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An approach to negative hypergeometric distribution by generating function for special numbers and polynomials

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Abstract: The aim of this paper is to not only provide a definition of a new family of special numbers and polynomials of higher-order with their generating functions, but also to investigate their fundamental properties in the spirit of probabilistic distributions. By applying generating functions methods, we derive miscellaneous novel identities and formulas involving the Chu–Vandermonde-type convolution formulas, combinatorial sums, Bernstein basis functions, and the other well-known special numbers and polynomials. Moreover, we provide a computational algorithm which returns special values of these numbers and polynomials. In addition, we show that our new identities and formulas are connected with the interpolation functions of the Apostol-type numbers and polynomials. Finally, we present some theoretical and applied details on probabilistic distributions arising from the aforementioned Chu–Vandermonde-type convolution formulas.

Key words: Generating functions, Stirling numbers, Apostol–Bernoulli numbers, Apostol–Euler numbers, Catalan numbers, combinatorial sums, binomial coefficients, Chu–Vandermonde convolution formula, probability distribution

1. Introduction

Throughout this paper, we consider the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ defined by the following generating functions, respectively:

$$F(t,\lambda)=\frac{2}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(\lambda)\frac{t^n}{n!}, \tag{1.1}$$

and

$$F(t,x,\lambda)=\frac{2(1+\lambda t)^x}{\lambda(1+\lambda t)-1}=\sum_{n=0}^{\infty}Y_n(x;\lambda)\frac{t^n}{n!} \tag{1.2}$$

(*cf.* [28]). Recently, many applications of these numbers and polynomials have been studied and investigated by different authors (*cf.* [11, 13, 28–30, 34]). Recently, Khan et al. [11] constructed 2-variable of the polynomials $Y_n(x;\lambda)$. They gave quasimonomial properties of these polynomials on the Weyl group structure.

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The main motivation of this article is to give some applications and investigations which have not been examined so far by any authors. In this paper, the higher-order expansions of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$ are firstly constructed. Secondly, with the help of these extensions, many identities including binomial coefficients and especially Vandermonde-type identities are generalized. Furthermore, a generalized version of the probability functions for the negative hypergeometric distribution is achieved by using the newly defined generalized Vandermonde-type identities.

The following notations and definitions are used in this paper:

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. Let $\log z$ denote the principal branch of the multivalued function $\log z$ with the imaginary part $\Im(\log z)$ constrained by the interval $(-\pi, \pi]$. We assume that

$$0^n = \begin{cases} 1, & (n = 0) \\ 0, & (n \in \mathbb{N}) \end{cases}$$

and

$$\binom{z}{v} = \frac{z(z-1)\cdots(z-v+1)}{v!} \; (v \in \mathbb{N},\, z \in \mathbb{C})$$

and

$$\binom{z}{0} = 1,$$

(*cf.* [1–15, 17–34]).

In order to give the results of this paper, we need some definitions, relations, and formulas for special numbers and polynomials including Stirling numbers, Apostol–Bernoulli numbers, Apostol–Euler numbers, and Catalan numbers with their generating functions.

The Apostol–Bernoulli numbers $\mathcal{B}_n^{(k)}(\lambda)$ of order k are defined by the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^k = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(\lambda) \frac{t^n}{n!}, \tag{1.3}$$

where $|t| < 2\pi$ when $\lambda = 1$ and $|t| < |\log(\lambda)|$ when $\lambda \neq 1$ (*cf.* [1–15, 17–34]; see also the references cited therein).

Substituting $k = 1$ into (1.3), we have

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n^{(1)}(\lambda),$$

where $\mathcal{B}_n(\lambda)$ denotes the Apostol–Bernoulli numbers (*cf.* [1–15, 17–34]; see also the references cited therein).

Substituting $\lambda = 1$ and $k = 1$ into (1.3), we have

$$B_n = \mathcal{B}_n(1),$$

where B_n denotes the Bernoulli numbers (*cf.* [1–15, 17–34]; see also the references cited therein).

Note that the Apostol–Bernoulli numbers of order k are interpolated by the zeta-type functions as follows:

$$\zeta(\lambda, -m, k) = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n n^m = -\frac{\mathcal{B}_{m+1}^{(k)}(\lambda)}{m+1} \tag{1.4}$$

(*cf.* [1, 32, 33]; see also the references cited therein).

The Apostol–Euler numbers $\mathcal{E}_n^{(k)}(\lambda)$ of order k are defined by the following generating function:

$$\left(\frac{2}{\lambda e^t+1}\right)^k=\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(\lambda)\frac{t^n}{n!}, \tag{1.5}$$

where $|t|<|\log(-\lambda)|$ (*cf.* [1–15, 17–33]; see also the references cited therein).

Setting $k=1$ in (1.5), we have

$$\mathcal{E}_n(\lambda)=\mathcal{E}_n^{(1)}(\lambda),$$

where $\mathcal{E}_n(\lambda)$ denotes the Apostol–Euler numbers (*cf.* [1–15, 17–33]; see also the references cited therein).

When $\lambda=1$, the Apostol–Euler numbers reduce to the Euler numbers as follows:

$$E_n=\mathcal{E}_n(1)$$

(*cf.* [1–15, 17–33]; see also the references cited therein).

The Stirling numbers of the first kind $s(n,v)$ are defined by

$$(x)_n=\sum_{v=0}^{n}s(n,v)x^v, \tag{1.6}$$

where $(x)_n$ denotes the falling factorial given by

$$(x)_n=x(x-1)(x-2)\dots(x-n+1)$$
$$(x)_0=1$$

and the generating function of the numbers $s(n,k)$ is given by

$$\frac{(\log(1+t))^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{t^n}{n!} \tag{1.7}$$

(*cf.* [10, 17, 33]; see also the references cited therein).

The Catalan numbers C_n are defined by the following generating function:

$$\frac{1-\sqrt{1-4t}}{2t}=\sum_{n=0}^{\infty}C_nt^n$$

where $0<|t|\leq\frac{1}{4}$ and $C_0=1$ (*cf.* [15]). The explicit formula and the recurrence relation for the Catalan numbers are given as follows, respectively:

$$C_n=\frac{1}{n+1}\binom{2n}{n}, \tag{1.8}$$

where $n\geq 0$, and

$$\frac{C_n}{C_{n-1}}=\frac{4n-2}{n+1}, \tag{1.9}$$

where $n \geq 1$ (*cf.* [2] and [15, pp. 109-110]).

The rest of the present paper is given as follows:

In Section 2, we provide the definition of a new family of special numbers and polynomials of higher-order with their generating functions. Furthermore, we investigate their fundamental properties by deriving explicit formulas, recurrence relations, and a computational algorithm. In Section 3, by making use of generating functions and their functional equations, we derive various identities and relations with the inclusion of not only these numbers, but also the Apostol-type numbers, the Stirling numbers of the first kind, the Bernstein basis functions, and combinatorial sums. In Section 4, we provide some derivative formulas arising from the generating functions. In Section 5, by using functional equations of the generating function, we give Chu–Vandermonde type convolution formulas. Additionally, we also give some relations among special values of these formulas including the Catalan numbers and combinatorial sums. In Section 6, by deriving a negative hypergeometric distribution, we give some observations and investigations associated with the aforementioned Chu–Vandermonde type convolution formulas.

2. The numbers $Y_n^{(k)}(\lambda)$ and the polynomials $Y_n^{(k)}(x;\lambda)$

Our motivation in this section is to define the higher-order of the recently defined family of special numbers and polynomials unifying the Apostol-type numbers and polynomials. Moreover, we provide a computational algorithm to compute special values of these numbers and polynomials.

Let k be a nonnegative integer and λ be a real or complex number. By taking these assumptions in consideration, we define the numbers $Y_n^{(k)}(\lambda)$ and the polynomials $Y_n^{(k)}(x;\lambda)$ by means of the following generating functions, respectively:

$$\mathcal{F}(t,k;\lambda) = \left(\frac{2}{\lambda(1+\lambda t)-1}\right)^k = \sum_{n=0}^{\infty} Y_n^{(k)}(\lambda)\frac{t^n}{n!}, \tag{2.1}$$

and

$$\mathcal{F}(t,x,k;\lambda) = \mathcal{F}(t,k;\lambda)(1+\lambda t)^x = \sum_{n=0}^{\infty} Y_n^{(k)}(x;\lambda)\frac{t^n}{n!}. \tag{2.2}$$

Substituting $x=0$ into (2.2), we have

$$Y_n^{(k)}(\lambda) = Y_n^{(k)}(0;\lambda). \tag{2.3}$$

Notice that the numbers $Y_n^{(k)}(\lambda)$ and the polynomials $Y_n^{(k)}(x;\lambda)$ are the higher-order of the numbers $Y_n(\lambda)$ and the polynomials $Y_n(x;\lambda)$, respectively. Namely, if we set $k=1$ in (2.1) and (2.2), the functions $\mathcal{F}(t,k;\lambda)$ and $\mathcal{F}(t,x,k;\lambda)$ reduce to the $F(t,\lambda)$ and $F(t,x,\lambda)$, respectively. Hence,

$$Y_n(\lambda) = Y_n^{(1)}(\lambda),$$

and

$$Y_n(x;\lambda) = Y_n^{(1)}(x;\lambda).$$

Theorem 2.1

$$Y_n^{(k)}(\lambda) = (-1)^n \binom{n+k-1}{n} \frac{2^k n! \lambda^{2n}}{(\lambda-1)^{k+n}}. \tag{2.4}$$

Proof By using (2.1), we have

$$\sum_{n=0}^{\infty} Y_n^{(k)}(\lambda)\frac{t^n}{n!} = \frac{2^k}{(\lambda-1)^k\left(\frac{\lambda^2}{\lambda-1}t+1\right)^k}.$$

Assume that $\left|\frac{\lambda^2}{\lambda-1}t\right| < 1$. By using negative binomial series expansion in the above equation, we get

$$\sum_{n=0}^{\infty} Y_n^{(k)}(\lambda)\frac{t^n}{n!} = \frac{2^k}{(\lambda-1)^k}\sum_{n=0}^{\infty}(-1)^n\binom{n+k-1}{n}\left(\frac{\lambda^2}{\lambda-1}\right)^n t^n. \tag{2.5}$$

By comparing the coefficients of t^n on both sides of the above equation, we obtain the desired result. □

Substituting $k=1$ into (2.4) yields the following known explicit formula:

$$Y_n(\lambda) = (-1)^n \frac{2n!}{\lambda-1}\left(\frac{\lambda^2}{\lambda-1}\right)^n$$

(*cf.* [28]).

By making use of (2.4), few values of the numbers $Y_n^{(k)}(\lambda)$ are computed by

$$\begin{aligned} Y_0^{(2)}(\lambda) &= \frac{4}{(\lambda-1)^2}, Y_1^{(2)}(\lambda) = -\frac{8\lambda^2}{(\lambda-1)^3}, \\ Y_2^{(2)}(\lambda) &= \frac{24\lambda^4}{(\lambda-1)^4}, Y_3^{(2)}(\lambda) = -\frac{96\lambda^6}{(\lambda-1)^5}, \dots \end{aligned}$$

$$\begin{aligned} Y_0^{(3)}(\lambda) &= \frac{8}{(\lambda-1)^3}, Y_1^{(3)}(\lambda) = -\frac{24\lambda^2}{(\lambda-1)^4}, \\ Y_2^{(3)}(\lambda) &= \frac{96\lambda^4}{(\lambda-1)^5}, Y_3^{(3)}(\lambda) = -\frac{480\lambda^6}{(\lambda-1)^6}, \dots . \end{aligned}$$

By using (2.4), we also obtain a recurrence relation for the numbers $Y_n^{(k)}(\lambda)$ by the following theorem:

Theorem 2.2 *Let n be a positive integer and k be a nonnegative integer. By setting*

$$Y_0^{(k)}(\lambda) = \frac{2^k}{(\lambda-1)^k},$$

the following recurrence relation holds true:

$$Y_n^{(k)}(\lambda) = \frac{\lambda^2}{1-\lambda}(n+k-1)\,Y_{n-1}^{(k)}(\lambda).$$

We give another recurrence relation for the numbers $Y_n^{(k)}(\lambda)$ by the following theorem:

Theorem 2.3 *Let*

$$Y_0^{(k)}(\lambda) = \frac{2^k}{(\lambda - 1)^k},$$

and let n *be positive integer. Then we have*

$$\sum_{j=0}^{k} (-1)^{k-j} (n)_j \binom{k}{j} \lambda^{2j} (1-\lambda)^{k-j} Y_{n-j}^{(k)}(\lambda) = 0. \tag{2.6}$$

Proof From (2.1), we have

$$2^k = \left(\lambda^2 t + \lambda - 1\right)^k \sum_{n=0}^{\infty} Y_n^{(k)}(\lambda) \frac{t^n}{n!}.$$

Using binomial theorem yields

$$2^k = \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} \lambda^{2j} (\lambda - 1)^{k-j} Y_n^{(k)}(\lambda) \frac{t^{n+j}}{n!}.$$

Thus,

$$\begin{aligned} 2^k &= \sum_{n=0}^{\infty} \sum_{j=0}^{k} (n)_j \binom{k}{j} \lambda^{2j} (\lambda - 1)^{k-j} Y_{n-j}^{(k)}(\lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} (n)_j \binom{k}{j} \lambda^{2j} (1-\lambda)^{k-j} Y_{n-j}^{(k)}(\lambda) \frac{t^n}{n!}. \end{aligned}$$

From the above equation, we get the assertion of the theorem. □

A relation between the numbers $Y_n^{(k)}(\lambda)$ and the polynomials $Y_n^{(k)}(x;\lambda)$ is given by the following theorem:

Theorem 2.4 *Let* n *be a nonnegative integer. Then we have*

$$Y_n^{(k)}(x;\lambda) = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j} (x)_{n-j} Y_j^{(k)}(\lambda). \tag{2.7}$$

Proof It follows from equations (2.1) and (2.2) that

$$\sum_{n=0}^{\infty} Y_n^{(k)}(x;\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (x)_n \lambda^n \frac{t^n}{n!} \sum_{n=0}^{\infty} Y_n^{(k)}(\lambda) \frac{t^n}{n!}.$$

Using the Cauchy product rule and equalizing the coefficients of the variable $\frac{t^n}{n!}$ in the previous equation yields the assertion of the theorem. □

By combining (2.7) with (2.4) and (1.6), we arrive at the following corollary:

Corollary 2.5

$$Y_n^{(k)}(x;\lambda) = \lambda^n \left(\frac{2}{\lambda-1}\right)^k \sum_{j=0}^{n}\sum_{v=0}^{n-j} (-1)^j j! \binom{n}{j}\binom{j+k-1}{j}\left(\frac{\lambda}{\lambda-1}\right)^j s(n-j,v)\, x^v.$$

By making use of Corollary 2.5, we provide a computational algorithm (Algorithm 1) for computing the polynomials $Y_n^{(k)}(x;\lambda)$.

Algorithm 1 Let n and k be nonnegative integers and $\lambda \in \mathbb{C}$. This algorithm will return the polynomials $Y_n^{(k)}(x;\lambda)$.

```
procedure HIGHER_Y_APOSTOL_TYPE_POLY(n: nonnegative integer, k: nonnegative integer, λ)
    Local variables: j, v, Y
    j, v, Y ← 0                                        ▷ initial zero value for all
    if n = 0 && k = 0 then
        return 1
    else
        for all j in {0, 1, 2, ..., n} do
            for all v in {0, 1, 2, ..., n − j} do
                Y ← Y+Power(−1, j)∗Factorial(j)∗Binomial_Coef(n, j)∗Binomial_Coef(j + k − 1, j)
                    ↪ ∗Power(λ/(λ − 1), j)∗Stirling_Num_First(n − j, v)∗Power(x, v)
            end for
        end for
        return Power(λ, n)∗Power(2/(λ − 1), k)∗Y
    end if
end procedure
```

Some values of the polynomials $Y_n^{(k)}(x;\lambda)$, computed with the help of Algorithm 1, are given as follows:

$$\begin{aligned}
Y_1^{(k)}(x;\lambda) &= \lambda\left(\frac{2}{\lambda-1}\right)^k\left(x-\frac{k\lambda}{\lambda-1}\right),\\
Y_2^{(k)}(x;\lambda) &= \lambda^2\left(\frac{2}{\lambda-1}\right)^k\left(x^2-\left(1+\frac{2k\lambda}{\lambda-1}\right)x+k(k+1)\left(\frac{\lambda}{\lambda-1}\right)^2\right),\\
Y_3^{(k)}(x;\lambda) &= \lambda^3\left(\frac{2}{\lambda-1}\right)^k\left(x^3-3\left(1+\frac{k\lambda}{\lambda-1}\right)x^2+\left(2+\frac{3k\lambda}{\lambda-1}+3k(k+1)\left(\frac{\lambda}{\lambda-1}\right)^2\right)x\right.\\
&\quad\left.-k(k+1)(k+2)\left(\frac{\lambda}{\lambda-1}\right)^3\right),\ldots
\end{aligned}$$

Remark 2.6 *Note that since (2.3) holds true, the values of the numbers $Y_n^{(k)}(\lambda)$ can be computed by substituting x=0 into the polynomials $Y_n^{(k)}(x;\lambda)$ obtained by the Algorithm 1.*

3. Some identities and relations for the numbers $Y_n^{(k)}(\lambda)$

In this section, by making use of generating functions and their functional equations, we derive some identities and relations with the inclusion of not only the numbers $Y_n^{(k)}(\lambda)$, but also the Apostol-type numbers, the Stirling numbers of the first kind, the Bernstein basis functions, and combinatorial sums.

Theorem 3.1

$$\sum_{j=0}^{n}(-1)^j Y_j(\lambda) Y_{n-j}(\lambda) = \frac{4\lambda^{2n}(1+(-1)^n)(n+1)!}{(n+2)(\lambda-1)^{n+2}}.$$

Proof

$$\sum_{j=0}^{n}(-1)^j Y_j(\lambda) Y_{n-j}(\lambda) = \frac{4(-1)^n n!}{(\lambda-1)^2}\left(\frac{\lambda^2}{\lambda-1}\right)^n \sum_{j=0}^{n}(-1)^j \frac{1}{\binom{n}{j}}. \tag{3.1}$$

Substituting the following well-known identity (*cf.* [7, Eq. (5.13)], [27, Eq. (13)]):

$$\sum_{j=0}^{n}(-1)^j \frac{1}{\binom{n}{j}} = (1+(-1)^n)\frac{n+1}{n+2}$$

into Equation (3.1) and after some elementary calculations, we arrive at the desired result. □

Theorem 3.2

$$\sum_{j=0}^{n} Y_j(\lambda) Y_{n-j}(\lambda) = \frac{(-1)^n(n+1)!}{2^{n-2}(\lambda-1)^2}\left(\frac{\lambda^2}{\lambda-1}\right)^n \sum_{j=0}^{n}\frac{2^j}{j+1}.$$

Proof

$$\sum_{j=0}^{n} Y_j(\lambda) Y_{n-j}(\lambda) = \frac{4(-1)^n n!}{(\lambda-1)^2}\left(\frac{\lambda^2}{\lambda-1}\right)^n \sum_{j=0}^{n}\frac{1}{\binom{n}{j}}. \tag{3.2}$$

Substituting the following well-known identity (*cf.* [35]):

$$\sum_{j=0}^{n}\frac{1}{\binom{n}{j}} = \frac{n+1}{2^n}\sum_{j=0}^{n}\frac{2^j}{j+1}$$

into Equation (3.2) and after some elementary calculations, we arrive at the desired result. □

Recall that the Bernstein basis functions are defined by

$$B_j^k(\lambda) = \binom{k}{j}\lambda^j(1-\lambda)^{k-j}, \tag{3.3}$$

(*cf.* [18]). By substituting (3.3) into (2.6), we derive a relation between the Bernstein basis functions and the numbers $Y_n^{(k)}(\lambda)$ by the following theorem:

Theorem 3.3 *If n is a positive integer, then we have*

$$\sum_{j=0}^{k}(-1)^{k-j}(n)_j \lambda^j B_j^k(\lambda) Y_{n-j}^{(k)}(\lambda) = 0.$$

Theorem 3.4

$$Y_v^{(k)}(\lambda) = (-1)^{k+1} 2^k \lambda^v \sum_{m=0}^{v} \frac{s(v,m)\,\mathcal{B}_{m+1}^{(k)}(\lambda)}{m+1}. \tag{3.4}$$

Proof Replacing $1+\lambda t$ by $e^{\log(1+\lambda t)}$ in (2.1), for $\left|\lambda e^{\log(1+\lambda t)}\right| < 1$, we have

$$\begin{aligned}\sum_{v=0}^{\infty} Y_v^{(k)}(\lambda)\frac{t^v}{v!} &= \frac{2^k}{\left(\lambda e^{log(1+\lambda t)} - 1\right)^k} \\ &= (-1)^k 2^k \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n e^{n\log(1+\lambda t)} \\ &= (-1)^k 2^k \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n \sum_{m=0}^{\infty} \frac{[n\log(1+\lambda t)]^m}{m!}.\end{aligned}$$

Combining (1.7) with the above equation yields:

$$\begin{aligned}\sum_{v=0}^{\infty} Y_v^{(k)}(\lambda)\frac{t^v}{v!} &= (-1)^k 2^k \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n \sum_{m=0}^{\infty} n^m \sum_{v=0}^{\infty} s(v,m)\frac{(\lambda t)^v}{v!} \\ &= (-1)^k 2^k \sum_{v=0}^{\infty} \left(\sum_{m=0}^{v} s(v,m) \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n n^m\right) \frac{(\lambda t)^v}{v!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^v}{v!}$ on both sides of the above equation, we get

$$Y_v^{(k)}(\lambda) = (-1)^k 2^k \lambda^v \sum_{m=0}^{v} s(v,m) \sum_{n=0}^{\infty} \binom{n+k-1}{n} \lambda^n n^m. \tag{3.5}$$

Combining (1.4) with (3.5) yields the assertion of the theorem. □

Remark 3.5 *Substituting $k=1$ into (3.4), we get Theorem 9 in [34].*

Theorem 3.6

$$Y_m^{(k)}(-\lambda) = (-1)^{m+k} \lambda^m \sum_{n=0}^{m} \mathcal{E}_n^{(k)}(\lambda)\, s(m,n). \tag{3.6}$$

Proof When we replace $1-\lambda t$ by $e^{\log(1-\lambda t)}$ in (2.1), for $\left|\lambda e^{\log(1-\lambda t)}\right| < 1$, we have

$$\begin{aligned}\sum_{m=0}^{\infty} Y_m^{(k)}(-\lambda)\frac{t^m}{m!} &= \frac{2^k(-1)^k}{\left(\lambda e^{log(1-\lambda t)} + 1\right)^k} \\ &= (-1)^k \sum_{n=0}^{\infty} \mathcal{E}_n^{(k)}(\lambda) \frac{[\log(1-\lambda t)]^n}{n!}.\end{aligned}$$

Combining (1.7) with the above equation yields:

$$\sum_{m=0}^{\infty} Y_m^{(k)}(-\lambda)\frac{t^m}{m!} = (-1)^k \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} (-\lambda)^m \mathcal{E}_n^{(k)}(\lambda)\, s(m,n)\right) \frac{t^m}{m!}.$$

After equalizing the coefficients of the variable $\frac{t^m}{m!}$ in the previous equation with the necessary calculations yields the assertion of the theorem. □

Remark 3.7 *Substituting $k=1$ into (3.6), we get Theorem 10 in [34].*

4. Formulas arising from derivatives of the functions $\mathcal{F}(t,k;\lambda)$

In this section, we obtain some derivative formulas and recurrence formulas for the numbers $Y_n^{(k)}(\lambda)$.

Theorem 4.1 *Let $n,k,v\in\mathbb{N}_0$. Then we have*

$$Y_{n+v}^{(k)}(\lambda)=\frac{(-1)^v(k)^{(v)}\lambda^{2v}}{2^v}Y_n^{(k+v)}(\lambda). \tag{4.1}$$

where

$$(x)^{(n)}=x(x+1)(x+2)\dots(x+n-1).$$

Proof Differentiating the functions $\mathcal{F}(t,k;\lambda)$ with respect to variable t, we obtain the following derivative formula:

$$\frac{\partial}{\partial t}\{\mathcal{F}(t,k;\lambda)\}=-\frac{k}{2}\lambda^2\mathcal{F}(t,k+1;\lambda).$$

Therefore, iterating the above derivation v times for the variable t yields the following partial differential equation:

$$\frac{\partial^v}{\partial t^v}\{\mathcal{F}(t,k;\lambda)\}=\frac{(-1)^v(k)^{(v)}\lambda^{2v}}{2^v}\mathcal{F}(t,k+v;\lambda).$$

Combining (2.1) with the above differential equation yields the assertion of the theorem. □

Combining (2.4) with (4.1), we also get the following corollary:

Corollary 4.2 *Let $n,v,k\in\mathbb{N}$. Then we have*

$$\binom{n+v+k-1}{n+v}(n+v)!=\binom{n+v+k-1}{n}(k)^{(v)}n!. \tag{4.2}$$

Theorem 4.3

$$\frac{d}{d\lambda}Y_n^{(k)}(\lambda)=-\frac{k}{2}\left(2\lambda nY_{n-1}^{(k+1)}(\lambda)+Y_n^{(k+1)}(\lambda)\right).$$

Proof Differentiating the functions $\mathcal{F}(t,k;\lambda)$ with respect to variable λ, we obtain the following derivative formula:

$$\frac{\partial}{\partial\lambda}\{\mathcal{F}(t,k;\lambda)\}=-\frac{k}{2}(2\lambda t+1)\mathcal{F}(t,k+1;\lambda).$$

From (2.1), we thus have

$$\sum_{n=0}^{\infty}\frac{d}{d\lambda}Y_n^{(k)}(\lambda)\frac{t^n}{n!}\quad=\quad-\frac{k}{2}(2\lambda t+1)\sum_{n=0}^{\infty}Y_n^{(k+1)}(\lambda)\frac{t^n}{n!}$$

$$
\begin{aligned}
&= -k\lambda t\sum_{n=0}^{\infty} Y_n^{(k+1)}(\lambda)\frac{t^n}{n!} - \frac{k}{2}\sum_{n=0}^{\infty} Y_n^{(k+1)}(\lambda)\frac{t^n}{n!} \\
&= -k\lambda\sum_{n=0}^{\infty} nY_{n-1}^{(k+1)}(\lambda)\frac{t^n}{n!} - \frac{k}{2}\sum_{n=0}^{\infty} Y_n^{(k+1)}(\lambda)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(-k\lambda nY_{n-1}^{(k+1)}(\lambda) - \frac{k}{2}Y_n^{(k+1)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}
$$

After equalizing the coefficients of the variable $\frac{t^n}{n!}$ in the previous equation with the necessary calculations yields the assertion of the theorem. □

5. Chu–Vandermonde-type convolution formula arising from functional equations of the generating function for the numbers $Y_n^{(k)}(\lambda)$

In this section, we give Chu–Vandermonde-type convolution formulas derived from functional equations of the generating function $\mathcal{F}(t,k;\lambda)$. We also give some special values of these formulas including the Catalan numbers and combinatorial sums.

Let $m \in \mathbb{N}$ and $k_1, k_2, \ldots, k_m \in \mathbb{N}$. Using (2.1) yields the following functional equation:

$$\mathcal{F}(t,k_1+k_2+\cdots+k_m;\lambda) = \mathcal{F}(t,k_1;\lambda)\,\mathcal{F}(t,k_2;\lambda)\ldots\mathcal{F}(t,k_m;\lambda).$$

By rearranging the above functional equation, we get

$$\sum_{n=0}^{\infty} Y_n^{(k_1+k_2+\cdots+k_m)}(\lambda)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} Y_n^{(k_1)}(\lambda)\frac{t^n}{n!}\right)\cdots\left(\sum_{n=0}^{\infty} Y_n^{(k_m)}(\lambda)\frac{t^n}{n!}\right).$$

Applying the Cauchy product rule in the above equation, we obtain

$$
\begin{aligned}
\sum_{n=0}^{\infty} Y_n^{(k_1+k_2+\cdots+k_m)}(\lambda)\frac{t^n}{n!} &= \sum_{v_1+v_2+\cdots+v_{m-1}=n} \frac{Y_{v_{m-1}}^{(k_m)}(\lambda)}{(v_{m-1})!}\frac{Y_{v_{m-2}}^{(k_{m-1})}(\lambda)}{(v_{m-2})!}\cdots \\
&\quad \times\frac{Y_{v_1}^{(k_1)}(\lambda)}{v_1!}\frac{Y_{n-v_1-v_2-\cdots-v_{m-1}}^{(k_2)}(\lambda)}{(n-v_1-v_2-\cdots-v_{m-1})!}t^n,
\end{aligned}
$$

where

$$\sum_{v_1+v_2+\cdots+v_{m-1}=n}$$

denotes

$$\sum_{v_{m-1}=0}^{n}\ \sum_{v_{m-2}=0}^{n-v_{m-1}}\cdots\sum_{v_1=0}^{n-v_2-v_3-\cdots-v_{m-1}}.$$

By comparing the coefficients of t^n in the above equation, we arrive at the following theorem:

Theorem 5.1 *Let* $m \in \mathbb{N}$, $k_1, k_2, \ldots, k_m \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then we have*

$$Y_n^{(k_1+k_2+\cdots+k_m)}(\lambda) = \sum_{v_1+v_2+\cdots+v_{m-1}=n} \mathcal{C}^n_{v_1,v_2,\ldots,v_{m-1}} Y^{(k_m)}_{v_{m-1}}(\lambda)\, Y^{(k_{m-1})}_{v_{m-2}}(\lambda) \cdots Y^{(k_1)}_{v_1}(\lambda)\, Y^{(k_2)}_{n-v_1-v_2-\cdots-v_{m-1}}(\lambda),$$

where

$$\begin{aligned}\mathcal{C}^n_{v_1,v_2,\ldots,v_{m-1}} &= \binom{n}{v_1, v_2, \ldots, n-v_1-\cdots-v_{m-1}} \\ &= \frac{n!}{v_1! v_2! \ldots (n-v_1-\cdots-v_{m-1})!}.\end{aligned}$$

Remark 5.2 *Substituting* $m=2$ *into Theorem 5.1, we have*

$$Y_n^{(k_1+k_2)}(\lambda) = \sum_{v_1=0}^{n} \binom{n}{v_1} Y^{(k_1)}_{v_1}(\lambda)\, Y^{(k_2)}_{n-v_1}(\lambda). \tag{5.1}$$

Additionally, if we set $m=3$ *in Theorem 5.1, we have*

$$Y_n^{(k_1+k_2+k_3)}(\lambda) = \sum_{v_2=0}^{n} \sum_{v_1=0}^{n-v_2} \binom{n}{v_2} \binom{n-v_2}{v_1} Y^{(k_1)}_{v_1}(\lambda)\, Y^{(k_2)}_{n-v_1-v_2}(\lambda)\, Y^{(k_3)}_{v_2}(\lambda).$$

Theorem 5.3 *Let* $m \in \mathbb{N}$, $k_1, k_2, \ldots, k_m \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then we have*

$$\begin{aligned}\binom{k_1+k_2+\cdots+k_m+n-1}{n} &= \sum_{v_1+v_2+\cdots+v_{m-1}=n} \binom{k_m+v_{m-1}-1}{v_{m-1}} \binom{k_{m-1}+v_{m-2}-1}{v_{m-2}} \cdots \\ &\times \binom{k_1+v_1-1}{v_1} \binom{k_2+n-v_1-v_2-\cdots-v_{m-1}-1}{n-v_1-v_2-\cdots-v_{m-1}}.\end{aligned}$$

Proof By combining (2.4) with Theorem 5.1 yields the Chu–Vandermonde-type convolution formula. Therefore, the details of the proof is omitted. □

Next, we present some special cases of Theorem 5.3 by the following corollaries:

Corollary 5.4 *Let* $k_1, k_2 \in \mathbb{N}$ *and* $n \in \mathbb{N}_0$. *Then we have*

$$\binom{k_1+k_2+n-1}{n} = \sum_{v_1=0}^{n} \binom{k_1+v_1-1}{v_1} \binom{k_2+n-v_1-1}{n-v_1}. \tag{5.2}$$

Proof Substituting $m=2$ into Theorem 5.3 yields the assertion of this corollary. Particularly, combining (5.1) with (2.4) yields

$$\sum_{v_1=0}^{n} \binom{n}{v_1} Y^{(k_1)}_{v_1}(\lambda)\, Y^{(k_2)}_{n-v_1}(\lambda) = (-1)^n n! \left(\frac{2}{\lambda-1}\right)^{k_1+k_2} \binom{k_1+k_2+n-1}{n} \left(\frac{\lambda^2}{\lambda-1}\right)^n. \tag{5.3}$$

By combining (2.4) with (5.3), we obtain

$$\sum_{v_1=0}^{n}\binom{n}{v_1}\left((-1)^{v_1}\binom{k_1+v_1-1}{v_1}\frac{2^{k_1}v_1!\lambda^{2v_1}}{(\lambda-1)^{k_1+v_1}}\right)\left((-1)^{n-v_1}\binom{k_2+n-v_1-1}{n-v_1}\frac{2^{k_2}(n-v_1)!\lambda^{2(n-v_1)}}{(\lambda-1)^{k_2+n-v_1}}\right)$$
$$= (-1)^n n!\left(\frac{2}{\lambda-1}\right)^{k_1+k_2}\binom{k_1+k_2+n-1}{n}\left(\frac{\lambda^2}{\lambda-1}\right)^n.$$

Doing straightforward calculations in the previous equation yields the desired result. □

Remark 5.5 *Substituting* $k_1=a+1$ *and* $k_2=r+1$ *into* (5.2) *yields* Eq. (1.78) *in* [6], *substituting* $k_1=k_2=r+1$ *into* (5.2) *yields* Eq. (1.79) *in* [6], *and also substituting* $k_1=r+1$ *and* $k_2=n+r+1$ *into* (5.2) *yields* Eq. (1.82) *in* [6].

Substituting $m=3$ into Theorem 5.3, we also obtain the following corollary:

Corollary 5.6 *Let* $k_1,k_2,k_3\in\mathbb{N}$ *and let* $n\in\mathbb{N}_0$. *Then we have*

$$\binom{k_1+k_2+k_3+n-1}{n} = \sum_{v_2=0}^{n}\sum_{v_1=0}^{n-v_2}\binom{k_1+v_1-1}{v_1}\binom{k_2+n-v_1-v_2-1}{n-v_1-v_2}\binom{k_3+v_2-1}{v_2}.$$

Remark 5.7 *The well-known Chu–Vandermonde identity is given as follows:*

$$\binom{x+a}{k}=\sum_{j=0}^{k}\binom{x}{j}\binom{a}{k-j} \tag{5.4}$$

(*cf.* [5, 10, 26]). *Since some of the obtained combinatorial sums are analogues of (5.4), we call them Chu–Vandermonde-type convolution formulas.*

5.1. Some applications related to the Chu–Vandermonde-type convolution formulas

Here, we give some applications related to the obtained Chu-Vandermonde type convolution formulas and provide some combinatorial sums.

Substituting $k_1=k_2=n$ into (5.2), and since

$$\binom{3n-1}{n}=\frac{2}{3}\binom{3n}{n},$$

we obtain the following corollary:

Corollary 5.8 *Let* $n\in\mathbb{N}_0$. *Then we have*

$$\binom{3n}{n}=\frac{3}{2}\sum_{j=0}^{n}\binom{n+j-1}{j}\binom{2n-j-1}{n-j}.$$

Substituting $k_1=n+1$ and $k_2=n$ into (5.2), we obtain the following corollary:

Corollary 5.9 *Let $n \in \mathbb{N}_0$. Then we have*

$$\binom{3n}{n} = \sum_{j=0}^{n} \binom{n+j}{j} \binom{2n-j-1}{n-j}. \tag{5.5}$$

Corollary 5.10 *Let $n \in \mathbb{N}_0$. Then we have*

$$C_n = \frac{1}{n+1}\binom{3n}{n} - \frac{1}{n+1}\sum_{j=0}^{n-1} \binom{n+j}{j} \binom{2n-j-1}{n-j}. \tag{5.6}$$

Proof By using (5.5), we have

$$\binom{3n}{n} - \binom{2n}{n} = \sum_{j=0}^{n-1} \binom{n+j}{j} \binom{2n-j-1}{n-j}.$$

Multiplying both sides of the above equation by $\frac{1}{n+1}$ and using (1.8) yields the desired results. □

In [26, Corollary 6.6], Simsek gave the following formula for the Catalan numbers:

$$C_n = \frac{2n+1}{n+1} \sum_{j=n}^{2n} (-1)^{n+j} \binom{2n}{j} \frac{1}{j+1}.$$

Combining the above equation with (5.6), we arrive at a combinatorial sum, including binomial coefficients, by the following theorem:

Theorem 5.11 *Let $n \in \mathbb{N}_0$. Then we have*

$$\binom{3n}{n} = \sum_{j=n}^{2n} (-1)^{n+j} \binom{2n}{j} \frac{2n+1}{j+1} + \sum_{j=0}^{n-1} \binom{n+j}{j} \binom{2n-j-1}{n-j}.$$

6. Probabilistic distributions arising from the Chu–Vandermonde-type formulas

In this section, we give some theoretical and applied details on probabilistic distributions arising from the aforementioned Chu–Vandermonde type convolution formulas.

By using Theorem 5.3, we arrive at the following probability functions for negative hypergeometric-type distribution with the parameters $k_1, k_2, \ldots, k_m$ and n with the random variables $(v_1, v_2, \ldots, v_n)$:

$$\begin{aligned} & f\left(v_1, \ldots, v_n; k_1 + \cdots + k_m + n - 1, n, k_1, \ldots, k_n\right) \\ = \; & \frac{\binom{k_m+v_{m-1}-1}{v_{m-1}}\binom{k_{m-1}+v_{m-2}-1}{v_{m-2}}\cdots\binom{k_1+v_1-1}{v_1}\binom{k_2+n-v_1-v_2-\cdots-v_{m-1}-1}{n-v_1-v_2-\cdots-v_{m-1}}}{\binom{k_1+k_2+\cdots+k_m+n-1}{n}}. \end{aligned} \tag{6.1}$$

By Theorem 5.3, we see that

$$\sum_{v_1+v_2+\cdots+v_{m-1}=n} f\left(v_1, \ldots, v_n; k_1 + \cdots + k_m + n - 1, n, k_1, \ldots, k_n\right) = 1.$$

Substituting $n = 2$ into (6.1), (6.1) reduces to (6.2). That is, this gives us the following well-known probability function for negative hypergeometric distribution:

$$f(v_1, k_1 + k_2 + n - 1, n, k_1) = \frac{\binom{v_1+k_1-1}{v_1}\binom{k_2+n-v_1-1}{n-v_1}}{\binom{k_1+k_2+n-1}{n}}, \tag{6.2}$$

where $k_1 + k_2 + n - 1$ is the population size, n is the number of success states in the population, k_1 is the number of failures, v_1 is the number of observed successes for $0 \leq v_1 \leq n$; $0 \leq k_1 \leq n$ (*cf.* [9, 16]).

Moment-generating function for the probability function $f(v_1, \ldots, v_n; k_1 + \cdots + k_m + n - 1, n, k_1, \ldots, k_n)$ is given by

$$M(t; k_1, \ldots, k_m, n) = \sum_{v_1+v_2+\cdots+v_{m-1}=n} e^{t(v_1+v_2+\cdots+v_{m-1})} f(v_1, \ldots, v_n; k_1 + \cdots + k_m + n - 1, n, k_1, \ldots, k_n).$$

The j-th moment is the j-th derivative of $M(t; k_1, \ldots, k_m, n)$ computed at $t = 0$. That is

$$\frac{d^j}{dt^j}\left\{M(t; k_1, \ldots, k_m, n)\right\}\Big|_{t=0} = \mu_j.$$

In the special case of the above equation when $m = 2$, we have

$$\begin{aligned} \mu_1 &= \frac{nk_1}{k_1 + k_2}, \\ \mu_2 &= \frac{nk_1(n(1 + k_1) + k_2)}{(k_1 + k_2)(1 + k_1 + k_2)}, \end{aligned}$$

(*cf.* [9, 16]).

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