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Unpredictable solutions of linear differential and discrete equations

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Abstract: The existence and uniqueness of unpredictable solutions in the dynamics of nonhomogeneous linear systems of differential and discrete equations are investigated. The hyperbolic cases are under discussion. The presence of unpredictable solutions confirms the existence of Poincaré chaos. Simulations illustrating the chaos are provided.

Key words: Unpredictable solutions, Poincaré chaos, linear nonhomogeneous systems

1. Introduction and preliminaries

The concept of unpredictable functions was introduced in [1]. The work in [2] was devoted to the investigation of sufficient conditions for the existence of unpredictable solutions of retarded quasilinear differential equations in the case that all eigenvalues of the matrix of coefficients admit negative real parts.

In this paper we investigate the existence and uniqueness of unpredictable solutions of linear differential and discrete equations in which unpredictable perturbations are used (Theorems 2.2 and 3.1). The present study has two principal novelties with respect to the previous results in the field. The first one is that we consider the hyperbolic cases such that the eigenvalues of the matrix of coefficients can admit positive real parts in the case of differential equations. The second one is that we propose a simpler and more comprehensible proof. Additionally, we consider new properties of unpredictable functions (Lemmas 1.4 and 1.5). An example of a piecewise constant unpredictable function is constructed. Moreover, a continuous function as a solution of a linear nonhomogeneous scalar equation has been found, which approximates an unpredictable function asymptotically. These functions are used in numerical simulations, which confirm the theoretical result of the paper.

It was shown in [1, 2] that the existence of an unpredictable solution implies Poincaré chaos so that irregularity is observable in the solutions. In such a case, the chaos is present in the dynamics where unpredictable functions are considered as points moving by shifts of the time argument [8], and the irregularity of the solutions is a consequence of the chaotic dynamics. The reason for the appearance of Poincaré chaos is the fact that the topology of uniform convergence on compact sets of the real axis is metrizable [8].

In the remaining parts of the paper, we will make use of the usual Euclidean norm for vectors and the norm induced by the Euclidean norm for square matrices.

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Definition 1.1 *[2] A uniformly continuous and bounded function* $\vartheta : \mathbb{R} \to \mathbb{R}^m$ *is unpredictable if there exist positive numbers* ϵ_0 *,* δ *and sequences* $\{t_n\}$ *,* $\{u_n\}$ *both of which diverge to infinity such that* $\|\vartheta(t+t_n)-\vartheta(t)\| \to 0$ *as* $n \to \infty$ *uniformly on compact subsets of* $\mathbb{R}$ *and* $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \epsilon_0$ *for each* $t \in [u_n - \delta, u_n + \delta]$ *and* $n \in \mathbb{N}$*.*

Definition 1.2 *[2] A bounded sequence* $\{\kappa_i\}$*,* $i \in \mathbb{Z}$*, in* $\mathbb{R}^m$ *is called unpredictable if there exist a positive number* ϵ_0 *and the sequences* $\{\zeta_n\}, \{\eta_n\}$*,* $n \in \mathbb{N}$*, of positive integers both of which diverge to infinity such that* $\|\kappa_{i+\zeta_n} - \kappa_i\| \to 0$ *as* $n \to \infty$ *for each* i *in bounded intervals of integers and* $\|\kappa_{\zeta_n+\eta_n} - \kappa_{\eta_n}\| \geq \epsilon_0$ *for each* $n \in \mathbb{N}$.

Definitions 1.1 and 1.2 will be utilized in Sections 2 and 3, respectively. Next, let us recall the definition of a Poisson stable function [8], adapted to our case.

Definition 1.3 *A continuous and bounded function* $\vartheta : \mathbb{R} \to \mathbb{R}^m$ *is positively Poisson stable if there exists a sequence* $\{t_n\}$*,* $t_n \to \infty$ *as* $n \to \infty$*, such that* $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ *as* $n \to \infty$ *uniformly on compact subsets of* $\mathbb{R}$.

By comparing Definitions 1.1 and 1.3 one can see that any unpredictable function is Poisson stable. The Poisson stable function is a specification of a Poisson stable point considered for dynamical systems in [8] and [7].

It is worth noting that in the literature a large number of results have been obtained for periodic, quasiperiodic, and almost periodic solutions of differential equations due to the established mathematical methods and important applications. On the other hand, recurrent and Poisson stable solutions are also crucial for the theory of differential equations [3, 8]. This study can revive the interests of specialists in differential equations theory for two reasons. The first one is related to the verification of the unpredictability, which requests a more sophisticated technique than for recurrent and Poisson stable solutions. Therefore, the problem of the existence of unpredictable solutions is a challenging one. Secondly unpredictable solutions can be investigated for various types of differential equations such as partial differential equations, evolution equations, and hybrid systems. Thus, a new approach for chaos extension in many types of dynamics is suggested in our study.

The following lemmas can be useful for applications of our results.

Lemma 1.4 *Suppose that* $\phi(t) : \mathbb{R} \to \mathbb{R}$ *is an unpredictable function. Then the function* $\phi^3(t)$ *is unpredictable.*

Proof One can find numbers $\epsilon_0 > 0$, $\delta > 0$ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\|\phi(t+t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t+t_n) - \phi(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. It is easy to check that $\|\phi^3(t+t_n) - \phi^3(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$, since it follows from the uniform continuity of the cubic function on a compact set.

Fix a natural number n. Let us show that for $t \in [u_n - \delta, u_n + \delta]$ the inequality $\|\phi(t+t_n) - \phi(t)\| \geq \epsilon_0$ implies $\|\phi^3(t+t_n) - \phi^3(t)\| \geq \epsilon_0^3/4$.

We have that

$$\begin{aligned} |\phi^3(t+t_n) - \phi^3(t)| &= \frac{1}{2}|\phi(t+t_n) - \phi(t)| \left[\phi^2(t+t_n) + \phi^2(t) + (\phi(t+t_n) + \phi(t))^2\right] \\ &\geq \frac{1}{2}\left(\phi^2(t+t_n) + \phi^2(t)\right)\epsilon_0. \end{aligned}$$

Consider the function $F(a,b) = a^2 + b^2$ for $|a-b| \geq \epsilon_0$. The minimum of F occurs at the points (a,b) with $|a| = |b| = \epsilon_0/2$. Therefore, $|\phi^3(t+t_n) - \phi^3(t)| \geq \epsilon_0^3/4$ for $t \in [u_n - \delta, u_n + \delta]$. □

Lemma 1.5 *If the function $\phi(t) : \mathbb{R} \to \mathbb{R}$ is unpredictable, then the function $\phi(t) + C$, where C is a constant, is also unpredictable.*

Proof There exist positive numbers ϵ_0, δ and sequences $\{t_n\}$, $\{u_n\}$, both of which diverge to infinity such that $\|\phi(t+t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t+t_n) - \phi(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. Let us denote $\omega(t) = \phi(t) + C$. Then we have that $\|\omega(t+t_n) - \omega(t)\| = \|\phi(t+t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\omega(t+t_n) - \omega(t)\| = \|\phi(t+t_n) - \phi(t)\| \geq \epsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. Therefore, the function $\phi(t) + C$ is unpredictable. □

2. Linear differential equations with unpredictable solutions

Let us consider the following system of linear differential equations:

$$x'(t) = Ax(t) + g(t), \tag{2.1}$$

where $x \in \mathbb{R}^p$ and the function $g : \mathbb{R} \to \mathbb{R}^p$ is uniformly continuous and bounded. Moreover, we assume that all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts.

Assume that the following condition is valid.

(C) $\Re e\lambda_i < 0$, $i = 1, 2, \ldots, r$, and $\Re e\lambda_i > 0$, $i = r+1, r+2, \ldots, p$, $1 \leq r < p$, where λ_i, $i = 1, \ldots, p$, are the eigenvalues of the matrix A and $\Re e\lambda_i$ denotes the real part of λ_i.

One can find a nonsingular matrix B such that the substitution $x = By$ transforms system (2.1) into the equation

$$y'(t) = B^{-1}ABy(t) + B^{-1}g(t), \tag{2.2}$$

with the block diagonal matrix of coefficients [5]. Therefore, we assume without loss of generality that the matrix A in system (2.1) is block diagonal such that $A = \operatorname{diag}(A_-, A_+)$, where the eigenvalues of the matrices A_- and A_+ possess negative and positive real parts, respectively. There exist numbers $K \geq 1$ and $\alpha > 0$ such that $\|e^{A_- t}\| \leq Ke^{-\alpha t}$ for $t \geq 0$ and $\|e^{A_+ t}\| \leq Ke^{\alpha t}$ for $t \leq 0$.

From equation (2.2), it implies that the following auxiliary assertion is needed.

Lemma 2.1 *If the function $g(t)$ is unpredictable, then the function $f(t) = B^{-1}g(t)$ is also unpredictable.*

The proof of the lemma immediately follows from the inequalities $\|f(t+t_n) - f(t)\| \leq \|B^{-1}\|\|g(t+t_n) - g(t)\|$ and $\|f(t+t_n) - f(t)\| \geq \dfrac{1}{\|B\|}\|g(t+t_n) - g(t)\|$.

In what follows we will denote $g(t) = (g_-(t), g_+(t))$, where the vector-functions $g_-(t)$ and $g_+(t)$ are of dimensions r and $p - r$, respectively.

As is known from the theory of differential equations [5], system (2.1) admits a unique bound on $\mathbb{R}$ solution $\varphi(t) = (\varphi_-(t), \varphi_+(t))$,

$$\varphi_-(t) = \int_{-\infty}^{t} e^{A_-(t-s)} g_-(s) ds, \quad \varphi_+(t) = -\int_{t}^{\infty} e^{A_+(t-s)} g_+(s) ds, \tag{2.3}$$

if the function $g(t)$ is bounded on $\mathbb{R}$. One can confirm that $\sup_{t\in\mathbb{R}} \|\varphi(t)\| \le \frac{2M_g K}{\alpha}$, where $M_g = \sup_{t\in\mathbb{R}} \|g(t)\|$. Moreover, $\varphi(t)$ is periodic, quasiperiodic, or almost periodic if the perturbation function $g(t)$ is respectively of the same type.

The following theorem is concerned with the unpredictable solution of system (2.1).

Theorem 2.2 *Assume that $g(t)$ is an unpredictable function and condition (C) is valid. Then system (2.1) possesses a unique unpredictable solution. Additionally, if all eigenvalues of matrix A have negative real parts, then the unpredictable solution is uniformly asymptotically stable.*

Proof The boundedness of $g(t)$ implies that system (2.1) admits a unique bounded solution $\varphi(t) = (\varphi_-(t), \varphi_+(t))$, which satisfies (2.3), and it is uniformly continuous since its derivative is bounded. Moreover, the bounded solution is uniformly asymptotically stable provided that all eigenvalues of matrix A have negative real parts. Hence, it is sufficient to prove that $\varphi(t)$ is an unpredictable function.

The function $\varphi(t)$ is uniformly continuous since its derivative is bounded. According to the Poisson stability of $g(t)$, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\|g(t+t_n) - g(t)\| \to 0$ uniformly on compact subsets of $\mathbb{R}$. One can easily find that

$$\begin{aligned} \|\varphi_-(t+t_n) - \varphi_-(t)\| &= \left\| \int_{-\infty}^{t} e^{A_-(t-s)} [g_-(s+t_n) - g_-(s)] ds \right\| \\ &\le \int_{-\infty}^{t} K e^{-\alpha(t-s)} \|g_-(s+t_n) - g_-(s)\| ds \end{aligned}$$

and

$$\begin{aligned} \|\varphi_+(t+t_n) - \varphi_+(t)\| &= \left\| \int_{t}^{\infty} e^{A_+(t-s)} [g_+(s+t_n) - g_+(s)] ds \right\| \\ &\le \int_{t}^{\infty} K e^{\alpha(t-s)} \|g_+(s+t_n) - g_+(s)\| ds. \end{aligned}$$

Fix an arbitrary positive number ϵ and a closed interval $[a,b]$, $-\infty < a < b < \infty$, of the real axis. We will show that for sufficiently large n it is true that $\|\varphi(t+t_n) - \varphi(t)\| < \epsilon$ on $[a,b]$. Let us choose numbers $c < a$, $d > b$, $\xi > 0$ such that $\frac{2M_g K}{\alpha} e^{-\alpha(a-c)} \le \frac{\epsilon}{4}$, $\frac{2M_g K}{\alpha} e^{-\alpha(d-b)} \le \frac{\epsilon}{4}$, and $\frac{K\xi}{\alpha} \le \frac{\epsilon}{4}$.

Consider n sufficiently large such that $\|g(t+t_n) - g(t)\| < \xi$ for $t \in [c,d]$. Then we have for all $t \in [a,b]$

that

$$\begin{aligned}
\|\varphi_-(t+t_n)-\varphi_-(t)\| &\le \int_{-\infty}^{c} Ke^{-\alpha(t-s)}\|g_-(s+t_n)-g_-(s)\|ds \\
&+\int_{c}^{t} Ke^{-\alpha(t-s)}\|g_-(s+t_n)-g_-(s)\|ds \\
&\le \int_{-\infty}^{c} 2M_gKe^{-\alpha(t-s)}ds+\int_{c}^{t} K\xi e^{-\alpha(t-s)}ds \\
&< \frac{2M_gK}{\alpha}e^{-\alpha(a-c)}+\frac{K\xi}{\alpha} \\
&\le \frac{\epsilon}{2}
\end{aligned}$$

and similarly one can show that

$$\begin{aligned}
\|\varphi_+(t+t_n)-\varphi_+(t)\| &\le \int_{t}^{d} Ke^{\alpha(t-s)}\|g_+(s+t_n)-g_+(s)\|ds \\
&+\int_{d}^{\infty} Ke^{\alpha(t-s)}\|g_+(s+t_n)-g_+(s)\|ds \\
&\le \int_{t}^{d} K\xi e^{\alpha(t-s)}ds+\int_{d}^{\infty} 2M_gKe^{\alpha(t-s)}ds \\
&< \frac{K\xi}{\alpha}+\frac{2M_gK}{\alpha}e^{-\alpha(d-b)} \\
&\le \frac{\epsilon}{2}.
\end{aligned}$$

Thus, for sufficiently large n, it is true that

$$\|\varphi(t+t_n)-\varphi(t)\| \le \|\varphi_+(t+t_n)-\varphi_+(t)\|+\|\varphi_-(t+t_n)-\varphi_-(t)\| < \epsilon$$

for $t\in[a,b]$.

Next, we will show the existence of a sequence $\{\bar{u}_n\}, \bar{u}_n \to \infty$ as $n\to\infty$, and positive numbers $\bar{\epsilon}_0, \delta$ such that $\|\varphi(t+t_n)-\varphi(t_n)\| \ge \bar{\epsilon}_0$ for $t\in[\bar{u}_n-\delta, \bar{u}_n+\delta]$.

According to the uniform continuity of $g(t)$, there exists a positive number $\bar{\delta}$, which does not depend on the sequences $\{t_n\}$ and $\{u_n\}$, such that the inequalities

$$\|g(t+t_n)-g(t_n+u_n)\| \le \frac{\epsilon_0}{4\sqrt{p}}$$

and

$$\|g(t)-g(u_n)\| \le \frac{\epsilon_0}{4\sqrt{p}}$$

are valid for every $t\in\left[u_n-\bar{\delta}, u_n+\bar{\delta}\right]$ and $n\in\mathbb{N}$.

Fix an arbitrary natural number n, and suppose that $g(t)=(g_1(t), g_2(t), ..., g_p(t))$, where each $g_k(t)$, $k=1,2,\ldots,p$, is a real valued function. It can be verified that there exists an integer j_n, $1\le j_n\le p$, such that

$$|g_{j_n}(t_n+u_n)-g_{j_n}(u_n)| \geq \frac{\epsilon_0}{\sqrt{p}}.$$

Hence, we have

$$\begin{aligned} |g_{j_n}(t+t_n)-g_{j_n}(t)| &\geq |g_{j_n}(t_n+u_n)-g_{j_n}(u_n)|-|g_{j_n}(t+t_n)-g_{j_n}(t_n+u_n)| \\ &-|g_{j_n}(t)-g_{j_n}(u_n)| \\ &\geq \frac{\epsilon_0}{2\sqrt{p}} \end{aligned} \tag{2.4}$$

for $t \in \left[u_n-\bar{\delta}, u_n+\bar{\delta}\right]$.

One can confirm that there exist numbers $s_1^n, s_2^n, ..., s_p^n$ in the interval $[u_n-\bar{\delta}, u_n+\bar{\delta}]$ such that the equation

$$\Big\| \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} (g(s+t_n)-g(s))ds \Big\| = 2\bar{\delta}\Big[\sum_{i=1}^{p}(g_i(s_i^n+t_n)-g_i(s_i^n))^2\Big]^{1/2}$$

is valid. Using inequality (2.4) we obtain that

$$\Big\| \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} (g(s+t_n)-g(s))ds \Big\| \geq 2\bar{\delta}|g_{j_n}(s_{j_n}^n+t_n)-g_{j_n}(s_{j_n}^n)| \geq \frac{\bar{\delta}\epsilon_0}{\sqrt{p}}.$$

By means of the equation

$$\begin{aligned} \varphi(t_n+u_n+\bar{\delta})-\varphi(u_n+\bar{\delta}) &= \varphi(t_n+u_n-\bar{\delta})-\varphi(u_n-\bar{\delta})+\int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} A[\varphi(s+t_n)-\varphi(s)]ds \\ &+\int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} [g(s+t_n)-g(s)]ds, \end{aligned}$$

one can obtain that

$$\|\varphi(t_n+u_n+\bar{\delta})-\varphi(u_n+\bar{\delta})\| \geq \frac{\bar{\delta}\epsilon_0}{\sqrt{p}}-(1+2\bar{\delta}\|A\|)\sup_{t\in[u_n-\bar{\delta},u_n+\bar{\delta}]}\|\varphi(t+t_n)-\varphi(t)\|.$$

Thus, $\sup_{t\in[u_n-\bar{\delta},u_n+\bar{\delta}]}\|\varphi(t+t_n)-\varphi(t)\| \geq \dfrac{\bar{\delta}\epsilon_0}{2(1+\bar{\delta}\|A\|)\sqrt{p}}$.

Now, suppose that $\sup_{t\in[u_n-\bar{\delta},u_n+\bar{\delta}]}\|\varphi(t+t_n)-\varphi(t)\| = \|\varphi(t_n+\bar{u}_n)-\varphi(\bar{u}_n)\|$ for some $\bar{u}_n \in [u_n-\bar{\delta}, u_n+\bar{\delta}]$, and let us denote

$$\bar{\epsilon}_0 = \frac{\bar{\delta}\epsilon_0}{4(1+\bar{\delta}\|A\|)\sqrt{p}}$$

and

$$\delta = \frac{\bar{\delta}\alpha\epsilon_0}{8M_g(1+\bar{\delta}\|A\|)\left(\alpha+2K\|A\|\right)\sqrt{p}}.$$

If $t \in [\bar{u}_n - \delta, \bar{u}_n + \delta]$, then we have

$$\begin{aligned}\|\varphi(t+t_n) - \varphi(t)\| &\geq \|\varphi(t_n + \bar{u}_n) - \varphi(\bar{u}_n)\| - \left| \int_{\bar{u}_n}^{t} \|A\| \varphi(s+t_n) - \varphi(s)\| ds \right| \\ &- \left| \int_{\bar{u}_n}^{t} \|g(s+t_n) - g(s)\| ds \right| \\ &\geq \frac{\bar{\delta}\epsilon_0}{2(1+\bar{\delta}\|A\|)\sqrt{p}} - \frac{4\delta M_g K \|A\|}{\alpha} - 2\delta M_g \\ &= \bar{\epsilon}_0.\end{aligned}$$

Hence, $\|\varphi(t+t_n) - \varphi(t)\| \geq \bar{\epsilon}_0$ for each t from the intervals $[\bar{u}_n - \delta, \bar{u}_n + \delta]$, $n \in \mathbb{N}$. One can confirm that the sequence $\{\bar{u}_n\}$ diverges to infinity. Consequently, $\varphi(t)$ is the unique unpredictable solution of system (2.1). □

The next section is devoted to unpredictable solutions of linear discrete equations.

3. Linear discrete equations with unpredictable solutions

Let us take into account the discrete equation

$$z_{i+1} = D z_i + \phi_i, \tag{3.1}$$

where $i \in \mathbb{Z}$, $D \in \mathbb{R}^{q \times q}$ is a nonsingular matrix, and $\{\phi_i\}$ is a bounded sequence. In this section the following condition is required.

(D) $D = \text{diag}(D_-, D_+)$, where D_- and D_+ are respectively $k \times k$ and $(q-k) \times (q-k)$ matrices with $0 \leq k \leq q$ such that $\|D_-\| < 1$ and $\|D_+\| > 1$.

We will denote $\phi_i = (\phi_i^-, \phi_i^+)$, where ϕ_i^-, ϕ_i^+ are respectively k and $q-k$ dimensional.

According to the results of [6], equation (3.1) possesses a unique bounded solution $\psi_i = (\psi_i^-, \psi_i^+)$, $i \in \mathbb{Z}$, which satisfies the relations

$$\psi_i^- = \sum_{j=-\infty}^{i} D_-^{i-j} \phi_{j-1}^- \tag{3.2}$$

and

$$\psi_i^+ = -\sum_{j=i}^{\infty} D_+^{i-j-1} \phi_j^+. \tag{3.3}$$

One can verify for each $i \in \mathbb{Z}$ that $\|\psi_i^-\| \leq \dfrac{M_\phi}{1 - \|D_-\|}$ and $\|\psi_i^+\| \leq \dfrac{M_\phi}{\|D_+\| - 1}$, where $M_\phi = \sup_{i \in \mathbb{Z}} \|\phi_i\|$.

The following theorem is concerned with the existence of an unpredictable solution of equation (3.1).

Theorem 3.1 *If $\{\phi_i\}$, $i \in \mathbb{Z}$, is an unpredictable sequence and the condition (D) is valid, then the equation (3.1) possesses a unique unpredictable solution.*

Proof Because the sequence $\{\phi_i\},\ i\in\mathbb{Z}$, is unpredictable, according to Definition 1.2, there exist a positive number ϵ_0 and sequences $\{\zeta_n\},\{\eta_n\},\ n\in\mathbb{N}$, of positive integers both of which diverge to infinity such that $\|\phi_{i+\zeta_n}-\phi_i\|\to 0$ as $n\to\infty$ on each bounded intervals of integers and $\|\phi_{\zeta_n+\eta_n}-\phi_{\eta_n}\|\geq\epsilon_0$ for each $n\in\mathbb{N}$. We will show that the unique bounded solution $\psi_i=(\psi_i^-,\psi_i^+),\ i\in\mathbb{Z}$, of (3.1) is unpredictable.

It is easy to check that

$$\psi^-_{i+\zeta_n}-\psi^-_i=\sum_{j=-\infty}^{i+\zeta_n}D_-^{i+\zeta_n-j}\phi^-_{j-1}-\sum_{j=-\infty}^{i}D_-^{i-j}\phi^-_{j-1}=\sum_{j=-\infty}^{i}D_-^{i-j}(\phi^-_{j+\zeta_n-1}-\phi^-_{j-1}),$$

and

$$\psi^+_{i+\zeta_n}-\psi^+_i=-\sum_{j=i+\zeta_n}^{\infty}D_+^{i+\zeta_n-j-1}\phi^+_j+\sum_{j=i}^{\infty}D_+^{i-j-1}\phi^+_j=\sum_{j=i}^{\infty}D_+^{i-j-1}(\phi^+_j-\phi^+_{j+\zeta_n}).$$

Fix an arbitrary positive number ϵ and let a and b be integers with $a<b$. We will show that for sufficiently large n it is true that $\|\psi_{i+\zeta_n}-\psi_i\|<\epsilon$ for $a\leq i\leq b$. Let us choose integers $c<a,\ b<d,\ \xi>0$ such that $\dfrac{1}{1-\|D_-\|}(2M_\phi\|D_-\|^{a-c}+\xi)<\dfrac{\epsilon}{2}$ and $\dfrac{1}{\|D_+\|-1}(\xi+2M_\phi\|D_+\|^{d-b+1})<\dfrac{\epsilon}{2}$.

Consider n sufficiently large such that $\|\psi_{i+\zeta_n}-\psi_i\|<\xi$ for $c\leq i\leq d$. Then we have for $a\leq i\leq b$ that

$$\begin{aligned}\|\psi^-_{i+\zeta_n}-\psi^-_i\|&=\|\sum_{j=-\infty}^{c}D_-^{i-j}(\phi^-_{j+\zeta_n-1}-\phi^-_{j-1})+\sum_{j=c}^{i}D_-^{i-j}(\phi^-_{j+\zeta_n-1}-\phi^-_{j-1})\|\\&\leq\frac{2M_\phi\|D_-\|^{a-c}}{1-\|D_-\|}+\frac{\xi}{1-\|D_-\|}<\frac{\epsilon}{2},\end{aligned}$$

and similarly,

$$\begin{aligned}\|\psi^+_{i+\zeta_n}-\psi^+_i\|&=\|\sum_{j=i}^{d}D_+^{i-j-1}(\phi^+_j-\phi^+_{j+\zeta_n})+\sum_{j=d}^{\infty}D_+^{i-j}(\phi^+_j-\phi^+_{j+\zeta_n})\|\\&\leq\frac{\xi}{\|D_+\|-1}+\frac{2M_\phi\|D_+\|^{b-d+1}}{\|D_+\|-1}<\frac{\epsilon}{2}.\end{aligned}$$

Thus, $\|\psi_{i+\zeta_n}-\psi_i\|\leq\|\psi^+_{i+\zeta_n}-\psi^+_i\|+\|\psi^-_{i+\zeta_n}-\psi^-_i\|<\epsilon$ for $a\leq i\leq b$, and hence, $\|\psi_{i+\zeta_n}-\psi_i\|\to 0$ as $n\to\infty$ on each bounded interval of integers.

On the other hand, the equation

$$\psi_{\zeta_n+\eta_n+1}-\psi_{\eta_n+1}=D\left(\psi_{\zeta_n+\eta_n}-\psi_{\eta_n}\right)+\phi_{\zeta_n+\eta_n}-\phi_{\eta_n}$$

implies that

$$\|\psi_{\zeta_n+\eta_n+1}-\psi_{\eta_n+1}\|\geq\epsilon_0-\|D\|\|\psi_{\zeta_n+\eta_n}-\psi_{\eta_n}\|.$$

Therefore, the inequality

$$(1+\|D\|)\max\{\|\psi_{\zeta_n+\eta_n+1}-\psi_{\eta_n+1}\|,\|\psi_{\zeta_n+\eta_n}-\psi_{\eta_n}\|\}\geq\epsilon_0$$

holds.

Let us define the sequence $\overline{\eta}_n$, $n \in \mathbb{N}$, such that $\overline{\eta}_n = \eta_n$ if $\|\psi_{\zeta_n+\eta_n} - \psi_{\eta_n}\| \geq \|\psi_{\zeta_n+\eta_n+1} - \psi_{\eta_n+1}\|$ and $\overline{\eta}_n = \eta_n + 1$ otherwise. Accordingly, we have that

$$\|\psi_{\zeta_n+\overline{\eta}_n} - \psi_{\overline{\eta}_n}\| \geq \frac{\epsilon_0}{1+\|D\|}$$

for each $n \in \mathbb{N}$. It is clear that $\overline{\eta}_n \to \infty$ as $n \to \infty$. Consequently, the bounded solution $\psi_i, i \in \mathbb{Z}$, of (3.1) is unpredictable. □

4. Examples

It was shown in [2] that the presence of an unpredictable function is inevitably accompanied by Poincaré chaos. Consequently, we can look for a confirmation of the results for unpredictability observing irregularity in simulations. The approach is effective for asymptotically stable unpredictable solutions, and it is just as illustrative for hyperbolic systems with unstable solutions. In the latter case we rely on the fact that any solution becomes unpredictable ultimately.

In this section, first of all, we will show the construction of an unpredictable function using the dynamics of the logistic map in a similar way to the method applied in [2].

Let us take into account the logistic map

$$\lambda_{i+1} = F_\mu(\lambda_i), \tag{4.1}$$

where $i \in \mathbb{Z}$ and $F_\mu(s) = \mu s(1-s)$. The interval $[0,1]$ is invariant under the iterations of (4.1) for $\mu \in (0,4]$ [4].

Using the results of [9], it was shown in Theorem 4.1 [1] that the logistic map (4.1) possesses an unpredictable solution for each $\mu \in [3+(2/3)^{1/2}, 4]$.

Let $\{\rho_i\}$, $i \in \mathbb{Z}$, be an unpredictable solution of the logistic map (4.1) with $\mu = 3.92$ inside the unit interval $[0,1]$, and consider the function

$$\Theta(t) = \int_{-\infty}^{t} e^{-5(t-s)/2}\Omega(s)ds, \tag{4.2}$$

where $\Omega(t)$ is the piecewise constant function defined on the real axis through the equation $\Omega(t) = \rho_i$ for $t \in [i, i+1)$, $i \in \mathbb{Z}$.

It is worth noting that $\Theta(t)$ is the unique globally exponentially stable solution of the differential equation

$$v'(t) = -\frac{5}{2}v(t) + \Omega(t).$$

Additionally, one can confirm that the function $\Theta(t)$ is bounded on the whole real axis such that

$$\sup_{t\in\mathbb{R}} |\Theta(t)| \leq \frac{2}{5},$$

and it is uniformly continuous since its derivative is bounded.

Because the sequence $\{\rho_i\}$ is unpredictable, there exist a positive number ϵ_0 and sequences $\{\zeta_n\}, \{\eta_n\}$ both of which diverge to infinity such that $|\rho_{i+\zeta_n} - \rho_i| \to 0$ as $n \to \infty$ for each i in bounded intervals of integers and $|\rho_{\zeta_n+\eta_n} - \rho_{\eta_n}| \geq \epsilon_0$ for each $n \in \mathbb{N}$.

Fix an arbitrary positive number ϵ and arbitrary real numbers α, β with $\beta > \alpha$. Let N be a sufficiently large natural number satisfying

$$N \geq \frac{2}{5} \ln\left(\frac{6}{5\epsilon}\right).$$

There exists a natural number n_0 such that for each $n \geq n_0$ the inequality

$$|\rho_{i+\zeta_n} - \rho_i| < \frac{5\epsilon}{6}$$

holds for $i = \lfloor\alpha\rfloor - N, \lfloor\alpha\rfloor - N + 1, ..., \lfloor\beta\rfloor$, where $\lfloor\alpha\rfloor$ and $\lfloor\beta\rfloor$ respectively denote the largest integers which are not greater than α and β. Accordingly, if $n \geq n_0$, then we have

$$|\Omega(t + \zeta_n) - \Omega(t)| < \frac{5\epsilon}{6} \tag{4.3}$$

for $t \in \big[\lfloor\alpha\rfloor - N, \lfloor\beta\rfloor + 1\big)$.

Fix a natural number $n \geq n_0$. Using the relations

$$\Theta(t) = e^{-5(t-\lfloor\alpha\rfloor+N)/2}\Theta\left(\lfloor\alpha\rfloor - N\right) + \int_{\lfloor\alpha\rfloor-N}^{t} e^{-5(t-s)/2}\Omega(s)ds$$

and

$$\Theta\left(t + \zeta_n\right) = e^{-5(t-\lfloor\alpha\rfloor+N)/2}\Theta\left(\lfloor\alpha\rfloor - N + \zeta_n\right) + \int_{\lfloor\alpha\rfloor-N}^{t} e^{-5(t-s)/2}\Omega(s + \zeta_n)ds$$

together with inequality (4.3), we obtain for $t \in \big[\lfloor\alpha\rfloor - N, \lfloor\beta\rfloor + 1\big)$ that

$$\begin{aligned}
|\Theta(t + \zeta_n) - \Theta(t)| &\leq e^{-5(t-\lfloor\alpha\rfloor+N)/2}\left|\Theta(\lfloor\alpha\rfloor - N + \zeta_n) - \Theta(\lfloor\alpha\rfloor - N)\right| \\
&+ \int_{\lfloor\alpha\rfloor-N}^{t} e^{-5(t-s)/2}\left|\Omega(s + \zeta_n) - \Omega(s)\right| ds \\
&\leq \frac{4}{5}e^{-5(t-\lfloor\alpha\rfloor+N)/2} + \frac{\epsilon}{3}\left(1 - e^{-5(t-\lfloor\alpha\rfloor+N)/2}\right).
\end{aligned}$$

The last inequality implies that $|\Theta(t+\zeta_n)-\Theta(t)| < \epsilon$ for $t \in [\lfloor\alpha\rfloor, \lfloor\beta\rfloor + 1]$. Hence, $|\Theta(t + \zeta_n) - \Theta(t)| \to 0$ as $n \to \infty$ uniformly on the interval $[\alpha, \beta]$.

On the other hand, one can confirm for each $n \in \mathbb{N}$ that $|\Omega(t + \zeta_n) - \Omega(t)| \geq \epsilon_0$ for $t \in [\eta_n, \eta_n + 1)$. For fixed $n \in \mathbb{N}$, using the equation

$$\Theta(t + \zeta_n) - \Theta(t) = \Theta(\zeta_n + \eta_n) - \Theta(\eta_n) - \frac{5}{2}\int_{\eta_n}^{t} \left(\Theta(s + \zeta_n) - \Theta(s)\right) ds + \int_{\eta_n}^{t} \left(\Omega(s + \zeta_n) - \Omega(s)\right) ds$$

we attain that

$$\begin{aligned}|\Theta(\zeta_n+\eta_n+1)-\Theta(\eta_n+1)| &\ge \left|\int_{\eta_n}^{\eta_n+1}(\Omega(s+\zeta_n)-\Omega(s))\,ds\right|-|\Theta(\zeta_n+\eta_n)-\Theta(\eta_n)|\\ &-\frac{5}{2}\left|\int_{\eta_n}^{\eta_n+1}(\Theta(s+\zeta_n)-\Theta(s))\,ds\right|.\end{aligned}$$

Therefore, one can verify that

$$\sup_{t\in[\eta_n,\eta_n+1]}|\Theta(t+\zeta_n)-\Theta(t)|\ge\frac{2\epsilon_0}{9}.$$

Thus, there exists a sequence $\{u_n\}$ with $\eta_n\le u_n\le\eta_n+1$, $n\in\mathbb{N}$, such that

$$|\Theta(u_n+\zeta_n)-\Theta(u_n)|\ge\frac{2\epsilon_0}{9}.$$

For $t\in[u_n-\delta,u_n+\delta]$, where $\delta=\epsilon_0/36$, we have

$$\begin{aligned}|\Theta(t+\zeta_n)-\Theta(t)| &\ge|\Theta(u_n+\zeta_n)-\Theta(u_n)|-\frac{5}{2}\left|\int_{u_n}^{t}|\Theta(s+\zeta_n)-\Theta(s)|\,ds\right|\\ &-\left|\int_{u_n}^{t}|\Omega(s+\zeta_n)-\Omega(s)|\,ds\right|\\ &\ge\frac{\epsilon_0}{9}.\end{aligned}$$

It is clear that $u_n\to\infty$ as $n\to\infty$. Consequently, the function $\Theta(t)$ is unpredictable.

Example 4.1 *Consider the system*

$$\begin{aligned}x_1'&=-2x_1+2x_2-50\Theta(t)\\ x_2'&=x_1-3x_2+5\Theta^3(t),\end{aligned}\tag{4.4}$$

where $\Theta(t)$ is the unpredictable function defined by (4.2). The eigenvalues of the matrix of coefficients of system (4.4) are -2 and -0.5. One can confirm that the perturbation function $(-50\Theta(t),5\Theta^3(t))$ is unpredictable in accordance with Lemma 1.4. By the main result of our paper, there is an asymptotically stable unpredictable solution $(\varphi_1(t),\varphi_2(t))$ of system (4.4). Consequently, any solution of the equation behaves irregularly ultimately. This is seen from the simulation of the solution with $x_1(0)=0,18$, $x_2(0)=0,01$ in Figure 1.

The next example is devoted to a system of differential equations whose matrix of coefficients admits both positive and negative eigenvalues.

Example 4.2 *Let us take into account the system*

$$\begin{aligned}y_1'&=-1000y_1+0.23y_2+120x_2^3(t)+160\\ y_2'&=6y_1+0.000002y_2-0.1x_1(t)+20,\end{aligned}\tag{4.5}$$

where $(x_1(t),x_2(t))$ is the solution of (4.4) depicted in Figure 1. The eigenvalues of the matrix of coefficients of system (4.5) are -1000 and 0.00138. The perturbation function $(120x_2^3(t)+160,-0.1x_1(t)+20)$ is unpredictable

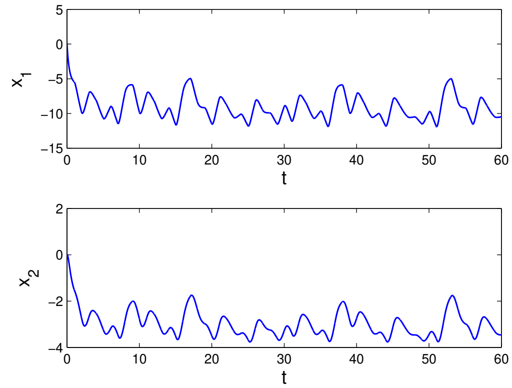

Figure 1. The time series of the x_1 and x_2 coordinates of system (4.4) with the initial conditions $x_1(0) = 0,18$, $x_2(0) = 0,01$. The figure manifests the irregular behavior of the solution.

by Lemmas 1.4 and 1.5. According to the result of Theorem 2.2, system (4.5) possesses a unique unpredictable solution. The simulation results for system (4.5) corresponding to the initial conditions $y_1(0) = 0$ *and* $y_2(0) = 0.1$ *are shown in Figure 2. The time series of both* y_1 *and* y_2 *coordinates in the figure confirm the presence of irregularity in the dynamics of system (4.5).*

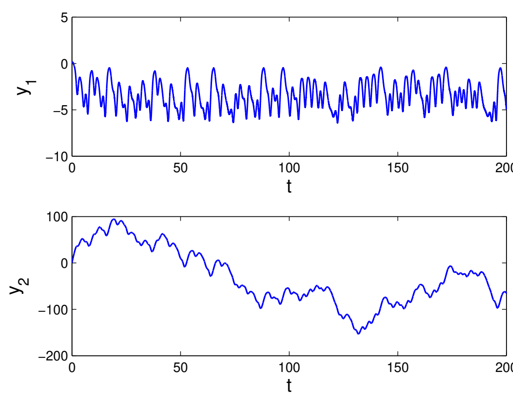

Figure 2. The time series for the y_1 and y_2 coordinates of system (4.5) with the initial conditions $y_1(0) = 0,\ y_2(0) = 0.1$. The irregular behavior of the solution reveals the presence of an unpredictable solution in the dynamics of (4.5).

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References

[1] Akhmet M, Fen MO. Poincaré chaos and unpredictable functions. Communications in Nonlinear Science and Numerical Simulation 2017; 48: 85-94.

[2] Akhmet M, Fen MO. Non-autonomous equations with unpredictable solutions. Communications in Nonlinear Science and Numerical Simulation 2018; 59: 657-670.

[3] Bener PR. Recurrent solutions to systems of ordinary differential equations. Journal of Differential Equations 1969; 5: 271-282.

[4] Hale J, Koçak H. Dynamics and Bifurcations. New York, NY, USA: Springer-Verlag 1991.

[5] Hale JK. Ordinary Differential Equations. Malabar, FL, USA: Krieger Publishing Company, 1990.

[6] Lakshmikantham V, Trigiante D. Theory of Difference Equations: Numerical Methods and Applications. New York, NY, USA: Marcel Dekker, 2002.

[7] Nemytskii VV, Stepanov VV. Qualitative Theory of Differential Equations. Princeton, NJ, USA: Princeton University Press, 1960.

[8] Sell GR. Topological Dynamics and Ordinary Differential Equations. London, UK: Van Nostrand Reinhold Company, 1971.

[9] Shi Y, Yu P. On chaos of the logistic maps. Dynamics of Continuous, Discrete and Impulsive Systems Series-B 2007; 14: 175–95.