorem is proved □

Now we consider the two and three-dimensional cases.

Theorem 2.8 *In the cases $d = 2$ and $d = 3$ the results of Theorem 2.1(a) − (d) continue to hold if the condition (1.4) is replaced by $q \in L_2(F)$.*

Proof If $q \in L_2(F)$, then we have

$$\sum_{\gamma \in \Gamma(k\pm)} | q_\gamma |^2 < \infty. \tag{2.41}$$

On the other hand, it is clear that if $d = 2, 3$ and (2.1) holds, then

$$\sum_{\gamma_1 \in \Gamma(k+)} \left(\frac{1}{| \gamma + t |^2 - | \gamma + \gamma_1 + t |^2} \right)^2 < \infty. \tag{2.42}$$

The inequalities (2.41) and (2.42) and the Schwarz inequality for l_2 imply that

$$\left| A(\gamma) e^{i\langle \gamma + t, x \rangle} \right| < \infty, \ \forall x, \tag{2.43}$$

where $A(\gamma)$ is defined in (2.7). Moreover, one can easily verify by using (2.21) that if (2.1) holds, then for fixed $\gamma_1, \gamma_1, ..., \gamma_{s-1}$ from $\Gamma(k+)$ the relation

$$\sum_{\gamma_s \in \Gamma(k+)} \left(\frac{1}{| \gamma + t |^2 - | \gamma + \gamma_1 + \gamma_2 + ... + \gamma_s + t |^2} \right)^2 = O(s^{-1}) \tag{2.44}$$

is satisfied. The relations (2.42)–(2.44) continue to hold if γ is replaced by b_j for $j = 1, 2, ..., s$ and (2.2)–(2.4) are satisfied. Therefore, instead of (1.4) and (2.21) using (2.41) and (2.44), respectively and repeating the proof of Theorem 2.1 $(a) - (d)$ by using the Schwarz inequality for l_2, we get the proof of the theorem □

3. On the root functions for the multiple eigenvalues

In this chapter, we consider the root functions of $L_t(q)$ corresponding to the multiple eigenvalues (2.2) and find necessary and sufficient conditions on the potential which provide some root functions to be eigenfunctions. For this we introduce the following notation.

Notation 3.1 *The integers $p_1 \geq p_2 \geq ...$ defined in (2.3), in general, are not different from each other. There are p different numbers among them denoted by $n_1 > n_2 > ... > n_p$, where $p \leq m$ and m is defined in (2.2). Then the vectors $b_1, b_2, ..., b_m$ defined in (2.2) belong to $\Gamma(k, n_1)$, $\Gamma(k, n_2), ...$ and $\Gamma(k, n_p)$ which are the points of the lattice Γ lying on the parallel hyperplanes $P(k, n_1)$, $P(k, n_2), ...$ and $P(k, n_p)$, where $P(k, n_l) = \{x + n_l v_k : x \in P(k)\}$ and $P(k)$ is the hyperplane generated by the vectors $v_1, v_2, ..., v_{k-1}, v_{k+1}, ..., v_d$. We redenote the elements of $\{b_1, b_2, ..., b_m\} \cap P(k, n_l)$ by $b_{l,1}, b_{l,2}, ..., b_{l,s_l}$. Let $H(n_l)$ and $H(n_l, j)$ be respectively the spaces spanned by*

$$\left\{ e^{i\langle a + n v_k + t, x \rangle} : a \in \Gamma(k), n > n_l \right\} \ \& \ \left\{ e^{i\langle a + n v_k + t, x \rangle} : a \in \Gamma(k), n > n_l \right\} \cup \left\{ e^{i\langle b_{l,j} + t, x \rangle} \right\}.$$

Remark 3.2 *By Notation 1, $b_{l,j}$ belongs to $P(k,n_l)$ and hence has the form*

$$b_{l,j}=a_{l,j}+n_l v_k,\ \ a_{l,j}\in\Gamma(k). \tag{3.1}$$

Besides, comparing Notations 1 with the notations of (2.3) and (2.4) we see that $s_1=s$ and $b_i\in\Gamma(k,n_1)$ for $i=1,2,...,s,$ that is, $b_i=a_i+n_1v_k,\ a_i\in\Gamma(k)$ for $i=1,2,...,s.$ Then by (2.2) and (3.1) we have the equalities

$$\lambda=\mid a_i+n_1v_k+t\mid^2=\mid a_{1,j}+n_1v_k+t\mid^2, \tag{3.2}$$

for all $j=1,2,...,s$ and $i=1,2,...,s$.

Now we prove the following statements by using the definitions of $\Gamma(k+)$, $S(k+)$, $H(n_l)$, $H(n_l,j)$, Theorem 2.1(a) and the approach of the proof of Theorem 2.2 of [9].

Theorem 3.3 *If $q\in S(k+)$, then the following hold:*

(a) *The operator $L_t(q)$ is invariant in $H(n_l)$ and $H(n_l,j)$ for all l and j.*

(b) *For all l and $\gamma\in\Gamma(k,n_l)$ the number of linearly independent root functions (eigenfunctions and associated functions) of the operator $L_t(\varepsilon q)$ corresponding to the eigenvalue $\mid\gamma+t\mid^2$ and lying in $H(n_l)$ does not depend on $\varepsilon\in\mathbb{C}$. The statement continues to hold if $\Gamma(k,n_l)$ and $H(n_l)$ are replaced by $\Gamma(k,n_l)\cup\{b_{j,l}\}$ and $H(n_l,j)$, respectively.*

(c) *For all l and j the operator $L_t(q)$ has a root function of the form*

$$\varphi_{l,j}(x)=e^{i\langle b_{l,j}+t,x\rangle}+\sum_{n=n_l+1}^{\infty}\left(\sum_{a\in\Gamma(k)}c(a,n)e^{i\langle a+nv_k+t,x\rangle}\right), \tag{3.3}$$

where $c(a,n)=(\varphi_{l,j},e^{i\langle a+nv_k+t,x\rangle})$.

Proof (a) Let Q be the subset of the lattice Γ such that $q_\gamma\neq 0$ if $\gamma\in Q$. By (1.3) we have $Q\subset\Gamma(k+)$. Therefore, if $f\in H(n_l,j)$, then $qf\in H(n_l)\subset H(n_l,j)$, that yields (a).

(b) Now from (a) and Theorem 2.1(a) arguing as in the proof of Theorem 2.2 of [9] we get the proof of (b). Namely, we argue as follows. Let D be a small disk with the center $\mid\gamma+t\mid^2$, where $\gamma\in\Gamma(k,n_l)$, and contain no other eigenvalues of $L_t(0)$. By Theorem 2.1(a) D contains no eigenvalues of $L_t(\varepsilon q)$ except $\mid\gamma+t\mid^2$. Therefore, the projection of $L_t(\varepsilon q)$ defined by the contour integration over the boundary of D depends continuously on ε which implies the proof of (b) for the space $H(n_l)$. The proof for $H(n_l,j)$ is the same.

(c) By Notation 1 the operator $L_t(0)$ has respectively n and $n+1$ linearly independent eigenfunctions corresponding to the eigenvalue λ in $H(n_l)$ and $H(n_l,j)$, where $n=s_1+s_2+...+s_{l-1}$. Hence, it follows from (b) that the operator $L_t(q)$ has a root function φ such that $\varphi\in H(n_l,j)$ but $\varphi\notin H(n_l)$, where $H(n_l,j)$ is the orthogonal sum of $H(n_l)$ and the one-dimensional space generated by $e^{i\langle b_{j,l}+t,x\rangle}$. It means that the projection of φ onto the one-dimensional space is nonzero, that is, $(\varphi,e^{i\langle b_{l,j}+t,x\rangle})$ is not zero. Therefore, without loss of generality, the latter number can be assumed to be 1. □

Now we find a necessary and sufficient condition on the potential q for which $\varphi_{l,j}$ is an eigenfunction. We say that φ is the l-th associated function of $L_t(q)$ if

$$\left(L_t(q)-\lambda I\right)^l\varphi\neq 0\ \ \&\ \ \left(L_t(q)-\lambda I\right)^{l+1}\varphi=0.$$

In other words, φ is called the first associated function if

$$(L_t(q) - \lambda I)\,\varphi = \Psi \tag{3.4}$$

and Ψ is an eigenfunction. If (3.4) holds and Ψ is the $(l-1)$-th associated function then we say that φ is the l-th associated function.

Theorem 3.4 (a) *The functions* $\varphi_{1,j}$ *for* $j = 1, 2, ...s$ *defined in* (3.3) *are the eigenfunctions of the operator* $L_t(q)$.

(b) *If* $l > 1$, *then the functions* $\varphi_{l,j}$ *for* $j = 1, 2, ...s_l$ *are either the eigenfunctions or the* n*-th associated functions of* $L_t(q)$, *where* $n < l$.

Proof (a) As it has been noted in the proof of Theorem 3.3(c) the operator $L_t(0)$ has $s_1 = s$ linearly independent eigenfunctions in $H(n_2)$ corresponding to the eigenvalue λ defined in (2.2). Then by Theorem 3.3(b), the operator $L_t(q)$ has s linearly independent root functions in $H(n_2)$. On the other hand, by Theorem 2.5, $L_t(q)$ has at least s linearly independent eigenfunctions in $H(n_2)$. Therefore, $\varphi_{1,j}$ for $j = 1, 2, ...s$ are the eigenfunctions and $H(n_2)$ does not contain an associated function.

(b) First let us consider the case $l = 2$. Using the relations $\varphi_{2,j} \in H(n_2, j)$ and $q \in S(k+)$ one can readily see that $(-\Delta + (q-\lambda)\,I)\,\varphi_{2,j} \in H(n_2)$. On the other hand, $H(n_2)$ does not contain associated functions (see the end of the proof of (a)). Therefore, $(-\Delta + (q-\lambda)\,I)\,\varphi_{2,j}$ is either an eigenfunction or zero. It means that $\varphi_{2,j}$ is either an eigenfunction or the first associated function. Similarly, using the induction method and taking the relation $(-\Delta + (q-\lambda)\,I)\,\varphi_{l,j} \in H(n_l)$ into account we obtain that $\varphi_{l,j}$ is either an eigenfunction or the n-th associated function, where $n < l$ □

Now we will consider in detail the root functions $\varphi_{2,j}$ corresponding to the vectors of the plane $P(k, n_2)$. By Theorem 3.4(b) the root functions corresponding to the vectors of the plane $P(k, n_2)$ are either eigenfunctions or the first associated functions. For simplicity of notations we omit the indices and denote the arbitrary vector from $\{b_{2,j} : j = 1, 2, ..., s_2\}$ by γ and the corresponding root function by φ. Thus, $\gamma = \delta + n_2 v_k$, $\delta \in \Gamma(k)$ and φ is an eigenfunction corresponding to the eigenvalue $\lambda = \mid \delta + n_2 v_k + t \mid^2$ if and only if the following equality holds

$$(\Delta + \lambda)\varphi = q\varphi, \tag{3.5}$$

where λ is defined in (2.2) and (3.2). Using the decompositions

$$\varphi(x) = e^{i\langle \delta + n_2 v_k + t, x\rangle} + \sum_{n=n_2+1}^{\infty} \left(\sum_{a\in\Gamma(k)} c(a,n) e^{i\langle a + n v_k + t, x\rangle} \right)$$

(see (3.3)) and

$$q(x) = \sum_{m=1}^{\infty} \sum_{u\in\Gamma(k)} q_{u+mv_k} e^{i\langle u + m v_k, x\rangle},$$

we see that (3.5) holds if and only if the following system of equations holds

$$(\lambda - \mid a + n v_k + t \mid^2) c(a,n) = q_{a-\delta+(n-n_2)v_k} + \sum_{m=1}^{\infty} \left(\sum_{u\in\Gamma(k)} c(a-u, n-m) q_{u+mv_k} \right) \tag{3.6}$$

for all $n \geq n_2+1$ and $a \in \Gamma(k)$. Now taking $n_2+1, n_2+2, ..., n_1-1, n_1$ instead of n in (3.6) and taking into account that $c(a-u, n-m) = 0$ for $n-m < n_2+1$ we obtain

$$\begin{aligned}
&(\lambda- \mid a+(n_2+1)\,v_k+t \mid^2)c(a,n_2+1) = q_{a-\delta+v_k},\\
&(\lambda- \mid a+(n_2+2)\,v_k+t \mid^2)c(a,n_2+2) = q_{a-\delta+2v_k} + \sum_{u\in\Gamma(k)} c(a-u,n_2+1)q_{u+v_k},\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.7)\\
&(\lambda- \mid a+(n_1-1)\,v_k+t \mid^2)c(a,n_1-1) = q_{a-\delta+(n_1-1-n_2)v_k}+\\
&\quad\sum_{m=1}^{n_1-n_2-2}\sum_{u\in\Gamma(k)} c(a-u,n_1-1-m)q_{u+mv_k},\\
&(\lambda- \mid a+n_1v_k+t \mid^2)c(a,n_1) = q_{a-\delta+(n_1-n_2)v_k}\\
&\quad+\sum_{m=1}^{n_1-n_2-1}\sum_{u\in\Gamma(k)} c(a-u,n_1-m)q_{u+mv_k}.
\end{aligned}$$

From the first equation we express $c(a, n_2+1)$ for $a \in \Gamma(k)$ in terms of the Fourier coefficients of the potential. In the right-hand side of the second equation only $c(a, n_2+1)$ for $a \in \Gamma(k)$ takes part. Therefore, from the second equation we express $c(a, n_2+2)$ for $a \in \Gamma(k)$ in terms of the Fourier coefficients of the potential. In this way, from the third, forth.... and (n_1-n_2-1)-th equations, we express $c(a, n_2+3)$, $c(a, n_2+4), ...$ and $c(a, n_1-1)$ for $a \in \Gamma(k)$ in terms of the Fourier coefficients of the potential. Thus, the right-hand side of the last equation is a polynomial of the Fourier coefficients for fixed u. On the other hand, if $a = a_i$ for $i = 1, 2, ..., s$, then by (3.2) the left-hand side of the last equation of (3.7) is zero. Therefore, we obtain the following equalities:

$$q_{a_i-\delta+(n_1-n_2)v_k} + \sum_{m=1}^{n_1-n_2-1}\left(\sum_{u\in\Gamma(k)} c(a_i-u,n_1-m)q_{u+mv_k}\right) = 0,\ \forall i = 1,2,...,s, \tag{3.8}$$

where $c(a_i-u, n_1-m)$ is explicitly expressed by the Fourier coefficients of the potential q. The equalities (3.8) can be written in the form

$$\sum_{u\in\Gamma(k)} Q(b_i,u) = 0,\ \forall i = 1,2,...,s, \tag{3.9}$$

where $Q(b_i, u)$ is a polynomial of the Fourier coefficients of the potential q and $b_i = a_i + n_1v_k$ (see Remark 3.2), for $i = 1, 2, ..., s$ are the vectors defined by (2.2) and lying on the plane $P(k, n_1)$. Thus, we have the following.

Theorem 3.5 *The function $\varphi_{2,j}$, defined in (3.3), is an eigenfunction if and only if (3.9) holds.*

Proof We have proved that if $\varphi_{2,j}$ is an eigenfunction, then (3.9). holds. Now suppose that $\varphi_{2,j}$ is not an eigenfunction. Then, by Theorem 3.4, it is the first associated function and

$$(\Delta+ \mid b_{2,j}+t \mid^2)\varphi_{2,j} = q\varphi_{2,l} + \Psi, \tag{3.10}$$

where $\Psi \in span\{\ \varphi_{1,j} : j = 1, 2, ..., s\}$. It means that

$$\Psi = \sum_{i \in E} c_i \varphi_{1,i},$$

where E is a nonempty subset of $\{\ 1, 2, ..., s\}$ and $c_i \neq 0$ for all $i \in E$. Using (3.10) instead of (3.5), arguing as in the proof of (3.7) and taking into account that $\varphi_{1,i} \in H(n_1, i)$, we get the system of equations whose first $n_1 - n_2 - 1$ equations coincide with the first $n_1 - n_2 - 1$ equations of (3.7) and the last equation has the form

$$(\lambda - \mid a + n_1 v_k + t \mid^2) c(a, n_1) = q_{a-\delta+(n_1-n_2)v_k} + \sum_{m=1}^{n_1-n_2-1} \sum_{u \in \Gamma(k)} c(a-u, n-m) q_{u+mv_k} + (\Psi, e^{i(a+n_1 v_k)}). \quad (3.11)$$

As in the case (3.7) the terms $c(a-u, n-m)$ in the right-hand side of (3.11) are obtained from the first $n_1 - n_2 - 1$ equations of (3.7). Therefore, arguing as in the proof of (3.8) we get

$$q_{a_i-\delta+(n_1-n_2)v_k} + \sum_{m=1}^{n_1-n_2-1} \sum_{u \in \Gamma(k)} c(a_i - u, n_1 - m) q_{u+mv_k} = c_i \neq 0,\ \forall i \in E,$$

; hence, the equality (3.9) does not hold for $i \in E$ □

The investigation of the root functions $\varphi_{l,j}$ corresponding to the vectors of $P(k, n_l)$ for $l > 2$ can be considered in a similar way. However, the multiplicities of the large eigenvalues of the multidimensional Schrödinger operator $L_t(q)$ are very large numbers. For example, if the period lattice of the potential q is $2\pi\mathbb{Z}^d$ then the multiplicity of the eigenvalue $|\gamma|^2$ is of order $|\gamma|^{d-2}$, where $\gamma \in \mathbb{Z}^d$ and hence approaches infinity as $|\gamma| \to \infty$ for $d > 2$. Therefore, the detailed investigation of the root functions $\varphi_{l,j}$ for all l is technically very complicated. In order to avoid eclipsing the essence of this paper by technical details, we consider only the case $l = 2$. Moreover, in the one-dimensional case there are only two root functions corresponding to the double eigenvalues $(2\pi n)^2$ $(n \neq 0)$ and $(2\pi n + \pi)^2$; hence, we do not need to consider the root functions corresponding to the vectors lying in $P(k, n_l)$ for $l > 2$. The one-dimensional case for the potential q from $L_2[0,1]$ was considered in [8, 9]. The case $q(x) = Ae^{2\pi i r x}$, where $A \in \mathbb{C}$ and $r \in \mathbb{Z}$, was investigated in detail in [10].

Now we consider the more general case $q \in L_1[0,1]$ by using some formulas obtained in the papers [15, 16]. In [15], we investigated the one-dimensional operators $L_t(q)$ for $t \neq 0, \pi$ corresponding to the boundary value problems (1.12) for the potential q satisfying (1.11). Let us consider the case $t = 0$. The case $t = \pi$ is similar. Since for $q \in L_1[0,1]$ Fourier decomposition (1.9) does not hold, one cannot immediately use Theorems 3.5 and 4.2 of the paper [9]. Therefore, we consider this case by using the results of the papers [15, 16]. Namely, we use the following. In Theorem 2.1 of [15] we proved that $(2\pi n)^2$ for $n \neq 0$ is the double eigenvalue of $L_0(q)$. Instead of the decomposition (1.9) we use the formula

$$((2\pi n)^2 - (2\pi(n+p))^2)(\Psi, e^{i2\pi(n+p)x}) = \sum_{m \in \mathbb{N}} q_m (\Psi, e^{i2\pi(n+p-m)x}) \quad (3.12)$$

which was proved in [15, 16] (see Lemma 1 of [16] and formula (2.3) of [15]), where Ψ is an eigenfunction corresponding to the eigenvalue $(2\pi n)^2$.

Theorem 3.6 *Suppose* (1.11) *holds. Let* Ψ *be an eigenfunction of* $L_0(q)$ *corresponding to the eigenvalue* $(2\pi n)^2$, *where* $n \in \mathbb{N}$.

(a) *At least one of the numbers*

$$(\Psi, e^{-i2\pi nx}),\ (\Psi, e^{i2\pi nx}) \tag{3.13}$$

is not zero. If the first one is not zero then Ψ *has the form*

$$e^{-i2\pi nx} + \sum_{p\in\mathbb{N}} c_p e^{i2\pi(-n+p)x}. \tag{3.14}$$

If the first one is zero and the second one is not zero, then Ψ *has the form*

$$e^{i2\pi nx} + \sum_{p\in\mathbb{N}} d_p e^{i2\pi(n+p)x}. \tag{3.15}$$

(b) *If the geometric multiplicity of the eigenvalue* $(2\pi n)^2$ *is two, then there exist linearly independent eigenfunctions* Ψ_{-n} *and* Ψ_n *having the forms* (3.14) *and* (3.15) *respectively.*

(c) *If the geometric multiplicity of the eigenvalue* $(2\pi n)^2$ *is one, then the operator* $L_0(q)$ *has an eigenfunction* Ψ *of the form* (3.15) *and an associated function* Φ *of the form* (3.14) *corresponding to the eigenfunction* $c\Psi$, *where* c *is a nonzero constant.*

Proof (a) Iterating (3.12) as was done in the proof of Theorem 2.4 in [15], we see that

$$(\Psi, e^{i2\pi(n+p)x}) = 0, \tag{3.16}$$

for all $p < -2n$. Therefore, if the first one of the numbers in (3.13) is not zero then Ψ has the form (3.14). In the same way we conclude that if the first one of the numbers in (3.13) is zero and the second one is not zero then (3.16) holds for $p < 0$ and Ψ has the form (3.15). These arguments also show that if both of the numbers in (3.13) are zero then (3.16) holds for all $p \in \mathbb{Z}$, that is, Ψ is the zero function. It means that Ψ is not an eigenfunction, which contradicts the assumption. Thus, at least one of the numbers in (3.13) is not zero.

(b) If both linearly independent eigenfunctions have the form (3.14), then some multiple of their difference has the form (3.15), since there are three possibilities: case (3.14), case (3.15) and the zero function. If both linearly independent eigenfunctions have the form (3.15), then their difference is zero, which contradicts the independence.

(c) By the definition of the associated function we have $\left(L_0(q) - (2\pi n)^2\right)\Phi = c\Psi$, where c is a nonzero number. Multiplying both sides by $e^{i2\pi(n+p)x}$ and then arguing as in the proof of (69) we obtain

$$(2\pi n)^2 - (2\pi(n+p))^2)(\Phi, e^{i2\pi(n+p)x}) = \sum_{m\in\mathbb{N}} q_m(\Phi, e^{i2\pi(n+p-m)x}) + c(\Psi, e^{i2\pi(n+p)x}). \tag{3.17}$$

If $p < -2n$ then as we noted in (a) the second term in the right side of (3.17) is zero. Therefore, iterating (3.17) we obtain that

$$(\Phi, e^{i2\pi(n+p)x}) = 0, \tag{3.18}$$

for all $p < -2n$. Now let $p = -2n$. Then the left side of (3.17) is zero and by (3.18) the first term in the right side of (3.17) is also zero. It implies that $(\Psi, e^{-i2\pi nx}) = 0$ and hence by (a) Ψ has the form (3.15). It remains to show that Φ has the form (3.14), that is, (3.18) is not true for $p = -2n$. If (3.18) is true for $p = -2n$, then arguing as above we obtain that (3.18) holds for $p < 0$. It, with (3.17) for $p = 0$, implies that $(\Psi, e^{i2\pi nx}) = 0$, which is a contradiction □

Let (3.14) be the eigenfunction Ψ. Then replacing p and Ψ by 0 and (3.14) in (3.12) and taking into account that $q_m = 0$ for $m \leq 0$ we obtain

$$q_{2n} + \sum_{p=1}^{2n-1} c_p q_{2n-p} = 0, \tag{3.19}$$

where $c_p = (\Psi, e^{i2\pi(-n+p)x})$. Replacing n with $-n$ in (3.12) and then using the equalities $c_0 = 1$ and $c_p = 0$ for $p < 0$ we obtain

$$((2\pi n)^2 - (2\pi(-n+p))^2)c_p = q_1 c_{p-1} + q_2 c_{p-2} + ... + q_{p-1} c_1 + q_p.$$

Iterating it we get

$$c_p = \frac{1}{4\pi^2 p(2n-p)} \left(q_p + \sum_{k=1}^{p-1} \sum_{n_1, n_2, \ldots, n_k} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{p-n(k)}}{b(n,p,k)} \right), \tag{3.20}$$

where $n(k) = n_1 + n_2 + ... + n_k$, $\{n_1, n_2, ..., n_k, p - n(k)\} \subset \mathbb{N}$ and

$$b(n,p,k) = \prod_{s=1}^{k} (4\pi^2 (p - n(s))(2n - p + n(s))).$$

Therefore, the equalities (3.19) and (3.20) give us the equality

$$q_{2n} + \sum_{p=1}^{2n-1} \frac{q_{2n-p}}{4\pi^2 p(2n-p)} \left(q_p + \sum_{k=1}^{p-1} \sum_{n_1, n_2 \ldots, n_k} \frac{q_{n_1} q_{n_1} \cdots q_{n_k} q_{p-n(k)}}{b(n,p,k)} \right) = 0. \tag{3.21}$$

Now let (3.14) and (3.15) be the associated function Φ and eigenfunction Ψ, respectively. Instead of (3.12) using (3.17) and repeating the above arguments by taking into account that $(\Psi, e^{i2\pi nx}) = 1$ and $(\Psi, e^{i2\pi(n+p)x}) = 0$ for $p < 0$, we get the equality obtained from (3.21) by replacing 0 with $-c$, where $c \neq 0$. Thus, we have proved the following.

Theorem 3.7 *Suppose the conditions in (1.11) hold. Then the geometric multiplicity of the eigenvalue* $(2\pi n)^2$ *for* $n \neq 0$ *of the operator* $L_0(q)$ *is two if and only if (3.21) holds. The similar result holds for the eigenvalue* $(2\pi n + \pi)^2$ *of* $L_\pi(q)$.

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