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Quasi-idempotent ranks of some permutation groups and transformation semigroups

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Abstract: Let S_n, A_n, I_n, T_n, and P_n be the symmetric group, alternating group, symmetric inverse semigroup, (full) transformations semigroup, and partial transformations semigroup on $X_n = \{1, \dots, n\}$, for $n \geq 2$, respectively. A non-idempotent element whose square is an idempotent in P_n is called a quasi-idempotent. In this paper first we show that the quasi-idempotent ranks of S_n (for $n \geq 4$) and A_n (for $n \geq 5$) are both 3. Then, by using the quasi-idempotent rank of S_n, we show that the quasi-idempotent ranks of I_n, T_n, and P_n (for $n \geq 4$) are 4, 4, and 5, respectively.

Key words: Symmetric/alternating group, full/partial transformations semigroup, symmetric inverse semigroup, quasi-idempotent, rank

1. Introduction

Let $X_n = \{1, \dots, n\}$ for $n \geq 2$, and let S_n, A_n, I_n, T_n, and P_n be the symmetric group (the group of all permutations), alternating group (the group of all even permutations), symmetric inverse semigroup (the semigroup of all partial one-to-one transformations), (full) transformations semigroup, and partial transformations semigroup on X_n, respectively.

Recall from Cayley's theorem for finite groups that every finite group is isomorphic to a subgroup of a symmetric group S_n. Hence, the finite symmetric group S_n and its subgroups have an important role in finite group theory and also in finite semigroup theory. Similarly, it is well known that every finite semigroup is isomorphic to a subsemigroup of a finite transformations semigroup T_n, and (Wagner–Preston theorem) that every finite inverse semigroup is isomorphic to a subsemigroup of a finite symmetric inverse semigroup I_n. Moreover, there is an isomorphism between the semigroup P_n and the subsemigroup P_n^* of the transformations semigroup consisting of all self-maps on $X_n \cup \{0\}$ for which $0\alpha = 0$. Hence, the importance of T_n and P_n to finite semigroup theory and the importance of I_n to finite inverse semigroup theory may be likened to the importance of symmetric group S_n to finite group theory.

An element $\alpha \in P_n$ is called an idempotent if $\alpha^2 = \alpha$. Following [4], an element $\alpha \in P_n$ is called a quasi-idempotent if $\alpha \neq \alpha^2 = \alpha^4$; that is, α is a non-idempotent element whose square is an idempotent. We denote the set of all quasi-idempotents in any subset U of any semigroup by $Q(U)$.

Let S be a semigroup, and let W be a nonempty subset of S. Then the subsemigroup generated by W, i.e. the smallest subsemigroup of S containing W, is denoted by $\langle W \rangle$. The rank of a finitely generated

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semigroup S, i.e. a semigroup generated by a finite subset, is defined by

$$\mathrm{rank}\,(S) = \min\{\,|W| : \langle W\rangle = S\}.$$

The important problem of finding the rank of a semigroup has long been studied in the literature and there are some studies that examined some special kinds of ranks such as idempotent-rank, nilpotent-rank, or (m,r)-rank (see, for example, [1, 3, 7]). In this paper we restrict our attention to another special kind of rank, the quasi-idempotent rank of S, which is defined by

$$\mathrm{qrank}\,(S) = \min\{\,|W| : \langle W\rangle = S,\, W \subseteq Q(S)\}.$$

For any $\alpha \in P_n$, the shift and fix of α are defined by

$$\begin{aligned}\mathrm{fix}\,(\alpha) &= \{x \in \mathrm{dom}\,(\alpha) : x\alpha = x\} \text{ and}\\ \mathrm{shift}\,(\alpha) &= \{x \in \mathrm{dom}\,(\alpha) : x\alpha \neq x\} = \mathrm{dom}\,(\alpha) \setminus \mathrm{fix}\,(\alpha),\end{aligned}$$

respectively. A permutation $\alpha \in S_n$ with $\mathrm{shift}\,(\alpha) = \{a_1, \ldots, a_k\}$ $(2 \leq k \leq n)$ is called a cycle of size k (k-cycle) and denoted by $\alpha = (a_1 \ldots a_k)$ if

$$a_i\alpha = a_{i+1} \quad (1 \leq i \leq k-1) \quad \text{and} \quad a_k\alpha = a_1.$$

In particular, a 2-cycle (a_1a_2) is called a transposition. The identity permutation ε on X_n is expressible as (a), for any $1 \leq a \leq n$, and called a 1-cycle. Also, a map $\alpha \in I_n$ with $\mathrm{dom}\,(\alpha) = X_n \setminus \{a_k\}$ and $\mathrm{shift}\,(\alpha) = \{a_1, \ldots, a_{k-1}\}$ $(2 \leq k \leq n)$ is called a chain of size k (k-chain) and denoted by $[a_1 \ldots a_k]$ if $a_i\alpha = a_{i+1}$ $(1 \leq i \leq k-1)$. Moreover, a map $\alpha \in I_n$ with $\mathrm{dom}\,(\alpha) = \mathrm{fix}\,(\alpha) = X_n \setminus \{a_k\}$ called a 1-chain and denoted by $[a_k]$. Two cycles $(a_1 \ldots a_k)$ and $(b_1 \ldots b_t)$ (and similarly two chains $[a_1 \ldots a_k]$ and $[b_1 \ldots b_t]$, or a cycle $(a_1 \ldots a_k)$ and a chain $[b_1 \ldots b_t]$), for $1 \leq k, t \leq n$, are said to be disjoint if the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_t\}$ are disjoint.

Recall that every permutation can be written as a product of disjoint cycles and also as a product of transpositions. Similarly, it is well known that every map in I_n can be written as a product of disjoint cycles (1-cycles are neglected in general) and chains (see, for example, [8]). Also recall that $S_2 = \langle (12)\rangle$, $S_3 = \langle (13), (23)\rangle$, $S_n = \langle (12), (12\ldots n)\rangle$ for $n \geq 3$, and that $A_3 = \langle (123)\rangle$ and A_n is generated by two elements:

$$(123) \text{ and } \begin{cases} (12...n), & \text{if } n \text{ is odd}\\ (23...n), & \text{if } n \text{ is even,}\end{cases}$$

for $n \geq 4$. Moreover, $T_2 = \langle (12), \|12\|\rangle$, $P_2 = \langle (12), \|12\|, [1]\rangle$, and $I_2 = \langle (12), [1]\rangle$; $I_n = \langle (12), (12\ldots n), [1]\rangle$, $T_n = \langle (12), (12\ldots n), \|12\|\rangle$, and $P_n = \langle (12), (12\ldots n), \|12\|, [1]\rangle$ for $n \geq 3$ where $\|12\| = \begin{pmatrix} 1 & 2 & 3 & \cdots & n\\ 2 & 2 & 3 & \cdots & n\end{pmatrix}$ (see, for example, [5]). Furthermore,

$$\mathrm{rank}\,(S_n) = \begin{cases} 1, & n = 2\\ 2, & n \geq 3\end{cases}, \quad \mathrm{rank}\,(A_n) = \begin{cases} 1, & n = 3\\ 2, & n \geq 4\end{cases},$$

$$\mathrm{rank}\,(I_n) = \mathrm{rank}\,(T_n) = \begin{cases} 2, & n = 2\\ 3, & n \geq 3\end{cases}, \text{ and } \mathrm{rank}\,(P_n) = \begin{cases} 3, & n = 2\\ 4, & n \geq 3\end{cases}.$$

For unexplained terms see [2, 5] for semigroup theory, and see [6] for group theory.

It was shown in [4, Lemma 2.1] that a non-idempotent element $\alpha \in I_n$ is a quasi-idempotent if and only if all its orbits are of size at most 2, and so we have $\alpha \in Q(I_n)$ if and only if α can be written as a product of some disjoint 1-cycles (1-cycles are neglected in general), 1-chains, and at least one 2-cycle or/and 2-chain. For example,

$$\alpha = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 4 & 3 & 6 \end{pmatrix} = (1)[2](34)[56] = [2](34)[56] \in Q(I_6).$$

In particular, $\alpha \in Q(S_n)$ if and only if α can be written as a product of some disjoint 2-cycles. Equivalently, $\alpha \in Q(S_n)$ if and only if the order of α is 2, and so $\alpha^{-1} = \alpha$, which is also called an involution more generally. As is well known, the involutions of a group have an important role for the group's structure and they are used for classification of finite simple groups. As emphasized in [6], although there appears to be almost nothing that can be said about the structure of a subgroup generated by two elements of given orders $m \geq 1$ and $n \geq 1$ in any case other than $m = n = 2$, two involutions in any group generate a dihedral subgroup. Moreover, involutions also have an important role for group presentation, since there is no need to use the inverse of any generators in relations since the inverse of any involution is itself. Finally, we easily conclude that $\alpha \in Q(A_n)$ if and only if α can be written as a product of a positive even number of disjoint 2-cycles.

With this background, we find the quasi-idempotent ranks of S_n (for $n \geq 2$) and A_n (for $n \geq 5$). Then, by using the quasi-idempotent rank of S_n, we find the quasi-idempotent ranks of I_n (for $n \geq 2$), T_n, and P_n (for $n \geq 3$), respectively.

2. Quasi-idempotent ranks of S_n and A_n

First, for convenience we define two new notations. Let $(b_1, b_2, \ldots, b_m)$ be an ordered m-tuple for any $m \in \mathbb{Z}^+$. Then, for $2 \leq m \leq n$, let

$$[[b_1, b_2, \ldots, b_m]] = \begin{cases} (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m}{2}} b_{\frac{m}{2}+1}) & \text{if } m \text{ is an even number} \\ (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-1}{2}} b_{\frac{m+3}{2}}) & \text{if } m \text{ is an odd number,} \end{cases}$$

and, for $4 \leq m \leq n$, let

$$[[b_1, b_2, \ldots, b_m]]^\sharp = \begin{cases} (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-2}{2}} b_{\frac{m+4}{2}}) & \text{if } m \text{ is an even number} \\ (b_1 b_m)(b_2 b_{m-1}) \cdots (b_{\frac{m-3}{2}} b_{\frac{m+5}{2}}) & \text{if } m \text{ is an odd number,} \end{cases}$$

where $(b_i\, b_j)$ denotes a 2-cycle for $1 \leq i, j \leq m$.

Theorem 2.1 *For $n \geq 4$, $S_n = \langle (12), \alpha, \beta \rangle$ for each $\alpha, \beta \in Q(S_n)$ with one of the following n-many forms:*

- $\alpha = [[1, \ldots, k+1]]\,[[k+2, \ldots, n]]$,
  $\beta = [[1, \ldots, k+2]]\,[[k+3, \ldots, n]]$ $\quad (1 \leq k \leq n-4$ *and* $n \geq 5)$;
- $\alpha = [[1, \ldots, n-2]](n-1\, n)$,
  $\beta = [[1, \ldots, n-1]]$;
- $\alpha = [[1, \ldots, n-1]]$,
  $\beta = [[1, \ldots, n]]$;
- $\alpha = [[1, \ldots, n]]$,
  $\beta = [[2, \ldots, n]]$;
- $\alpha = [[2, \ldots, n]]$,
  $\beta = (12)[[3, \ldots, n]]$.

Proof The result is clear since $\alpha\beta = (12\ldots n)$ for each case. □

Corollary 2.2 $\operatorname{qrank}(S_n) = \begin{cases} 1, & n=2 \\ 2, & n=3 \\ 3, & n \geq 4 \end{cases}$.

Proof The result is clear for $n = 2, 3$ since $S_2 = \langle (12) \rangle$ and $S_3 = \langle (13), (23) \rangle$. It is also clear for $n \geq 4$ that $\operatorname{qrank}(S_n) \geq 3$ since two involutions in any group generate a dihedral subgroup. Then the result follows from Theorem 2.1, as required. ■

Now notice that $Q(A_2) = Q(A_3) = \emptyset$, $Q(A_4) = \{(12)(34), (13)(24), (14)(23)\}$, and that

$$< Q(A_4) >= \{\varepsilon, (12)(34), (13)(24), (14)(23)\} \cong C_2 \times C_2,$$

the Klein-4 group. Therefore, A_n is not quasi-idempotent generated for $n \leq 4$, and so from now on we consider the case $n \geq 5$.

Theorem 2.3 *For $n \geq 5$, $A_n = \langle \alpha, \beta, \gamma \rangle$ where*

$$\alpha = \begin{cases} (13)\,[[4,\ldots,n]]^\sharp & \text{if } n \equiv 0 \text{ mod } 4 \\ (12)\,[[4,\ldots,n]] & \text{if } n \equiv 1,2 \text{ mod } 4 \\ (12)\,[[4,\ldots,n]]^\sharp & \text{if } n \equiv 3 \text{ mod } 4 \end{cases},$$

$$\beta = \begin{cases} (23)\,(\frac{n+2}{2}\ \frac{n+6}{2}) & \text{if } n \equiv 0 \text{ mod } 4 \\ (13)\,[[4,\ldots,n]] & \text{if } n \equiv 1,2 \text{ mod } 4 \\ (13)\,(\frac{n+3}{2}\ \frac{n+5}{2}) & \text{if } n \equiv 3 \text{ mod } 4 \end{cases}, \text{ and}$$

$$\gamma = \begin{cases} (1n)(23)\,[[4,\ldots,n-1]] & \text{if } n \equiv 0 \text{ mod } 4 \\ (14)(23)\,[[5,\ldots,n]] & \text{if } n \equiv 1 \text{ mod } 4 \\ (24)\,[[5,\ldots,n]] & \text{if } n \equiv 2 \text{ mod } 4 \\ (14)\,[[5,\ldots,n]] & \text{if } n \equiv 3 \text{ mod } 4 \end{cases}.$$

Proof First of all notice that

$$[[4,\ldots,n]]^\sharp = \begin{cases} (4n)(5\,n-1)\cdots(\frac{n}{2}\ \frac{n+8}{2}) & \text{if } n \equiv 0 \text{ mod } 4 \\ (4n)(5\,n-1)\cdots(\frac{n+1}{2}\ \frac{n+7}{2}) & \text{if } n \equiv 3 \text{ mod } 4 \end{cases}.$$

Then the result is clear since:

(*i*) if $n \equiv 0$ mod 4 we have $(\alpha\beta)^4 = (123)$ and $(\alpha\beta)^4\gamma(\alpha\beta) = (2\ldots n)$;

(*ii*) if $n \equiv 1$ mod 4 we have $\alpha\beta = (123)$ and $\beta\gamma = (1\ldots n)$;

(*iii*) if $n \equiv 2$ mod 4 we have $\alpha\beta = (123)$ and $\alpha\beta\alpha\gamma = (2\ldots n)$;

(*iv*) if $n \equiv 3$ mod 4 we have $(\alpha\beta)^4 = (123)$ and $\alpha\beta\gamma = (1\ldots n)$. ■

Corollary 2.4 *For $n \geq 5$* $\operatorname{qrank}(A_n) = 3$.

Proof Clearly $\operatorname{qrank}(A_n) \geq 3$ since $\alpha^2 = \varepsilon$ for each $\alpha \in Q(A_n)$ and two involutions in any group generate a dihedral subgroup. Then the result follows from Theorem 2.3. □

3. Quasi-idempotent ranks of I_n, T_n and P_n

Lemma 3.1 *Let α and β be the quasi-idempotents with one of the n-many forms given in Theorem 2.1. Then we have $I_2 = \langle (12), [12] \rangle$, $I_3 = \langle (12), (23), [12] \rangle$, and $I_n = \langle (12), [12], \alpha, \beta \rangle$ for $n \geq 4$.*

Proof It is clear that $I_2 = \langle (12), [12] \rangle$ and $I_3 = \langle (12), (23), [12] \rangle$ since $(12)[12] = [1]$ and $(23)(12) = (123)$. For $n \geq 4$ consider the quasi-idempotents $\alpha, \beta \in Q(S_n)$ with one of the n-many forms given in Theorem 2.1. Then we have $I_n = \langle (12), [12], \alpha, \beta \rangle$ since $(12)[12] = [1]$ and $\alpha\beta = (12 \ldots n)$, as required. □

Notice that T_2 and P_2 are not quasi-idempotent generated since $Q(T_2) = \{(12)\}$ and $\langle (12) \rangle \neq T_2$, and since $Q(P_2) = \{(12), [12], [21]\}$ and $\langle Q(P_2) \rangle \neq P_2$. However, as we will show in Lemmas 3.2 and 3.3, T_n and P_n are quasi-idempotent generated for each $n \geq 3$. For ease of notations, let

$$[123|2] = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 2 & 4 & \cdots & n \end{pmatrix} \in Q(T_n).$$

Lemma 3.2 *Let α and β be the quasi-idempotents with one of the n-many forms given in Theorem 2.1. Then we have $T_3 = \langle (12), (23), [123|2] \rangle$ and $T_n = \langle (12), [123|2], \alpha, \beta \rangle$ for $n \geq 4$.*

Proof First, clearly we have $T_3 = \langle (12), (23), [123|2] \rangle$ since $(23)(12) = (123)$ and $(23)[123|2] = \|12\|$. Now let $n \geq 4$. Since $(23)[123|2] = \|12\|$ we have $\langle S_n \cup \{[123|2]\} \rangle = T_n$, and so $T_n = \langle (12), [123|2], \alpha, \beta \rangle$ where α, β are quasi-idempotents in S_n with one of the n-many forms given in Theorem 2.1. □

Lemma 3.3 *Let α and β be the quasi-idempotents with one of the n-many forms given in Theorem 2.1. Then we have $P_3 = \langle (12), (23), [12], [123|2] \rangle$ and $P_n = \langle (12), [12], [123|2], \alpha, \beta \rangle$ for $n \geq 4$.*

Proof First clearly we have $P_3 = \langle (12), (23), [12], [123|2] \rangle$ since $(23)(12) = (123)$, $(12)[12] = [1]$ and $(23)[123|2] = \|12\|$. Now let $n \geq 4$. Since $(12)[12] = [1]$ and $(23)[123|2] = \|12\|$ we have $\langle I_n \cup \{[123|2]\} \rangle = P_n$, and so we have $P_n = \langle (12), [12], [123|2], \alpha, \beta \rangle$ where α, β are quasi-idempotents in S_n with one of the n-many forms given in Theorem 2.1. □

For the proof of Theorem 3.4 we use the following well-known property. Let S be a finite semigroup and let T be a subsemigroup of S such that $S \setminus T$ is an ideal of S. It is well known that if $S = \langle A \rangle$, for any $A \subseteq S$, then $T = \langle T \cap A \rangle$, and so any generating set of S must contain at least one extra element in addition to any generating set of T. Therefore, $\text{rank}\,(S) \geq \text{rank}\,(T) + 1$. Similarly, $\text{qrank}\,(S) \geq \text{qrank}\,(T) + 1$ when S and T are generated by their own quasi-idempotents.

Theorem 3.4 *We have*

$$\text{qrank}\,(I_n) = \begin{cases} 2, & n = 2 \\ 3, & n = 3 \\ 4, & n \geq 4 \end{cases}, \quad \text{qrank}\,(T_n) = \begin{cases} 3, & n = 3 \\ 4, & n \geq 4 \end{cases},$$

and

$$\text{qrank}\,(P_n) = \begin{cases} 4, & n = 3 \\ 5, & n \geq 4 \end{cases}.$$

Proof From Lemmas 3.1, 3.2, and 3.3, the result is clear for $n = 2, 3$. Now let $n \geq 4$. Recall that $\mathrm{qrank}\,(I_n) \geq \mathrm{qrank}\,(S_n) + 1$ since $I_n \setminus S_n$ is an ideal of I_n; $\mathrm{qrank}\,(T_n) \geq \mathrm{qrank}\,(S_n) + 1$ since $T_n \setminus S_n$ is an ideal of T_n; and $\mathrm{qrank}\,(P_n) \geq \mathrm{qrank}\,(T_n) + 1$ since $P_n \setminus T_n$ is an ideal of P_n. Thus, the result is also clear for $n \geq 4$ from Lemmas 3.1, 3.2, and 3.3. □

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