Turkish Journal of Mathematics

Volume 43 | Number 5 Article 24

1-1-2019

On the J-reflexive sequences

PARVIZ SADAT HOSSEINI

BAHMANN YOUSEFI

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2371 – 2376
© TÜBİTAK
doi:10.3906/mat-1904-183

On the J-reflexive sequences

Parviz SADAT HOSSEINI*, **Bahmann YOUSEFI**
Department of Mathematics, Faculty of Science, Payame Noor University, Tehran, Iran

Received: 30.04.2019 • **Accepted/Published Online:** 11.08.2019 • **Final Version:** 28.09.2019

Abstract: We call a sequence $\mathcal{T} = (T_n)_n$ of bounded operators on a Banach space X, J-reflexive if every bounded operator on X that leaves invariant, the J-sets of $\mathcal{T}$ is contained in the closure of $\{I, T_1, T_2, ...\}$ in the strong operator topology. We discuss some properties of J-reflexive sequences. We also give and prove some sufficient conditions under which an operator sequence is J-reflexive. Some examples are considered. Indeed, weakly J^{mix}-reflexivity is also defined. Finally, we extend the J-reflexive property in terms of subsets.

Key words: J-sets, J^{mix}-sets, reflexive operator, strongly bounded below

1. Introduction

One of the most challenging problems in operator theory is "invariant subspace problem", which asks whether every operator on a Hilbert space (more generally, a Banach space) admits a nontrivial invariant subspace. Here, "operator" means "continuous linear transformation" and "invariant subspace" means "closed linear manifold that the operator maps it to itself". A subspace is nontrivial if it is neither the zero subspace nor the whole space. An example constructed by Enflo [6] shows that for some Banach spaces there exist operators with only trivial invariant subspaces. For a Hilbert space, however, the invariant subspace problem remains open. There is a deep connection between invariant subspaces of an operator and its reflexivity. Reflexive operators are those that can be identified by their nontrivial invariant subspaces and they have been studied for a few decades. At first, Halmos [11] introduced a reflexive algebra of operators: an algebra $\mathcal{A}$ of operators is reflexive if it is equal to the algebra of bounded operators which leave invariant each subspace left invariant by every operator in $\mathcal{A}$. In fact, a reflexive operator algebra $\mathcal{A}$ is an operator algebra that has enough invariant subspaces to characterize it. In the last few decades various notions of reflexivity have been introduced and studied. Hadwin et al. in [10] introduced orbit-reflexivity: an operator T on a Hilbert space is orbit-reflexive if the only operators that leave invariant every norm closed T-invariant subset are contained in the strong closure of $\{I, T, T^2, ...\}$. For example compact operators, normal operators, contractions and weighted shifts on Hilbert spaces are orbit-reflexive [10]. In [9], the authors also introduced and studied the notion of null-orbit reflexivity, which is a slight perturbation of the notion of orbit-reflexivity. The class of null-orbit reflexive operators includes the classes of hyponormal, algebraic, compact, strictly block-upper (lower) triangular operators, and operators whose spectral radius is not 1. They also proved that every polynomially bounded operator on a Hilbert space is both orbit-reflexive and null-orbit-reflexive. For further references on these topics, see [7–10, 12–14].

*Correspondence: psadath@pnu.ac.ir
2010 *AMS Mathematics Subject Classification:* Primary 47A65, Secondary 47B99

In [18], we introduced the notion of J-reflexivity and discussed this property for operators. In this paper, we extend the J-reflexive property to the operator sequences. In fact, our purpose is to characterize operator sequences on Banach spaces that can be identified by their J-sets. First, we introduce some notations from [17].

Let X be a complex Banach space and denote by $B(X)$ the space of all bounded operators on X. For a sequence $\mathcal{T} = (T_n)_n$ in $B(X)$, the set $orb(\mathcal{T}) = \{T_n : n \geq 0\}$ is called the orbit of $\mathcal{T}$, where $T_0 = I$ is the identity operator. Also, for any $x \in X$, the set

$$orb(\mathcal{T}, x) = \{T_n x : n \geq 0\}$$

is called orbit of $\mathcal{T}$ under x. The J-set of $\mathcal{T}$ under x, $J(\mathcal{T}, x)$, is defined by:

$$J(\mathcal{T}, x) = \{y : \text{there exists a sequence}\,(x_n)_n \text{ in } X \text{ and a strictly increasing}$$

$$\text{sequence}\,(k_n)_n \text{ in } \mathbb{N} \text{ such that } x_n \to x \text{ and } T_{k_n} x_n \to y\}.$$

Also, the set

$$J^{mix}(\mathcal{T}, x) = \{y : \text{there exists a sequence}\,(x_n)_n \in X \text{ such that } x_n \to x$$

$$\text{and } T_n x_n \to y\}$$

is called J^{mix}-set of $\mathcal{T}$ under x.

If $\mathcal{T} = (T_n)_n$ is bounded (i.e, $\{\|T_n\| : n \geq 0\}$ is bounded), then it is easy to see that

$$J(\mathcal{T}, x) = \{y : \text{there exists a strictly increasing sequence of positive integers}$$

$$(k_n) \text{ such that } T_{k_n} x \to y\}.$$

For a good source for these topics we refer the reader to [1–5, 15, 16, 19].

The proof of the following lemma comes from [17].

Lemma 1.1 *Let $\mathcal{T} = (T_n)_n$ be in $B(X)$. If $\mathcal{T}$ is a Cauchy sequence and it has a subsequence converging to a bounded operator U in the strong operator topology, then for every x, both sets $J(\mathcal{T}, x)$ and $J^{mix}(\mathcal{T}, x)$ are equal to the singleton set $\{Ux\}$.*

2. On the property of J-reflexive sequences

In this section, we first define the J-reflexive property, and then some properties of J-reflexive sequences are investigated. Also, we state and prove sufficient conditions for an operator sequence to be J-reflexive. Finally, we express the J-reflexive property in terms of subsets.

From now on, for simplicity, we denote the closure of a subset A in the strong operator topology by $\overline{A}^{SOT}$.

Definition 2.1 *We call $\mathcal{T} = (T_n)_n \subseteq B(X)$ a J-reflexive sequence if every bounded operator that leaves invariant $J(\mathcal{T}, x)$ (for all x in X) is contained in the $\overline{orb(\mathcal{T})}^{SOT}$. J^{mix}-reflexivity can also be defined in a similar way.*

It is easy to see that if $\mathcal{T} = (T_n) \subseteq B(X)$, and for all $x \in X$, $J(\mathcal{T}, x)$ is an empty set or the whole space X or the singelton set $\{0\}$, then $\mathcal{T}$ is not J-reflexive. However, if $\mathcal{T} = (T_n)$ converges to the identity operator, then $J(\mathcal{T}, x) = \{x\}$ for all x and so $\mathcal{T}$ is J-reflexive.

Isomorphisms preserve the J-reflexive property, as follows.

Theorem 2.2 *Suppose that X and Y are Banach spaces and $S : X \to Y$ is an isomorphism. If $\mathcal{T}$ is a J-reflexive sequence on X, then $S\mathcal{T}S^{-1}$ is also a J-reflexive sequence on Y.*

Proof See Theorem 3.5, in [18]. □

Definition 2.3 *Let $\mathcal{T} = (T_n)_n$ be an operator sequence on X. A nonzero vector $x \in X$ is called periodic point for $\mathcal{T}$ if there exists $n \in \mathbb{N}$ such that the set*

$$P_{n,x} = \{m : T_n x = T_m x\}$$

is an infinite set.

Recall that the space of convergent sequences is usually denoted by c. This is a Banach space over $\mathbb{C}$ or $\mathbb{R}$ under the supremum norm.

Example 2.4 *Let $n \in \mathbb{N}$ and $P_n : c \to c$ be defined by*

$$P_n(x_1, x_2, ...) = (x_1, ..., x_{n-1}, 0, x_{n+1}, ...).$$

Then any vector $x = (x_i)$ in c such that x has at most finitely many nonzero x_i's is a periodic point for $(P_n)_n$.

In the following theorem, the sufficient conditions for the J-reflexivity of an operator sequence are given.

Theorem 2.5 *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ be bounded. Then*
a) If $\mathcal{T}$ has no periodic point and $x \in J(\mathcal{T}, x)$ for all $x \in X$, then $\mathcal{T}$ is J-reflexive.
b) If $\mathcal{T}$ converges to a surjective operator in $B(X)$, then $\mathcal{T}$ is J-reflexive.

Proof a) Suppose on the contrary that $\mathcal{T}$ is not J-reflexive. Then there exists an operator $S \in B(X) \backslash \overline{orb(\mathcal{T})}^{SOT}$ such that

$$S(J(\mathcal{T}, x)) \subseteq J(T, x)$$

for all $x \in X$. Therefore, there exist $x_0 \in X$ and $\delta > 0$ such that $B(Sx_0; \delta) \cap orb(\mathcal{T}, x_0)$ is a finite set. Since $x_0 \in J(\mathcal{T}, x_0)$, $Sx_0 \in J(\mathcal{T}, x_0)$, and this means that there exists a strictly increasing sequence of positive integers $(k_n)_n$ such that

$$Sx_0 = \lim_n T_{k_n} x_0.$$

This is possible only if x_0 is a periodic point for $\mathcal{T}$ which is a contradiction.
b) Suppose that $(T_n)_n$ converges to a surjective operator $U \in B(X)$. By Lemma 1.1, for all x, $J(\mathcal{T}, x) = \{Ux\}$, and therefore for every $W \in B(X)$ that leaves invariant the J-sets of $\mathcal{T}$ we have $WUx = Ux$ for all $x \in X$. Since U is surjective, it follows that $W = I$. Hence, $\mathcal{T}$ is J-reflexive. □

Example 2.6 *Let $\mathcal{S} = (S_n)$, where, $S_n : c \to c$ be defined by $S_n x = 2^{\frac{1}{n}} x$ for all $n \in \mathbb{N}$. Since for all x, $S_n x \to x$, thus $x \in J(\mathcal{S}, x)$. Indeed, $\mathcal{S}$ has no periodic point. Hence, $\mathcal{S}$ is a J-reflexive sequence by Theorem 2.5 (a).*

Example 2.7 *Suppose that U is any bounded surjective operator on l^∞, and define*

$$T_n x = (1 + \frac{1}{n})Ux.$$

Since $(T_n)_n$ converges to U, so by Theorem 2.5 (b), $(T_n)_n$ is a J-reflexive sequence.

The concept of J-reflexivity can also be expressed in terms of subsets as follows.

Definition 2.8 *Let $\mathcal{T} = (T_n) \subseteq B(X)$ and M be a subset of X. $\mathcal{T}$ is called M-J-reflexive if every $W \in B(X)$ that leaves invariant $J(\mathcal{T}, x)$ for all x in M, is contained in the $\overline{orb(\mathcal{T})}^{SOT}$. Also, M-J^{mix}-reflexivity can be defined by a similar method.*

It is clear that M-J-reflexivity implies J-reflexivity. The converse is true for bounded sequences and dense subsets.

Theorem 2.9 *Suppose that $\mathcal{T} = (T_n)$ in $B(X)$ is J-reflexive and bounded. If M is a dense subset of X, then $\mathcal{T}$ is M-J-reflexive.*

Proof See Theorem 3.13, in [18]. □

3. Weakly J^{mix}-reflexive sequences

In this section, we first define the weakly J^{mix}-reflexive property, and then we state and prove a sufficient condition for an operator sequence to be weakly J^{mix}-reflexive.

Recall that for $\mathcal{T} \subseteq B(X)$, by $\mathcal{T}'$ we mean the collection of all bounded operators on X which commutes with each member of $\mathcal{T}$.

Definition 3.1 *A sequence $\mathcal{T} = (T_n)$ in $B(X)$, is called weakly J^{mix}-reflexive if every operator in $\mathcal{T}'$ that leaves invariant $J^{mix}(\mathcal{T}, x)$ for all x in X, is contained in the $\overline{orb(\mathcal{T})}^{SOT}$.*

In the following theorem, a sufficient condition for the weakly J^{mix}-reflexive of an operator sequence is given.

Theorem 3.2 *Suppose that $\mathcal{T} = (T_n) \subseteq B(X)$. Then $\mathcal{T}$ is weakly J^{mix}-reflexive if $J(\mathcal{T}, 0)$ is the only J^{mix}-set that contains the zero vector.*

Proof Let W in $\mathcal{T}'$ leave invariant $J^{mix}(\mathcal{T},x)$ for all x in X. Then for any x, there exist sequences (x_n) and (z_n) in X such that the both sequences are convergent to x, and we have

$$W(\lim T_n x_n) = \lim T_n z_n.$$

Thus, $\lim WT_n x_n = \lim T_n z_n$, and so $\lim T_n(Wx_n - z_n) = 0$. This implies that $0 \in J^{mix}(\mathcal{T}, Wx - x)$, and by hypothesis we get $Wx = x$ for all x. Hence, $W = I$ and $\mathcal{T}$ is weakly J^{mix}-reflexive. □

Definition 3.3 *A sequence $\mathcal{T} = (T_n)_n$ of operators on a normed space X is called bounded below if each T_n is bounded below. We call $\mathcal{T}$ uniformly bounded below if there exists $c > 0$ such that $\|T_n x\| \geq c\|x\|$ for all n, and for all x. Also, $\mathcal{T}$ is called strongly bounded below if for every n there exists $c_n \geq 1$ such that $\|T_n x\| \geq c_n\|x\|$ for all x in X.*

Now, we want to consider a group of operator sequences called BSC-sequence that has been introduced in [17]. For the benefit of readers, we give it as follows:

Definition 3.4 *We mean by a BSC-sequence, a sequence $\mathcal{T}$ of operators on a normed space X that holds in the following conditions:*
(a) $\mathcal{T}$ is a uniformly bounded below sequence.
(b) $\mathcal{T}$ is a Cauchy sequence in the strong operator topology.
Furthermore, if $\mathcal{T}$ is a Cauchy sequence in the norm topology, then we say that $\mathcal{T}$ is a BC-sequence.

Theorem 3.5 *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ and $x, y \in X$ be such that $x \neq y$. If $\mathcal{T}$ is uniformly bounded below or is strongly bounded below, then we have*

$$J^{mix}(\mathcal{T},x) \ \cap J^{mix}(\mathcal{T},y) = \emptyset.$$

Proof See Theorem 2.9, in [17]. □

Theorem 3.6 *Let $\mathcal{T} = (T_n)_n \subseteq B(X)$ and $x, y \in X$ such that $x \neq y$. If $\mathcal{T}$ is a bounded BSC-sequence or BC-sequence, then $J(\mathcal{T},x) \cap J(\mathcal{T},y) = \emptyset$.*

Proof See Theorem 2.11, in [17]. □

Remark 3.7 *Suppose that $\mathcal{T} = (T_n)_n \subseteq B(X)$ is bounded, then for all x, we have*

$$J^{mix}(\mathcal{T},x) = \{y : T_n x \to y\}.$$

Thus, $J^{mix}(\mathcal{T},0) = \{0\}$. If $\mathcal{T}$ is uniformly bounded below or is strongly bounded below, then by Theorem 3.5, $J^{mix}(\mathcal{T},0)$ is the only J^{mix}-set that contains zero vector. Therefore, by Theorem 3.2, $\mathcal{T}$ is weakly J^{mix}-reflexive. Similarly, if $\mathcal{T}$ is a bounded BSC-sequence (or BC-sequence), then by Theorem 3.6 for all nonzero x, $0 \notin J(\mathcal{T},x)$. Since $J^{mix}(\mathcal{T},x) \subseteq J(\mathcal{T},x)$, we have $0 \notin J^{mix}(\mathcal{T},x)$ for all nonzero x. This means that $\mathcal{T}$ is weakly J^{mix}-reflexive.

References

[1] Ayadi A, Marzougui H., J-class abelian semigroups of matrices on $\mathbb{C}^n$ and hypercyclicity. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2014; 108: 557-566.

[2] Azimi MR, Müller, V. A note on J-sets of linear operators. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2011; 105: 449-453.

[3] Bayart F, Matheron É. Dynamics of Linear Operators. Newyork, NY, USA: Cambridge University Press, 2009.

[4] Costakis G, Manoussos A. J-class operators and hypercyclicity. Journal of Operator Theory 2012; 67: 101-119.

[5] Costakis G, Manoussos A. J-class weighted shifts on the space of bounded sequences of complex numbers. Integral Equations and Operator Theory 2008; 62: 149-158.

[6] Enflo P. On the invariant subspace problem for Banach spaces. Acta Mathematica 1987; 158: 213-313.

[7] Esterle J. Operators of Read's type are not orbit-reflexive. Integral Equations and Operator Theory 2009; 63: 591-593.

[8] Hadwin D, Ionascu I, McHugh M, Yousefi H. $\mathbb{C}$-orbit reflexive operators. Operators and Matrices 2011; 5: 511-527.

[9] Hadwin D, Ionascu I. Yousefi H. Null-orbit reflexive operators. Operators and Matrices 2012; 6: 567-576.

[10] Hadwin D. Nordgren E, Radjavi H, Rosenthal P. Orbit-reflexive operators. Journal of the London Mathematical Society 1986; 34 (2): 111-119.

[11] Halmos PR. Invariant subspaces. In: Abstract Spaces and Approximation– Proceedings of the Conference; Basel, Switzerland; 1968, pp. 26-30.

[12] McHugh M. Orbit-reflexivity. PhD, University of New Hampshire, New Hampshire, USA, 1995.

[13] Merlevéde F, Peligrad C, Peligrad M. Reflexive operator algebras on Banach spaces. Pacific Journal of Mathematics 2014; 267: 451-464.

[14] Müller V, Vršovský L. On orbit reflexive operators. Journal of the London Mathematical Society 2009; 79 (2): 497-510.

[15] Nasseri AB. J-class operators on certain Banach spaces. PhD, Technical University of Dortmund, Dortmund, Germany, 2013.

[16] Nasseri AB. Operators on l^∞ with totally disconnected spectrum and applications to J-class operators. Journal of Mathematical Analysis and Applications 2014; 410: 94-100.

[17] Sadat Hosseini P, Yousefi B. Fixed Points of BSC-Sequences. Communications of the Korean Mathematical Society 2017; 32: 899-908. doi: 10.4134/CKMS.c160253

[18] Sadat Hosseini P, Yousefi B. On the J-reflexive operators. Turkish Journal of Mathematics 2018; 42: 1795-1802. doi: 10.3906/mat-1707-9

[19] Tian G, Hou BN. Limits of J-class operators. Proceedings of the American Mathematical Society 2014; 142: 1663-1667.