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On Walker 4-manifolds with pseudo bi-Hermitian structures

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Abstract: (M_{2n}, g^w, D) is a 4-dimensional Walker manifold and this triple is also a pseudo-Riemannian manifold (M_{2n}, g^w) of signature $(++--)$ (or neutral), which is admitted a field of null 2-plane. In this paper, we consider bi-Hermitian structures $(\varphi_1,\ \varphi_2)$ on 4-dimensional Walker manifolds. We discuss when these structures are integrable and when the bi-Kähler forms are symplectic.

Key words: Almost complex structures, symplectic structures, almost Hermitian and Kähler structures, pseudobi-Hermitian structures, Walker manifold.

1. Introduction

Let M_{2n} be a manifold with a neutral metric which is a pseudo-Rieamnnian metric g of signature (n,n). Let $\Im_q^p(M_{2n})$ be the set of all tensor fields of type (p,q) on M_{2n}. Manifolds, tensor fields, and connections are assumed to be differentiable and of class C^∞.

The pair (M_{2n},φ) is called an almost complex manifold if the condition $\varphi^2=-I$ is hold, where I is a field of identity endomorphisms and φ is an affinor field $\varphi\in\Im_1^1(M_{2n})$. The affinor field φ is integrable if and only if there exists a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e., $\nabla\varphi=0$. Moreover, if the Nijenhuis tensor of such an affinor field φ defined by

$$N_\varphi(X,Y)=[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]+[X,Y]$$

is equivalent to the vanish, then the almost complex structure φ is called integrable. In this case, the almost complex manifold (M_{2n},φ) is called a complex manifold.

Let M_{2n} be a 4-dimensional complex manifold and φ_i, for $i=1,\ 2$, be two independent compatible integrable almost complex structures. Here $\varphi_1(x)\neq\varphi_2(x)$ for a point x in M_{2n}. Also, g metric is a Hermitian metric with respect to both complex structures φ_1 and φ_2, i.e.,

$g(\varphi_1X,\varphi_1Y)=g(X,Y)$ and $g(\varphi_2X,\varphi_2Y)=g(X,Y)$.

In this case, the quartet $(M_{2n},\ g,\varphi_1,\varphi_2)$ is called bi-Hermitian manifold. If $\varphi_1(x)\neq\varphi_2(x)$ everywhere on M_{2n}, a bi-Hermitian structure (g,φ_1,φ_2) is called strongly bi-Hermitian. The real function p is defined by

$$p=-\frac{1}{4}trace(\varphi_1\circ\varphi_2)$$

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or equivalently

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -2pI,$$

where p is the angle function of a bi-Hermitian structure and where I is the field of identity endomorphisms [1,13].

An almost Hermitian structure on a manifold M_{2n} consists of a nondegenerate 2-form w, an almost complex structure φ and a metric g satisfying the compatibility condition $w\left(\mathrm{X},\mathrm{Y}\right)=g(\varphi X,Y)$. If the 2-form w is closed, i.e., $dw=0$, a triple (g,φ,w) is called an almost Kähler structure. Also, the triple (g,φ,w) is called Kähler structure if the almost complex structure φ is integrable [4].

Let $(M_{2n},\ g,\varphi_1,\varphi_2)$ be a bi-Hermitian manifold. For such a structure we define 2-forms w_i setting $w_i\left(X,Y\right)=g\left(\varphi_i X,Y\right),\ i=1,2$. If the 2-forms w_i are closed $\left(dw_i=0\right)$, the bi-Hermitian structure is called bi-Kähler. Such bi-Hermitian structures have been studied by many authors (see, e.g. [1-3, 13]).

2. Walker metrics

Let M_{2n} be a 4-dimensional manifold and g^w be a neutral metric (or g^w is of signature $(++--)$. g^w is called Walker metric if there exists a 2- dimensional null distribution D on M_{2n}, which is parallel with respect to g^w. Such metrics are studied by Walker [15] and canonical form of the metric g^w is given by

$$g^w=\left({g^w}_{ij}\right)=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{2.1}$$

where a,b, and c are some functions depending on the coordinates $\left(x^1,x^2,x^3,x^4\right)$. Note that the parallel null 2-plane D is spanned locally by $\{\partial_1,\partial_2\}$, where $\partial_i=\frac{\partial}{\partial x^i}$ $(i=1,\ 2,\ 3,\ 4)$. Such Walker manifolds are intensively investigated (see, e.g. [4-12,14,15]).

3. Almost bi-Hermitian structures on a neutral 4-manifold

In this section, we consider 4-dimensional pseudo-Riemannian manifolds of neutral signature. For the next step, it is appropriate to state a neutral metric g and the almost complex structure φ in terms of an orthonormal frame $\{e_i\}$, $(i=1,\ 2,\ 3,\ 4\ )$ of vectors and its dual frame $\left\{e^j\right\}$, $(j=1,\ 2,\ 3,\ 4\ )$ of 1-forms. The metric g is given by

$$g=\left(g\left(e_i,e_j\right)\right)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.1}$$

Let $(M_{2n}\ ,g,\varphi_1,\varphi_2)$ be a bi-Hermitian manifold. From identity $\varphi_1\circ\varphi_2+\varphi_2\circ\varphi_1=-2pI$, two almost complex structures φ_1 and φ_2 can be written as:

$$\varphi_1=\left({\varphi_1}^i_j\right)=\begin{pmatrix} 0 & 3 & -2 & 2 \\ -3 & 0 & 2 & 2 \\ -2 & 2 & 0 & 3 \\ 2 & 2 & -3 & 0 \end{pmatrix}, \tag{3.2}$$

$$\varphi_2 = \left(\varphi_{2}{}_j^i\right) = \begin{pmatrix} 0 & 3 & 2 & 2 \\ -3 & 0 & 2 & -2 \\ 2 & 2 & 0 & 3 \\ 2 & -2 & -3 & 0 \end{pmatrix}. \tag{3.3}$$

According to g, φ_1, and φ_2, we have two kinds of Kähler forms on 4-manifolds which are given by

$$w_1\left(X,Y\right) = g\left(\varphi_1 X, Y\right)\ , w_2\left(X,Y\right) = g\left(\varphi_2 X, Y\right). \tag{3.4}$$

Equation (3.4) is equivalent to in matrix notations in the following equation

$$w_1 = \varphi_1{}^T g, w_2 = \varphi_2{}^T g, \tag{3.5}$$

where matrix φ^T is the transpose matrix of matrix φ . From (3.1)–(3.3) and (3.5), we can write

$$w_1 = \left(w_{1ij}\right) = \begin{pmatrix} 0 & -3 & 2 & -2 \\ 3 & 0 & -2 & -2 \\ -2 & 2 & 0 & 3 \\ 2 & 2 & -3 & 0 \end{pmatrix}, \tag{3.6}$$

$$w_2 = \left(w_{2ij}\right) = \begin{pmatrix} 0 & -3 & -2 & -2 \\ 3 & 0 & -2 & 2 \\ 2 & 2 & 0 & 3 \\ 2 & -2 & -3 & 0 \end{pmatrix}. \tag{3.7}$$

These Kähler forms in terms of the local orthonormal basis $\left\{e^j\right\}$ $(j = 1,\ 2,\ 3,\ 4\ )$ of 1-forms are written as:

$$w_1 = \sum_{i<j} w_{1ij}\ e^i \bigwedge e^j = -3e^1 \wedge e^2 + 2e^1 \wedge e^3 - 2e^1 \wedge e^4$$

$$-2e^2 \wedge e^3 - 2e^2 \wedge e^4 + 3e^3 \wedge e^4, \tag{3.8}$$

$$w_2 = \sum_{i<j} w_{2ij}\ e^i \bigwedge e^j = -3e^1 \wedge e^2 - 2e^1 \wedge e^3 - 2e^1 \wedge e^4$$

$$-2e^2 \wedge e^3 + 2e^2 \wedge e^4 + 3e^3 \wedge e^4. \tag{3.9}$$

4. Almost bi-Hermitian structures and bi-Kähler forms on Walker 4-manifolds

Let $(M_{2n}\ , g^w)$ be a Walker-4 manifold which is given in (2.1), where g^w is Walker metric and let $\{e_i\}$ and $\{\partial_i\}\,, (i = 1,\ 2,\ 3,\ 4)$ be two orthonormal frames. Also, matrix $A = \left(A_j^i\right)$ of the change of coordinates satisfies:

$$g = A^T g^w A, \tag{4.1}$$

where A^T is the transpose matrix of A.

Substituting (2.1) and (3.1) in (4.1), one of the matrices which we apply in the present analysis, we obtain as:

$$A=\left(A_j^i\right)=\left(\begin{array}{cccc} 0 & -\left(\frac{1-a}{2}\right) & 0 & \frac{1+a}{2} \\ \frac{1-b}{2} & c & -\left(\frac{1+b}{2}\right) & c \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{array}\right). \tag{4.2}$$

Also, matrix $A=\left(A_j^i\right)$ of the change of coordinates satisfies:

$$\varphi=A^{-1}\varphi^{'}A, \tag{4.3}$$

where A^{-1} is the inverse matrix of A and it is given by:

$$A^{-1}=\left(\begin{array}{cccc} 0 & 1 & c & \left(\frac{1+b}{2}\right) \\ -1 & 0 & -\left(\frac{1+a}{2}\right) & 0 \\ 0 & -1 & -c & \frac{1-b}{2} \\ 1 & 0 & -\left(\frac{1-a}{2}\right) & 0 \end{array}\right). \tag{4.4}$$

Substituting (3.2), (4.2), and (4.4) in (4.3), the almost complex structure in (3.2) is obtained as the following:

$$\varphi_1{}^{'}=\left(\varphi_1{}^{'i}{}_j\right)=\left(\begin{array}{cccc} -2 & 5 & 5c-2a & \frac{1}{2}(5b-a) \\ -1 & 2 & \frac{1}{2}(5b-a) & 2b-c \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -5 & -2 \end{array}\right). \tag{4.5}$$

Similarly, substituting (3.3), (4.2), and (4.4) in (4.3), the almost complex structure in (3.3) is obtained as the following:

$$\varphi_2{}^{'}=\left(\varphi_2{}^{'i}{}_j\right)=\left(\begin{array}{cccc} 2 & 5 & 5c+2a & \frac{1}{2}(5b-a) \\ -1 & -2 & \frac{1}{2}(5b-a) & -2b-c \\ 0 & 0 & -2 & 1 \\ 0 & 0 & -5 & 2 \end{array}\right). \tag{4.6}$$

A matrix $A=\left(A_j^i\right)$ of the change of coordinates for the tensor fields of type $(0,2)$ satisfies:

$$w=A^{T}w^{'}A, \tag{4.7}$$

where A^T is the transpose matrix of A.

Substituting (3.6) and (4.2) in (4.7), the bi-Kähler form in (3.6) is obtained as:

$$w_1{}^{'}=\left(w_1{}^{'}{}_{ij}\right)=\left(\begin{array}{cccc} 0 & 0 & -2 & -1 \\ 0 & 0 & 5 & 2 \\ 2 & -5 & 0 & \frac{1}{2}\left(-a-5b+4c\right) \\ 1 & -2 & -\frac{1}{2}\left(-a-5b+4c\right) & 0 \end{array}\right). \tag{4.8}$$

The bi-Kähler form in (4.8) is written in terms of the coordinate basis as follows:

$$w_1{}^{'}=\sum_{i<j}w_1{}^{'}{}_{ij}\,dx^i\bigwedge dx^j=-2dx^1\wedge dx^3-dx^1\wedge dx^4+5dx^2\wedge dx^3+$$

$$2dx^2 \wedge dx^4 + \frac{1}{2}(-a - 5b + 4c)dx^3 \wedge dx^4. \tag{4.9}$$

Similarly, substituting (3.7) and (4.2) in (4.7), we obtain the bi-Kähler form in (3.7) as:

$$w_2{}' = \left(w_2{}'_{ij}\right) = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 5 & -2 \\ -2 & -5 & 0 & -\frac{1}{2}(a + 5b + 4c) \\ 1 & 2 & \frac{1}{2}(a + 5b + 4c) & 0 \end{pmatrix}. \tag{4.10}$$

Also, in terms of the coordinate basis, the bi-Kähler form in (4.10) is written as follows:

$$w_2{}' = \sum_{i<j} w_2{}'_{ij}\ dx^i \bigwedge dx^j = 2dx^1 \wedge dx^3 - dx^1 \wedge dx^4 + 5dx^2 \wedge dx^3 -$$

$$2dx^2 \wedge dx^4 - \frac{1}{2}(a + 5b + 4c)dx^3 \wedge dx^4. \tag{4.11}$$

5. Integrability of $\varphi_1{}'$ and $\varphi_2{}'$ (bi-Hermitian structures)

The almost complex structure φ' is integrable if and only if

$$\left(N_{\varphi'}\right)^i_{jk} = \varphi'^m_j \partial_m \varphi'^i_k - \varphi'^m_k \partial_m \varphi'^i_j - \varphi'^i_m \partial_j \varphi'^m_k + \varphi'^i_m \partial_k \varphi'^m_j = 0. \tag{5.1}$$

From (4.5) and (5.1), the Nijenhuis tensor of $\varphi_1{}'$ in (4.5) has nonzero components as follows:

$$N^x_{xz} = -N^x_{zx} = 2a_y - 5c_y - \frac{25}{2}b_x + \frac{5}{2}a_x,$$

$$N^x_{xt} = -N^x_{tx} = -\frac{5}{2}b_y + \frac{1}{2}a_y - 10b_x + 5c_x,$$

$$N^y_{xz} = -N^y_{zx} = -10b_x + 5c_x - \frac{5}{2}b_y + \frac{1}{2}a_y,$$

$$N^y_{xt} = -N^y_{tx} = -8b_x + 4c_x - 2b_y + c_y + \frac{5}{2}b_x - \frac{1}{2}a_x,$$

$$N^x_{yz} = -N^x_{zy} = 25c_x - 10a_x + 20c_y - 8a_y - \frac{25}{2}b_y + \frac{5}{2}a_y,$$

$$N^y_{yz} = -N^x_{zy} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y - 2a_y,$$

$$N^x_{yt} = -N^x_{ty} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y - 2a_y,$$

$$N^y_{yt} = -N^y_{ty} = \frac{5}{2}b_y - \frac{1}{2}a_y + 10b_x - 5c_x,$$

$$N^x_{zt} = -N^x_{tz} = \left(\frac{25c}{2} - 5a\right)b_x + \left(-\frac{5c}{2} + 5b\right)a_x + \left(-\frac{25b}{2} + \frac{5a}{2}\right)c_x + \left(\frac{21b - a}{4} - 2c\right)a_y$$

$$+ \left(\frac{-25b + 5a}{4}\right)b_y + (-10b + 5c)\,c_y,$$

$$N^y_{zt} = -N^y_{tz} = \left(\frac{-11a - 25b}{4} + 10c\right)b_x + \left(\frac{5b - a}{4}\right)a_x + \left(\frac{5c}{2} - a\right)b_y + \left(b - \frac{c}{2}\right)a_y$$

$$+ \left(\frac{-5b + a}{2}\right)c_y + (-5c + 2a)\,c_x.$$

From these equations, we have:

Theorem 5.1 *The almost complex structure ${\varphi_1}'$ is integrable if and only if the following PDEs hold:*

$$2b_x - c_x = 0, 2b_y - c_y = 0,$$

$$5b_x - a_x = 0, 5b_y - a_y = 0. \tag{5.2}$$

From (4.6) and (5.1), the Nijenhuis tensor of ${\varphi_2}'$ in (4.6) has nonzero components as follows:

$$N^x_{xz} = -N^x_{zx} = -2a_y - 5c_y - \frac{25}{2}b_x + \frac{5}{2}a_x,$$

$$N^x_{xt} = -N^x_{tx} = -\frac{5}{2}b_y + \frac{1}{2}a_y + 10b_x + 5c_x,$$

$$N^y_{xz} = -N^y_{zx} = 10b_x + 5c_x - \frac{5}{2}b_y + \frac{1}{2}a_y,$$

$$N^y_{xt} = -N^y_{tx} = -8b_x - 4c_x + 2b_y + c_y + \frac{5}{2}b_x - \frac{1}{2}a_x,$$

$$N^x_{yz} = -N^x_{zy} = 25c_x + 10a_x - 20c_y - 8a_y - \frac{25}{2}b_y + \frac{5}{2}a_y,$$

$$N^y_{yz} = -N^x_{zy} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y + 2a_y,$$

$$N^x_{yt} = -N^x_{ty} = \frac{25}{2}b_x - \frac{5}{2}a_x + 5c_y + 2a_y,$$

$$N_{yt}^{y}=-N_{ty}^{y}=\frac{5}{2}b_y-\frac{1}{2}a_y-10b_x-5c_x,$$

$$N_{zt}^{x}=-N_{tz}^{x}=\left(\frac{25c}{2}+5a\right)b_x+\left(-\frac{5c}{2}-5b\right)a_x+\left(-\frac{25b}{2}+\frac{5a}{2}\right)c_x+$$

$$\left(\frac{11b+a}{4}+2c\right)a_y+\left(\frac{25b-5a}{4}\right)b_y+(10b+5c)\,c_y,$$

$$N_{zt}^{y}=-N_{tz}^{y}=\left(\frac{-11a-25b}{4}-10c\right)b_x+\left(\frac{5b-a}{4}\right)a_x+\left(\frac{5c}{2}+a\right)b_y+$$

$$\left(-\frac{c}{2}-b\right)a_y+\left(\frac{-5b+a}{2}\right)c_y+(-5c-2a)\,c_x.$$

From these equations, we have:

Theorem 5.2 *The almost complex structure ${\varphi_2}'$ is integrable if and only if the following PDEs hold:*

$$20b_x+10c_x-5b_y+a_y=0,$$

$$25b_x-5a_x+10c_y+4a_y=0. \tag{5.3}$$

From (5.2) and (5.3), we can write the following integrability conditions for almost bi-Hermitian–Walker structures.

Theorem 5.3 *The triple $(g^w,\varphi_1',\varphi_2')$ is bi-Hermitian–Walker structure if and only if the following PDEs hold:*

$$a_x=a_y=b_x=b_y=c_x=c_y=0. \tag{5.4}$$

6. Symplectic structures

In this section, we focus our attention on bi-Kähler forms $({w_1}',{w_2}')$ which are symplectics, i.e,

$$d{w_i}'=0 \qquad (i=1,\ 2)\,. \tag{6.1}$$

From (4.9), external differential of ${w_1}'$ is written as:

$$d{w_1}' = -\frac{1}{2}\left(a_1+5b_1-4c_1\right)dx^1\wedge dx^3\wedge dx^4-\frac{1}{2}(a_2+5b_2-4c_2)\,dx^2\wedge dx^3\wedge dx^4.$$

Therefore, we have:

Theorem 6.1 *The Kähler form in (4.9) is a symplectic form ($d{w_1}'=0$) if the following PDEs hold:*

$$a_1+5b_1-4c_1=0,$$

$$a_2+5b_2-4c_2=0. \tag{6.2}$$

From (4.11), external differential of $w_2{}'$ is written as:

$$dw_2{}' = -\frac{1}{2}(a_1 + 5b_1 + 4c_1)\, dx^1 \wedge dx^3 \wedge dx^4 - \frac{1}{2}(a_2 + 5b_2 + 4c_2)\, dx^2 \wedge dx^3 \wedge dx^4.$$

Therefore, we have:

Theorem 6.2 *The Kähler form in (4.11) is a symplectic form ($dw_2{}' = 0$) if the following PDEs hold:*

$$a_1 + 5b_1 + 4c_1 = 0,$$

$$a_2 + 5b_2 + 4c_2 = 0. \tag{6.3}$$

From Theorem 6.1 and Theorem 6.2, we can write the following theorem:

Theorem 6.3 *The quinary ($g^w, \varphi_1{}', \varphi_2{}', w_1{}', w_2{}'$) is bi-Kähler–Walker if and only if the following PDEs hold:*

$$a_1 + 5b_1 = 0, c_1 = 0,$$

$$a_2 + 5b_2 = 0, c_2 = 0. \tag{6.4}$$

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