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Modules in which semisimple fully invariant submodules are essential in summands

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Abstract: One of the useful generalization of extending notion is $FI$-extending property. A module is called $FI$-extending if every fully invariant submodule is essential in a direct summand. In this paper, we explore Weak $FI$-extending concept by considering only semisimple fully invariant submodules rather than all fully invariant submodules. To this end, we call such a module Weak $FI$-extending. We obtain that $FI$-extending modules are properly contained in this new class of modules. Amongst other structural properties, we also deal with direct sums and direct summands of Weak $FI$-extending modules.

Key words: Extending module, socle of a module, $C_{11}$-module, Weak $CS$-module, fully invariant, $FI$-extending.

1. Introduction

All rings are associative with unity and modules are unital right modules. We use $R$ to denote such a ring and $M$ to denote a right $R$-module. Recall that a module is called $CS$ (or extending) if every submodule is essential in a direct summand; equivalently, every complement submodule is a direct summand. Note that this condition has proved to be an important common generalization of the injective, semisimple and uniform module (i.e., every non zero submodule is essential in the module) notions (see [5, 10, 17]).

There have been a number of useful generalizations of the extending property, including the following:

(1) $M$ is a weak $CS$ module (or WCS)[11] if every semisimple submodule of $M$ is essential in a direct summand of $M$;

(2) $M$ is a $C_{11}$-module [12, 13] if each submodule of $M$ has a complement that is a direct summand of $M$;

(3) $M$ is an $FI$-extending module [1, 2] if every fully invariant submodule (i.e.; every submodule such that the image under all endomorphisms contained in itself) is essential in a direct summand of $M$.

For the aforementioned generalizations as well as different kind of recent developments in the theory, see [4, 8, 17].

In a similar manner to weak $CS$-modules [11], weak $C_{11}$-modules were introduced in [6] (see, also [18]). Recall that a module $M$ is a weak $C_{11}$ (or WC$C_{11}$-)module if each semisimple submodule of $M$ has a complement that is a direct summand.

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In [16], the authors attempt to obtain results on direct summands of an FI-extending module. Since their main results therein contain gaps, in the present study, we introduce and investigate a weak version of FI-extending concept by considering only semisimple fully invariant submodules rather than all fully invariant submodules of the module. We call such a module Weak FI-extending (or, WFI-extending). In Section 2, we give some equivalent characterizations as well as structural properties of WFI-extending modules in common with WC_{11}-modules and FI-extending modules. In contrast to extending modules, we show that a direct sum of WFI-extending modules is also a WFI-extending module. Moreover, we provide examples which make it clear that the class of FI-extending modules is properly contained in the class of WFI-extending modules.

Observe that any nonuniform prime ring is an FI-extending module and it has essential socle. It follows that, there exist WFI-extending modules which are not WC_{11}. We have then, for any module, the following implications:

\[
\begin{align*}
CS \implies C_{11} \implies FI\text{-extending} \\
\downarrow \quad \downarrow \quad \downarrow \\
WCS \implies WC_{11} \implies WFI\text{-extending}
\end{align*}
\]

No other implications can be added to this table in general. To see why this is the case, we refer to [13, p.1814], [14, Example 11], [3, Proposition 1.2], [11, Example 1.1], [20, counterexample 3] and Example 2.4.

Recall that, it is not known that whether direct summands of Weak CS, Weak C_{11}, and FI-extending modules is also Weak CS, Weak C_{11} and FI-extending respectively or not. Motivated by the latter problem, in Section 3, we focus our attention on direct summands of WFI-extending (and also FI-extending) modules with a conditional direct summand property. Our results yield as corollary that direct summands of an FI-extending module with C_{3} and essential socle are also FI-extending. Furthermore, we obtain that if a module is WFI-extending (FI-extending, WC_{11}, WCS, C_{11}) with summand intersection property (SIP) then so does a direct summand of it.

Let R be any ring and M a right R-module. If X \subseteq M, then X \leq M, Soc X and E(M) denote X as a submodule of M, the socle of X and the injective hull of M respectively. For any unexplained terminology, definitions, and notations, see [5, 10, 17].

2. Weak FI-extending modules

In this section, we give some equivalent conditions to WFI-extending modules. Also, we obtain some structural properties of WFI-extending modules in common with WC_{11}-modules and FI-extending modules. We provide examples which show that the new class of modules properly contains the class of FI-extending modules. Since WFI-extending modules are based on the class of semisimple fully invariant submodules of the module, it is better to start with their basic properties.

**Lemma 2.1** Let M be a right R-module. Then

(i) Any sum or intersection of semisimple fully invariant submodules of M is again a semisimple fully invariant submodule of M.

(ii) If \( M = \bigoplus_{i \in I} M_i \) and X is a semisimple fully invariant submodule of M, then \( X = \bigoplus_{i \in I} \pi_i(X) = \bigoplus_{i \in I} (M_i \cap X) \), where \( \pi_i \) is the i-th canonical projection homomorphism of M.
If $A$ is semisimple fully invariant in $B$ and $B$ is semisimple fully invariant in $M$ then $A$ is semisimple fully invariant in $M$.

**Proof** Immediate by definitions.

**Definition 2.2** A module $M$ is called Weak FI-extending (or WFI-extending) if every semisimple fully invariant submodule of $M$ is essential in a direct summand of $M$.

Observe that any $WC_{11}$-module and FI-extending module is WFI-extending. Moreover, any module which has zero socle (for example, a polynomial ring $R[x]$ over any ring $R$) is clearly a WFI-extending module. The following characterization shows that WFI-extending property behaves like FI-extending property in terms of complements and lifting homomorphisms.

**Proposition 2.3** Let $M$ be a module. The following conditions are equivalent.

(i) $M$ is WFI-extending,

(ii) Every semisimple fully invariant submodule of $M$ has a complement which is a direct summand,

(iii) For each semisimple fully invariant submodule $X$ of $M$, there exists a complement submodule $L$ of $M$ and a complement $K$ of $L$ such that $X$ is essential in $L$ and any homomorphism from $L \oplus K$ to $M$ lifts to $M$.

**Proof** (i)$\iff$(ii) Let $X$ be a semisimple fully invariant submodule of $M$. First assume that $M$ is WFI-extending. There exists $e^2 = e \in \text{End}(M)$ such that $X$ is essential in $eM$. Hence, $(1 - e)M$ is the desired complement. Conversely, let $cM$ be a complement of $X$, where $c^2 = c \in \text{End}(M)$. Let $x \in X$. Then $x = cx + (1 - c)x$. Since $X$ is fully invariant, $cx \in X \cap cM = 0$. Thus, $X \subseteq (1 - c)M$, and so $X$ is essential in $(1 - c)M$.

(i)$\iff$(iii) This equivalence is a direct consequence of [14, Lemma 2] (see, also [17, Lemma 3.97]).

Next we provide several WFI-extending modules which are not FI-extending.

**Example 2.4** (i) [14, Example 11]. There exists a commutative valuation domain $S$ such that every homomorphic image of $S$ is a self-injective ring. There exists an ideal $A$ of $S$ such that $S/A$ is a local ring which has zero socle. Let $T = S/A$ and $J$ be the unique maximal ideal of $T$. Let $R$ be the subring of $T \oplus T$ defined by $R = \{(t, t') | t - t' \in J\}$. Then $R_R$ is not a $C_{11}$-module by [13, Proposition 3.2 and Theorem 3.10]. By [3, Proposition 1.3], $R_R$ is not FI-extending. However, it is clear that $R_R$ is WFI-extending.

(ii) [2, Example 4.11]. Let $D$ be a simple domain which is not a division ring. Take $R = \begin{bmatrix} D & D \oplus D \\ 0 & D \end{bmatrix}$ then $I = \begin{bmatrix} 0 & D \oplus D \\ 0 & 0 \end{bmatrix}$ is an ideal of $R$ i.e., $I$ is a fully invariant submodule of $R_R$ which is not essential in a right ideal direct summand of $R$. It follows that $R_R$ is not FI-extending. Since $\text{Soc}(R_R) = 0$, $R_R$ is WFI-extending.

(iii) [18, Proposition 14]. Let $R$, $M$, and $K$ be as in [18, Proposition 14]. Note that $K_R$ corresponds to the tangent bundle of the related sphere. Furthermore, $K_R$ is an indecomposable module of uniform dimension.
bigger than one. Hence, \( K_R \) is not uniform. Let us take the trivial extension of \( R \) with \( K \), say \( T \). Thus, \[
T = \begin{bmatrix} R & K \\ 0 & R \end{bmatrix} = \left\{ \begin{bmatrix} r & x \\ 0 & r \end{bmatrix} \middle| r \in R, x \in K \right\}. \] Hence, \( T_T \) is a commutative indecomposable module which is not FI-extending [3, Proposition 1.3]. However, it is easy to see that \( T \) is WFI-extending.

There are more algebraic topological examples with the same type of last part of the former example. For the construction of these examples, see [9, Theorems 2.4 and 2.5]. Example 2.4 brings us the natural question, namely, when a WFI-extending module is an FI-extending module. Thus, we have the next result.

**Proposition 2.5** Let \( M \) be a WFI-extending module with essential socle. Then \( M \) is FI-extending.

**Proof** Let \( X \) be a fully invariant submodule of \( M \). If \( X = 0 \) then \( M \) will do. Assume \( X \neq 0 \). Note that \( \text{Soc} X \) is a fully invariant submodule of \( X \). By Lemma 2.1, \( \text{Soc} X \) is a semisimple fully invariant submodule of \( M \). By hypothesis, there exists a direct summand \( L \) of \( M \) such that \( \text{Soc} X \) is essential in \( L \). Now \( M = L \oplus L' \) for some submodule \( L' \) of \( M \). It is clear that \( \text{Soc} X \) is essential in \( X \). Hence, \( X \cap L' = 0 \). Thus, \( \text{Soc} X \oplus L' \leq X \oplus L' \) and \( \text{Soc} X \oplus L' \) is essential in \( M \) yield that \( X \oplus L' \) is essential in \( M \). Hence, the result follows by Proposition 2.3.

Next result shows that the WFI-extending property is inherited by fully invariant submodules as in FI-extending concept (see [2, Proposition 1.2]).

**Proposition 2.6** Let \( M \) be a WFI-extending module and \( X \) a fully invariant submodule of \( M \). Then \( X \) is WFI-extending.

**Proof** Let \( S \) be a semisimple fully invariant submodule of \( X \). By Lemma 2.1, \( S \) is semisimple fully invariant in \( M \). Hence, there is a direct summand \( D \) of \( M \) such that \( S \) is essential in \( D \). Let \( \pi : M \to D \) be the projection endomorphism. Then \( S = \pi(S) \leq \pi(X) \cap D = \pi(X) \). Hence, \( S \) is essential in \( \pi(X) \) and \( \pi(X) \) is a direct summand of \( X \).

**Proposition 2.7** Let \( M \) be a module. Then \( M \) is WFI-extending if and only if for each semisimple fully invariant submodule \( S \) of \( M \) there exists \( e = e^2 \in \text{End}(E(M)) \) such that \( S \) is essential in \( e(E(M)) \) and \( e(M) \subseteq M \).

**Proof** Assume that \( M \) is WFI-extending. There exists a direct summand \( X \) of \( M \) such that \( S \) is essential in \( X \). Now \( M = X \oplus Y \) for some submodule \( Y \) of \( M \). Hence, there exists injective hulls \( E(X) \) and \( E(Y) \) such that \( E(M) = E(X) \oplus E(Y) \). Let \( e : E(M) \to E(X) \) be the projection endomorphism. Then \( e(M) \leq M \) and \( S \) is essential in \( e(E(M)) \). Conversely, let \( S \) be a semisimple fully invariant submodule of \( M \). Then \( S \) is essential in \( M \cap e(E(M)) = e(M) \). However, \( e(M) \) is a direct summand of \( M \). Hence, \( M \) is WFI-extending.

Recall that in contrast to extending (or CS) modules, some of their generalizations \( C_{11} \), \( WC_{11} \), FI-extending modules behave better on direct sums [13, Theorem 2.4], [6, Theorem 1.20], [2, Theorem 1.3]. To this end, we show that a direct sum of WFI-extending modules enjoys with WFI-extending property.

**Theorem 2.8** Direct sums of modules with the WFI-extending property have again the WFI-extending property.
Proof Suppose the modules $M_i (i \in I)$ have the WFI-extending property. If $S$ is a semisimple fully invariant submodule of the direct sum $M = \bigoplus_{i \in I} M_i$, then $S = \bigoplus_{i \in I} (S \cap M_i)$ by Lemma 2.1. Clearly, $S \cap M_i$ is a semisimple fully invariant submodule of $M_i$ for each $i \in I$. By hypothesis, $S \cap M_i$ is contained as an essential submodule in a direct summand $D_i$ of $M_i$ for each $i \in I$. Then $D = \bigoplus_{i \in I} D_i$ is a direct summand of $M$ that is an essential extension of the submodule $S$. \hfill \Box

Corollary 2.9 If $M$ is a direct sum of FI-extending (e.g., $C_{11}$, $WC_{11}$, extending) modules, then $M$ is WFI-extending.

Proof Clear by Theorem 2.8. \hfill \Box

3. Direct summands of Weak FI-extending modules

In this section, we deal with direct summands of WFI-extending modules. Recall that, it is not known so far whether direct summands of Weak CS, Weak $C_{11}$, and FI-extending modules is again Weak CS, Weak $C_{11}$, and FI-extending respectively or not. In this trend, it is also an open problem whether direct summands of a WFI-extending module are WFI-extending or not. However, we provide some positive answers for the former question by adding some conditional direct summand properties on the module. First of all, we prove the following easy result which provides a decomposition of a WFI-extending module.

Lemma 3.1 Let $M$ be a WFI-extending module. Then $M = M_1 \oplus M_2$ where $M_1$ is a submodule of $M$ with essential socle and $M_2$ a submodule of $M$ with zero socle.

Proof Let $S$ denote the socle of $M$. Then $S$ is a semisimple fully invariant submodule of $M$. By assumption, there exist submodules $M_1$ and $M_2$ of $M$ such that $S$ is essential in $M_1$ and $M = M_1 \oplus M_2$. Now $S = \text{Soc} M = \text{Soc} M_1 \oplus \text{Soc} M_2$. Clearly, $\text{Soc} M_2 = 0$. Hence, $S = \text{Soc} M_1$. \hfill \Box

One might conjecture that whether the converse of Lemma 3.1 is true or not. However, the next example eliminates this possibility. Incidentally, we refer to [17, p. 257] and [15] for details on the construction of this interesting example.

Example 3.2 Let $R$ be the ring in [15, An example]. Let $M_1 = R$ and $M_2 = R/I$ where $I = \text{Soc} R$. Note that $I$ is essential in $M_1$ and $\text{Soc}(M_2) = 0$. Then the $R$-module $M = M_1 \oplus M_2$ is not WFI-extending. Assume the contrary that $M$ is WFI-extending. Let $N$ be a simple fully invariant submodule of $M$. Then there exists a direct summand $L$ of $M$ such that $N$ is essential in $L$. Hence, $N = \text{Soc} N = \text{Soc} L$. By [17, Lemma 5.30], either $\text{Soc} L = 0$ or $\text{Soc} L = \text{Soc} M$. We have that either $N = 0$ or $N = I \oplus 0$. In any case, we have a contradiction. It follows that $M$ is not WFI-extending.

Before proving our main results on direct summands of a WFI-extending module, we should give the following two observations which are basically related to our aim in this section. Firstly, observe that definitions of extending, $WC_{11}$, FI-extending and WFI-extending modules require direct summands but not uniqueness of them. Now, we obtain uniqueness up to isomorphism for the WFI-extending modules with a conditional direct summand property, $C_3$. Recall that a module $M$ is said to satisfy the $C_3$ condition if the sum of any two direct summands of $M$ with zero intersection is a direct summand of $M$ (see [17]).

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Proposition 3.3 Let $M$ be a WFI-extending module which satisfies $C_3$. If $X$ is a semisimple fully invariant submodule of $M$ then $X$ is essentially contained in direct summands of $M$ which are unique up to isomorphism.

Proof Assume that $X$ is essentially contained in direct summands $K$ and $L$ of $M$. Then $M = K \oplus L$. Now, let $\pi : M \to K$ be the canonical projection. By $C_3$ condition, $M \oplus K'$ is also a direct summand of $M$ which yields that $\pi(L)$ is a direct summand of $K$. Note that $\pi(L)$ is isomorphic to $L$ and $X = \text{Soc} L = \text{Soc} K$. Thus, $\text{Soc}(L \oplus K') = X \oplus \text{Soc} K' = \text{Soc} \pi(L) \oplus \text{Soc} K'$ and $\text{Soc}\pi(L)$ is in $X$. Hence, by the modular law, $X = X \cap (\text{Soc} \pi(L) \oplus \text{Soc} K') = \text{Soc}\pi(L) \oplus (X \cap \text{Soc} K') = \text{Soc}\pi(L)$ which gives that $X$ is contained in $\pi(L)$. It follows that $\pi(L)$ is essential in $K$. Since $\pi(L)$ is a direct summand of $K$, $\pi(L) = K \cong L$ which gives the result.

For the corresponding result on $WC_{11}$-modules to Proposition 3.3 (see [6, Proposition 1.7]). Our second observation is based on a structural property of fully invariant submodules. Let $M = M_1 \oplus M_2$ be a module. Let $N$ be a fully invariant submodule of $M_1$. Then $N \oplus M_2$ need not be a fully invariant submodule in $M$.

There are several counterexamples but we give the following easy case as an example to make things clear.

Example 3.4 Let $R$ be any ring and $M_1 = M_2 = R$. Let $I$ be any ideal of $R$ such that $I \neq R$, say $x \notin I$. Let $M = M_1 \oplus M_2$ be the $R$-module. Then $I$ is a fully invariant submodule of $M_1$. Now, define the $R$-homomorphism $f : M \to M$ by $f(a,b) = (b,a)$. Then, $f(0,x) = (x,0) \notin I \oplus M_2$. Thus, $f(I \oplus R) = f(I \oplus M_2) = M_2 \oplus I$ is not contained in $I \oplus M_2$, i.e., $I \oplus M_2$ is not a fully invariant submodule of $M$.

In [16], the authors mistakenly use the former situation. Since the proofs of [16, Theorems 2.5, 2.7 and Corollaries 2.8, 2.9 and 2.10] are based on the above structural property, their mentioned results on $FI$-extending modules are invalid.

Now, we return to obtain results on direct summands of a WFI-extending module.

Theorem 3.5 If the module $M = M_1 \oplus M_2$ has the WFI-extending property and $M_1$ is a semisimple fully invariant direct summand then both $M_1$ and $M_2$ have the WFI-extending property.

Proof It is clear that $M_1$ is WFI-extending. To prove that $M_2$ is WFI-extending, let $X$ be a semisimple fully invariant submodule of $M_2$. As $M_1$ is fully invariant in $M$, Hom$(M_1, M_2) = 0$. Thus $M_1 \otimes X$ is a semisimple fully invariant submodule of $M$. By hypothesis, there exists a direct summand $N$ of $M$ such that $M_1 \otimes X$ is essential in $N$. Now, $M = N \oplus W$ for some submodule $W$ of $M$. Thus, $N = N \cap (M_1 \otimes M_2) = M_1 \otimes (N \cap M_2)$ by the modular law. Since $M_1 \otimes X \leq N$ and $X \leq M_2$, $X \leq N \cap M_2$. Moreover, the fact that $M_1 \otimes X$ is essential in $N$ gives that $X$ is essential in $N \cap M_2$. Thus, $M = N \oplus W = (N \cap M_2) \oplus M_1 \oplus W$. Therefore, $M_2 = (N \cap M_2) \oplus ((M_1 \oplus W) \cap M_1)$ by the modular law. Thus, $X$ is essential in $N \cap M_2$ which is a direct summand of $M_2$. Thus, $M_2$ is WFI-extending.

Lemma 3.6 Let $M = M_1 \oplus M_2$. Then $M_1$ is WFI-extending module if and only if for every semisimple fully invariant submodule $N$ of $M_1$, there exists a direct summand $K$ of $M$ such that $M_2 \subseteq K$, $K \cap N = 0$, and $K + N$ is essential in $M$.

Proof Assume that $M_1$ is WFI-extending module. Let $N$ be a semisimple fully invariant submodule of $M_1$. By Lemma 2.1, there exists a direct summand $L$ of $M_1$ such that $N \cap L = 0$, and $N \oplus L$ is essential
in $M_1$. Clearly $L \oplus M_2$ is a direct summand of $M$, $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2) \oplus N$ is essential in $M$. Conversely, suppose that $M_1$ has the stated property. Let $H$ be a semisimple fully invariant submodule of $M_1$. By hypothesis, there exists a direct summand $K$ of $M$ such that $M_2 \subseteq K$, $K \cap H = 0$ and $K \oplus H$ is essential in $M$. Now, $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$ so that $K \cap M_1$ is a direct summand of $M$, and hence also of $M_1$, $H \cap (K \cap M_1) = 0$ and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$ which is essential in $M_1$. By Lemma 2.1, $M_1$ is WFI-extending.

**Theorem 3.7** Let a WFI-extending module $M = M_1 \oplus M_2$ be a direct sum of submodules $M_1$, $M_2$ such that $Soc M_2$ is essential in $M_2$ and for every direct summand $K$ of $M$ with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of $M$. Then $M_1$ is a WFI-extending module.

**Proof** Let $N$ be a semisimple fully invariant submodule of $M_1$. Then, it is easy to check that $N_2 = \sum_{\varphi \in \text{Hom}(M_1, M_2)} \varphi(N_1)$ is a fully invariant submodule of $M_2$. Now, the argument in [19, Theorem 3.1] yields that $N_1 \oplus N_2$ is a fully invariant submodule of $M$. Thus, $Soc (N_1 \oplus N_2) = N_1 \oplus Soc N_2$ is semisimple fully invariant in $M$, by Lemma 2.1. By hypothesis, there exists a direct summand $K'$ of $M$ such that $N_1 \oplus Soc N_2$ is essential in $K'$. Thus $M = K \oplus K'$ for some submodule $K$ of $M$ and $(N_1 \oplus Soc N_2) \cap K = 0$, $(N_1 \oplus Soc N_2) \oplus K$ is essential in $M$. Since $Soc M_2$ is essential in $M_2$, $N_1 \cap M_2 = 0$. It follows that $N_1 \oplus M_2 \oplus K$ is essential in $M$. Observe that $M_2 \oplus K$ is a direct summand of $M$. By Lemma 3.6, $M_1$ is WFI-extending.

Theorem 3.7 applies in the case that $M$ is a WFI-extending module satisfying condition $C_3$. Thus, we have at once:

**Corollary 3.8** If $M$ is a WFI-extending module satisfying $C_3$ and $M = M_1 \oplus M_2$ with $Soc M_2$ essential in $M_2$, then $M_1$ is WFI-extending.

**Proof** Immediate by Theorem 3.7.

Next result provides affirmative answer for the long standing open problem on direct summands of FI-extending [2] (and also WFI-extending) modules and brings new as well as correct version of [16, Corollary 2.10].

**Corollary 3.9** If $M$ is a WFI-extending module satisfying $C_3$ and $Soc M$ is essential in $M$, then any direct summand of $M$ is FI-extending.

**Proof** Since any direct summand of $M$ has essential socle, by Proposition 2.5 and Corollary 3.8 the result follows.

We apply former results to some more special cases including (relative) injectivity condition on one of the direct summand in the decomposition of the module.

**Corollary 3.10** Let a WFI-extending module $M = M_1 \oplus M_2$ be a direct sum of submodules $M_1$, $M_2$ such that, $Soc M_2$ is essential in $M_2$ and $M/M_1$ is $M_1$-injective. Then $M_1$ is a WFI-extending module.

**Proof** Since $M_2$ is isomorphic to $M/M_1$, $M_2$ is $M_1$-injective. Let $L$ be a direct summand of $M$ such that $L \cap M_2 = 0$. By [5, Lemma 7.5], there exists a submodule $H$ of $M$ such that $H \cap M_2 = 0$, $M = H \oplus M_2$ and $L \subseteq H$. Now $L$ is a direct summand of $H$ and hence $L \oplus M_2$ is a direct summand of $M = H \oplus M_2$. By Theorem 3.7, $M_1$ is a WFI-extending module.
Corollary 3.11 Let a module $M = M_1 \oplus M_2$ be a direct sum of a submodule $M_1$ and an injective submodule $M_2$ with essential socle. Then $M$ is WFI-extending if and only if $M_1$ is WFI-extending.

Proof If $M$ is WFI-extending, then $M_1$ is WFI-extending by Corollary 3.10. Conversely, if $M_1$ is WFI-extending then $M$ is WFI-extending by Theorem 2.8.

The conditional direct summand property, namely SIP, works well as a companion condition with extending properties e.g.; $C_{11}$, $WC_{11}$, and FI-extending. Recall that a module $M$ is said to have SIP if the intersection of every pair of direct summands is also a direct summand of the module (see [17]). Notice that SIP is inherited by direct summands of a module with SIP [17, Lemma 2.74]. Now, we have the following result for the WFI-extending modules which corresponds to [19, Theorem 3.1].

Theorem 3.12 Let $M$ be a WFI-extending module which has SIP. Then a direct summand of $M$ is also WFI-extending which has SIP.

Proof Let $M = M_1 \oplus M_2$. Let us show that $M_1$ is a WFI-extending module which has SIP. First note that $M_1$ has SIP. Now, let $N_1$ be a semisimple fully invariant submodule of $M_1$. Then it can be seen that $N_2 = \sum_{\varphi \in \text{Hom}(M_1, M_2)} \varphi(N_1)$ is a fully invariant submodule of $M_2$. On using the argument in [19, Theorem 3.1], $N_1 \oplus N_2$ is a fully invariant submodule of $M$. Now, let us think of $\text{Soc}(N_1 \oplus N_2)$. Observe that $\text{Soc}(N_1 \oplus N_2) = N_1 \oplus \text{Soc}N_2$ which is fully invariant in $N_1 \oplus N_2$. By Lemma 2.1, $N_1 \oplus \text{Soc}N_2$ is a semisimple fully invariant submodule of $M$. By hypothesis, there exists a direct summand $N$ of $M$ such that $N_1 \oplus \text{Soc}N_2$ is essential in $N$. On the other hand, $N_1 \oplus \text{Soc}N_2$ is also essential in $(M_1 \cap N) \oplus (M_2 \cap N)$ which yields that $N_1$ is essential in $M_1 \cap N$. Since $M$ has SIP, $M_1 \cap N$ is a direct summand of $M$. It follows that $M_1 \cap N$ is a direct summand of $M_1$. Hence, $M_1$ is WFI-extending.

Corollary 3.13 Let $M$ be an FI-extending (or $WC_{11}$, WCS, $C_{11}$) module which has SIP. Then a direct summand of $M$ is WFI-extending.

Proof Clear by Theorem 3.12.

It is well known that for any prime integer $p$; the $\mathbb{Z}$-modules $(\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^2)$ and $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ are not extending (see [13, 14]). Observe that both of these module are FI (and hence WFI-)extending and have finite uniform dimensions. Then we have the useful result on WFI-extending modules with finite uniform dimension.

Proposition 3.14 Let $R$ be a Dedekind domain and $M$ an $R$-module with finite uniform dimension. Then any direct summand of $M$ is WFI-extending.

Proof Let $M = M_1 \oplus M_2$ be the direct sum of submodules $M_1$ and $M_2$. Let us show that $M_1$ is WFI-extending. If $M_1$ is torsion-free, then $M_1$ has zero socle, in this case, $M_1$ is a WFI-extending module. Now, suppose that $M_1$ is not torsion-free. By [7, Theorem 9], it follows that $M_1 = N_1 \oplus N_2$, for some finitely generated module $N_1$ and injective submodule $N_2$. By [17, Theorem 4.12], $N_1$ is Weak CS; hence, it is WFI-extending. Now, $M_1$ is WFI-extending, by Theorem 2.8.

As an application of Theorem 2.8 to the direct summands of modules which are direct sum of uniform modules, we reach the following facts.
Corollary 3.15 Let \( M = U \oplus V \) be a direct sum of uniform modules \( U \) and \( V \). Then every direct summand of \( M \) is WFI-extending.

Proof Let \( D \) be a nonzero direct summand of \( M \). If \( D = M \) then \( D \) is WFI-extending by Corollary 2.9. If \( D \neq M \) then \( D \) is uniform; hence, it is WFI-extending.

Corollary 3.16 Let \( M \) be a \( \mathbb{Z} \)-module (i.e. Abelian group) such that \( M \) is a direct sum of uniform modules. Then any direct summand of \( M \) is WFI-extending.

Proof Let \( N \) be a direct summand of \( M \). Then \( N \) is also a direct sum of uniform modules by [17, Theorem 4.45]. Now, Corollary 2.9 gives the result.

References


