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On operator systems generated by reducible projective unitary representations of compact groups

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Abstract: We study reducible projective unitary representations $(U_g)_{g\in G}$ of a compact group G in separable Hilbert spaces H. It is shown that there exist the projections Q and P for which $\mathcal{V} = \overline{span(U_g Q U_g^*,\ g \in G)}$ is the operator system and $P\mathcal{V}P = \{\mathbb{C}P\}$. As an example, a bipartite Hilbert space $H = \mathfrak{H} \otimes \mathfrak{H}$ is considered. In this case, the action of $(U_g)_{g\in G}$ has the property of transforming separable vectors to entangled.

Key words: Operator systems, covariant resolutions of identity, reducible unitary representations of compact groups, quantum anticliques

1. Introduction

A subspace $\mathcal{V}$ consisting of bounded linear operators in a separable Hilbert space H is said to be an operator system [5] if it is self-adjoint ($V \in \mathcal{V}$ implies $V^* \in \mathcal{V}$) and the identity operator $I \in \mathcal{V}$. Recently, operator systems have attracted the interest of researchers in the context of both functional analysis and quantum information theory [1–4, 6, 7, 9, 10, 12]. It should be noted that operator systems are often called noncommutative operator graphs.

The Kraus representation of a quantum channel generates the operator system [7]. The possibility of transmitting quantum information via a channel with zero error is completely determined by the properties of the operator system corresponding to this channel [3, 4, 9, 10]. Moreover, it is hoped that the proximity of the two quantum channels [11] can be estimated using the corresponding operator systems.

Let G be a compact group with the Haar measure μ, $\mu(G) = 1$, and $\mathfrak{B}$ is the sigma-algebra generated by compact subsets of G, then the set of positive operators $\{M(B), B \in \mathfrak{B}\}$ in a Hilbert space H is said to be a resolution of identity if [8]

$$M(\emptyset) = 0,\ M(G) = I,$$

$$M(\cup_j B_j) = \sum_j M(B_j),\ B_k \cap B_l = \emptyset \ for\ k \neq l,\ B_j \in \mathfrak{B}, \tag{1.1}$$

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and a convergence in (1.1) is understood in the sense of strong operator topology. Let $(U_g)_{g\in G}$ be a projective unitary representation of G in H. Then $\{M(B), B \in \mathfrak{B}\}$ is said to be covariant with respect to the action of $(U_g)_{g\in G}$ if

$$U_g M(B) U_g^* = M(gB).$$

In [2], the study of operator systems generated by covariant resolutions of identity in the sense of

$$\mathcal{V} = \overline{span\{M(B), B \in \mathfrak{B}\}} \tag{1.2}$$

was initiated. It is known [8] that in finite dimensional spaces H, any covariant resolution of identity $\{M(B), B \in \mathfrak{B}\}$ has the form

$$M(B) = \int\limits_B U_g M_0 U_g^* d\mu(g),$$

where M_0 is some positive operator in H. In this case, (1.2) can be replaced with

$$\mathcal{V} = \overline{span(U_g M_0 U_g^*,\ g \in G)}. \tag{1.3}$$

A particularly interesting case is the bipartite Hilbert space $H = \mathfrak{H} \otimes \mathfrak{H}$. Then, a vector $v \in H$ is said to be separable if it can be represented in the form $v = v_1 \otimes v_2$. In the opposite case, v is known as entangled. Given a unit vector $v \in H$ denote $|v\rangle\langle v|$ an orthogonal projection to the subspace $\{\mathbb{C}v\}$. We call a projection Q to be separable if it can be represented as a sum of orthogonal projections $|v\rangle\langle v|$ for which v are separable vectors. In [3, 4] operator systems generated by unitary representations of the circle group as well as the Heisenberg–Weyl group G were studied in detail. Operator systems were constructed having the form (1.2) and (1.3) for which $M_0 = Q$ are separable projections. Moreover it was proved that for such $\mathcal{V}$ there are projections P satisfying the property

$$P\mathcal{V}P = \{\mathbb{C}P\}. \tag{1.4}$$

The projections P, $rankP \geq 2$, satisfying (1.4) are known as quantum anticliques for $\mathcal{V}$ [12]. In the present paper, how these ideas work in general case is shown.

2. Reducible projective unitary representation of groups in an arbitrary separable Hilbert space

In this section, we do not take into account a tensor product structure of Hilbert space H. Consider a reducible continuous projective unitary representation $G \ni g \to U_g$ in H. Then, there are countably many resolutions of H into the orthogonal sum

$$H = \oplus_j H_j, \tag{2.1}$$

such that the restrictions

$$U_g^{(j)} = U_g|_{H_j} \tag{2.2}$$

determine cyclic representations of G. It means that for any j $U_g H_j \subset H_j$, $g \in G$, and there exist a unit vector $v_j \in H_j$ such that the closer $\overline{span(U_g v_j,\ g \in G)} = H_j$. Since all irreducible representations of compact group are finite dimensional, the following statement holds true.

Proposition 2.1 *Suppose that at least one of the following conditions is satisfied*

- *all representations (2.1) and (2.2) are irreducible;*
- *$dimH_j < +\infty$ and cyclic vectors v_j has the property*

$$\int_G U_g \left|v_j\right\rangle \left\langle v_j\right| U_g^* d\mu(g) = \frac{1}{dimH_j} Q_j,$$

where Q_j is an orthogonal projection on H_j.

Then, the positive operator

$$Q = \sum_j dimH_j \left|v_j\right\rangle \left\langle v_j\right| \tag{2.3}$$

generates an operator system by the formula

$$\mathcal{V} = \overline{span(U_g Q U_g^*, \ g \in G)}. \tag{2.4}$$

Proof It immediately follows that $\mathcal{V}$ is self-adjoint. Hence, it remains to prove that $I \in \mathcal{V}$. Since $dimH_j < +\infty$, we get

$$\int_G U_g^{(j)} \left|v_j\right\rangle \left\langle v_j\right| U_g^{(j)*} d\mu(g) = \frac{1}{dimH_j} Q_j.$$

Thus,

$$\int_G U_g Q U_g^* d\mu(g) = I.$$

□

Now suppose that the spectral decompositions of $U_g^{(j)}$ contain the same projection $\left|h_j\right\rangle \left\langle h_j\right|$ for all $g \in G$. Recall that two vectors v and h in a finite dimensional space K are said to be unbiased if

$$\left| \left\langle v, h\right\rangle \right|^2 = \frac{1}{dimK}. \tag{2.5}$$

Proposition 2.2 *Suppose that vectors h^j and v_j are unbiased for all j. Then, the projection*

$$P = \sum_j \left|h^j\right\rangle \left\langle h^j\right|$$

is a quantum anticlique for (2.4).

Proof It follows from (2.5) that

$$\left|h^j\right\rangle \left\langle h^j\right| \left|v_j\right\rangle \left\langle v_j\right| \left|h^j\right\rangle \left\langle h^j\right| = \frac{1}{dimH_j} \left|h^j\right\rangle \left\langle h^j\right|.$$

Taking into account (2.3), we get

$$PQP = P$$

because $U_g P = P U_g$ are commuting for all $g \in G$. □

3. Operator systems in a bipartite Hilbert space

Here we shall give an explicit example showing how the techniques of the previous section works in a bipartite finite dimensional Hilbert space $H = \mathfrak{H} \otimes \mathfrak{H}$. Denote by $(|jk\rangle)$ the basis in H consisting of separable vectors, $1 \leq j, k \leq dim\mathfrak{H} < +\infty$. Together with $(|jk\rangle)$, we consider the basis in H consisting of entangled vectors

$$h_k^j = \frac{1}{\sqrt{d}} \sum_{s=1}^{d} e^{i\frac{2\pi ks}{d}} \left| s\ s + j\ mod\ dim\mathfrak{H} \right\rangle.$$

Denote by H_j the subspaces spanned by vectors h_k^j, $1 \leq k \leq dim\mathfrak{H}$.

In the following statement we claim that the conditions of Proposition 1 are satisfied.

Theorem 3.1 *Fix* $1 \leq m_0, n_0 \leq dim\mathfrak{H}$. *Suppose that* $G \ni g \to U_g$ *can be resolved in a sum of cyclic representations*

$$U_g = \oplus_j U_g|_{H_j}$$

with the cyclic vectors $v_j^{m_0} = |m_0\ m_0 + j\ mod\ dim\mathfrak{H}\rangle$ *and the projections* $|h_{n_0}^j\rangle \langle h_{n_0}^j|$ *are contained in the spectral decompositions of* U_g *for all* $g \in G$. *Then, the projection*

$$P = \sum_j |h_{n_0}^j\rangle \langle h_{n_0}^j|$$

is a quantum anticliques for the operator system

$$\mathcal{V} = \overline{span(U_g Q U_g^*,\ g \in G)}, \tag{3.1}$$

where

$$Q = \sum_j |v_j^{m_0}\rangle \langle v_j^{m_0}|.$$

Remark 3.2 *Since the projections* $U_g Q U_g^*$ *have infinite ranks in general, we need to take a closer look at (3.1) in the sense of strong operator topology to guarantee the inclusion* $I \in \mathcal{V}$.

Proof Proposition 1 implies that (3.1) is an operator system. Now it is enough to check that $h_{n_0}^j$ and $v_j^{m_0}$ are unbiased and the result follows from Proposition 2. □

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