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URTZI BUIJS

ANTONIO GARVIN

ANICETO MURILLO

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Rational maps from Euclidean configuration spaces to spheres

Urtzi BUIJS[1], **Antonio GARVÍN**[2,*], **Aniceto MURILLO**[1]
[1]Department of Algebra, Geometry and Topology, University of Malaga, Malaga, Spain
[2]Department of Applied Mathematics, University of Malaga, Malaga, Spain

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Abstract: In this note we give an algorithm to determine the rational homotopy type of the free and pointed mapping spaces $\mathrm{map}(F(\mathbb{R}^m,k),\mathbb{S}^n)$ and $\mathrm{map}^*(F(\mathbb{R}^m,k),\mathbb{S}^n)$. An explicit description of these spaces is given for $k=3$. The general case for n odd is also presented as an immediate consequence of the rational version of a classical result of Thom.

1. Introduction

We are interested in determining the rational homotopy type of the spaces $\mathrm{map}(F(\mathbb{R}^m,k),\mathbb{S}^n)$ and $\mathrm{map}^*(F(\mathbb{R}^m,k),\mathbb{S}^n)$ of free and pointed continuous maps from the configuration spaces of k particles in $\mathbb{R}^m$ to the n-dimensional sphere. These spaces are useful. For instance, since $F(\mathbb{R}^m,2)\simeq\mathbb{S}^{m-1}$, they include mapping spaces between spheres whose rational homotopy type have already been described in [4]. Also, recall that the generalized Randakumar and Ramana Rao problem* [1, 13], a strong generalization of the classical Borsuk–Ulam theorem, asks whether a convex m-dimensional polytope can be partitioned into k convex pieces on which $m-1$ continuous functions are equalized $(m,k\geq 2)$. Whenever k is a prime power, an affirmative answer [1, Thm. 1.2] follows from the nonexistence of Σ_k-equivariant maps $F(\mathbb{R}^m,k)\longrightarrow S(W_k^{\oplus m-1})$. Here, W_k is the hyperplane of $\mathbb{R}^k$ of equation $x_1+\cdots+x_k=0$ and $S(W_k^{\oplus m-1})$ is the unit sphere on the direct sum of $m-1$ copies of W_k. Observe that the symmetric group Σ_k naturally acts on both spaces by permuting coordinates and columns respectively, and $S(W_k^{\oplus m-1})$ is just a special Σ_k representation of $\mathbb{S}^{(m-1)(k-1)-1}$.

For $k=3$, we obtain the following decomposition in which, for simplicity in the notation, we denote $M(m,n)=\mathrm{map}(F(\mathbb{R}^m,3),\mathbb{S}^n)$ and $M^*(m,n)=\mathrm{map}^*(F(\mathbb{R}^m,3),\mathbb{S}^n)$.

Theorem 1 *(i) For n odd and any $m\geq 2$,*

$$M(m,n)\simeq_{\mathbb{Q}}\begin{cases}\mathbb{S}^n\times K(\mathbb{Q},n-(m-1))^3\times K(\mathbb{Q},n-2(m-1))^2, & \text{if } n>2(m-1),\\ \mathbb{S}^n\times K(\mathbb{Q},n-(m-1))^3, & \text{if } m-1<n<2(m-1),\\ \bigsqcup_{\mathbb{N}}\mathbb{S}^n, & \text{if } n=m-1,\\ \mathbb{S}^n, & \text{if } n<m-1.\end{cases}$$

*Correspondence: garvin@uma.es

*Nandakumar R. Fair partitions. Blog entry, http://nandacumar.blogspot.de/2006/09/cutting-shapes.html, 2006.

$$M^*(m,n) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Q}, n-(m-1))^3 \times K(\mathbb{Q}, n-2(m-1))^2, \text{ if } n > 2(m-1), \\ K(\mathbb{Q}, n-(m-1))^3, \quad \text{if } m-1 < n < 2(m-1), \\ \bigsqcup_{\mathbb{N}} *, \quad \text{if } n = m-1, \\ *, \quad \text{if } n < m-1. \end{cases}$$

(ii) For $n = 2$ and any $m \geq 2$,

$$M(m,2) \simeq_{\mathbb{Q}} \begin{cases} \mathbb{S}^2, \quad \text{if } m > 4, \\ \bigsqcup_{\mathbb{N}} \mathbb{S}^2, \quad \text{if } m = 4, \\ (\mathbb{S}^1)^3 \times \mathbb{S}^2 \sqcup \bigsqcup_{\mathbb{N}} (\mathbb{S}^1)^2 \times \mathbb{S}^3, \quad \text{if } m = 3, \\ X \sqcup \bigsqcup_{\mathbb{N}} \mathbb{S}^1 \times H_e \times K(\mathbb{Q}, 2)^3 \times \mathbb{S}^3, \quad \text{if } m = 2. \end{cases}$$

$$M^*(m,2) \simeq_{\mathbb{Q}} \begin{cases} *, \quad \text{if } m > 4, \\ \bigsqcup_{\mathbb{N}} *, \quad \text{if } m = 4, \\ \bigsqcup_{\mathbb{N}} (\mathbb{S}^1)^3, \quad \text{if } m = 3, \\ \bigsqcup_{\mathbb{N}} Y \times K(\mathbb{Q}, 2)^3, \quad \text{if } m = 2. \end{cases}$$

(iii) For n even greater than 2 and $m \geq 2$,

$$M(m,n) \simeq_{\mathbb{Q}} \begin{cases} \mathbb{S}^n, \quad \text{if } m > 2n, \\ \bigsqcup_{\mathbb{N}} \mathbb{S}^n, \quad \text{if } m = 2n, \\ K(\mathbb{Q}, 2n-m) \times \mathbb{S}^n, \quad \text{if } n+2 \leq m \leq 2n-1. \end{cases}$$

$$M^*(m,n) \simeq_{\mathbb{Q}} \begin{cases} *, \quad \text{if } m > 2n, \\ \bigsqcup_{\mathbb{N}} *, \quad \text{if } m = 2n, \\ K(\mathbb{Q}, 2n-m), \quad \text{if } n+2 \leq m \leq 2n-1. \end{cases}$$

Here H_e is the Heisenberg manifold, X is a rational space which is the total space in a rational fibration of the form

$$(\mathbb{S}^1)^2 \times K(\mathbb{Q}, 2)^3 \to X \to (\mathbb{S}^1)^3 \times \mathbb{S}^2,$$

and Y is the nilmanifold [10] whose minimal model is

$$(\Lambda(a_1, b_1, c_1, x_1, y_1), d), \quad dx_1 = a_1 b_1, \quad dy_1 = b_1 c_1,$$

with subscripts indicating degree (see Section 3 for details). As usual, $\simeq_{\mathbb{Q}}$ means "rationally equivalent to" and $\bigsqcup$ denotes disjoint union.

The method used in the proof may well serve as an algorithm to compute $\operatorname{map}(F(\mathbb{R}^m,k),\mathbb{S}^n)$ and $\operatorname{map}^*(F(\mathbb{R}^m,k),\mathbb{S}^n)$ for given integers $m,k\geq 2$ and $n\geq 1$. However, the general case for more than three particles for n even, does not produce such a straight decomposition. Nevertheless, whenever n is odd, $\mathbb{S}^n$ is rationally an H-space and the spaces $\operatorname{map}(F(\mathbb{R}^m,k),\mathbb{S}^n)$, $\operatorname{map}^*(F(\mathbb{R}^m,k),\mathbb{S}^n)$ can be easily decomposed as products of Eilenberg-MacLane spaces in view of the rational version, both free and pointed, of the classical work of Thom [9, 14], see Proposition 1:

Theorem 2 *Denote* $M(m,k,n)=\operatorname{map}(F(\mathbb{R}^m,k),\mathbb{S}^n)$ *and* $M^*(m,k,n)=\operatorname{map}^*(F(\mathbb{R}^m,k),\mathbb{S}^n)$. *Then:*

$$M(m,k,n)\simeq_{\mathbb{Q}}\begin{cases}\displaystyle\prod_{j=0}^{k-1}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}, & \text{if } n>(k-1)(m-1),\\ \displaystyle\bigsqcup_{\mathbb{N}}\left(\prod_{j=0}^{l-1}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}\right), & \text{if } n=l(m-1),\\ & 1\leq l\leq k-1,\\ \displaystyle\prod_{j=0}^{l}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}, & \text{if } l(m-1)<n<(l+1)(m-1),\\ & 1\leq l\leq k-2,\\ \mathbb{S}^n, & \text{if } n<m-1.\end{cases}$$

$$M^*(m,k,n)\simeq_{\mathbb{Q}}\begin{cases}\displaystyle\prod_{j=1}^{k-1}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}, & \text{if } n>(k-1)(m-1),\\ \displaystyle\bigsqcup_{\mathbb{N}}\left(\prod_{j=1}^{l-1}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}\right), & \text{if } n=l(m-1),\\ & 1\leq l\leq k-1,\\ \displaystyle\prod_{j=1}^{l}K(\mathbb{Q},n-j(m-1))^{\left[{k \atop k-j}\right]}, & \text{if } l(m-1)<n<(l+1)(m-1),\\ & 1\leq l\leq k-2,\\ *, & \text{if } n<m-1.\end{cases}$$

Here, as in [8], the brackets $\left[{k \atop k-j}\right]$ represent the unsigned Stirling numbers of the first kind.

As an illustrative example, for the case including the generalized Randakumar and Ramana Rao problem, we get directly:

Corollary 1 *If either* m *or* k *is an odd number, then:*

For $m\geq 3$,

$$\operatorname{map}(F(\mathbb{R}^m,k),\mathbb{S}^{(m-1)(k-1)-1})\simeq_{\mathbb{Q}}\prod_{j=0}^{k-2}K(\mathbb{Q},(k-(j+1))(m-1)-1)^{\left[{k \atop k-j}\right]},$$

$$\operatorname{map}^*(F(\mathbb{R}^2,k),\mathbb{S}^{(m-1)(k-1)-1})\simeq_{\mathbb{Q}}\prod_{j=1}^{k-2}K(\mathbb{Q},(k-(j+1))(m-1)-1)^{\left[{k \atop k-j}\right]}.$$

For $m = 2$,

$$\operatorname{map}(F(\mathbb{R}^2,k),\mathbb{S}^{k-2}) \simeq_{\mathbb{Q}} \bigsqcup_{\mathbb{N}} \left(\prod_{j=0}^{k-3} K(\mathbb{Q}, k-(2+j))^{\left[{k \atop k-j}\right]} \right),$$

$$\operatorname{map}^*(F(\mathbb{R}^2,k),\mathbb{S}^{k-2}) \simeq_{\mathbb{Q}} \bigsqcup_{\mathbb{N}} \left(\prod_{j=1}^{k-3} K(\mathbb{Q}, k-(2+j))^{\left[{k \atop k-j}\right]} \right). \qquad \square$$

2. Preliminaries

We will use basic results from rational homotopy theory for which [7] has become a standard reference. Via the classical adjoint functors between the categories of commutative differential graded algebras (CDGA's henceforth) over $\mathbb{Q}$ which is always assumed to be the ground field, and simplicial sets, given by piecewise linear forms and realization,

$$\text{SSets} \rightleftarrows \text{CDGA},$$

one has the notion of (Sullivan) model of a nonnecessarily connected space Z such that all its components are nilpotent [3]: By such a model we mean a cofibrant $\mathbb{Z}$-graded free commutative differential graded algebra whose simplicial realization has the same homotopy type as the Milnor simplicial approximation of the rationalization of Z.

If $(\Lambda W, d)$ is a model of Z in this sense and $u\colon \Lambda W \to \mathbb{Q}$ the model of a 0-simplex of Z, consider the differential ideal K_u generated by $A_1 \cup A_2 \cup A_3$, being

$$A_1 = W^{<0},\ A_2 = dW^0,\ A_3 = \{\alpha - u(\alpha) : \alpha \in W^0\}.$$

Then $(\Lambda W, d)/K_u$ is again a free commutative differential graded algebra of the form $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ in which $\overline{W}^1$ is a complement in W^1 of $d(W^0)$ up to identifications given by A_1 and A_3, see [3, S4] for details. Then [2, 4.3], $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ is a Sullivan model of the path component of Z containing the fixed 0-simplex.

In particular, if X is a nilpotent finite CW-complex and Y is a finite type CW-complex then the components of the free and pointed mapping spaces $\operatorname{map}(X,Y)$ and $\operatorname{map}^*(X,Y)$ are nilpotent [11] and the above applies. We briefly recall the Haefliger model [9] of these spaces and its components following the presentation in [2, 3].

Let B be a finite dimensional commutative differential graded algebra model of X and let $A = (\Lambda V, d)$ be a Sullivan model of Y. We denote by $B^\sharp$ the differential graded coalgebra dual of B, $B^\sharp = Hom(B,\mathbb{Q})$, with the grading $(B^\sharp)^{-n} = (B^\sharp)_n = Hom(B^n,\mathbb{Q})$. Consider the free commutative differential graded algebra $\Lambda(A \otimes B^\sharp)$ generated by the $\mathbb{Z}$-graded vector space $A \otimes B^\sharp$, with the differential d induced by the one on A and $B^\sharp$. Let $I \subset \Lambda(A \otimes B^\sharp)$ be the differential ideal generated by $1 \otimes 1 - 1$, and the elements of the form

$$a_1 a_2 \otimes \beta - \sum_j (-1)^{|a_2||\beta'_j|}(a_1 \otimes \beta'_j)(a_2 \otimes \beta''_j), \quad a_1, a_2 \in A,\ \beta \in B^\sharp,$$

where the coproduct on β is, $\Delta\beta = \sum_j \beta'_j \otimes \beta''_j$. The inclusion $V \otimes B^\sharp \hookrightarrow A \otimes B^\sharp$ induces an isomorphism of graded algebras

$$\rho\colon \Lambda(V \otimes B^\sharp) \xrightarrow{\cong} \Lambda(A \otimes B^\sharp)/I$$

and thus $\widetilde{d} = \rho^{-1} d\rho$ defines a differential in $\Lambda(V \otimes B^\sharp)$. We can do the same construction taking $(B_+)^\sharp$ (elements of $B^\sharp$ of negative degree) instead of $B^\sharp$, and taking $\bar{\Delta}$ (the reduced coproduct) instead of Δ.

Then [2, 3], the commutative differential graded algebra $(\Lambda(V \otimes B^\sharp), \widetilde{d})$ is a model of $\operatorname{map}(X,Y)$, and the commutative differential graded algebra $(\Lambda(V \otimes B^\sharp_+), \widetilde{d})$ is a model of $\operatorname{map}^*(X,Y)$.

Now, let $\varphi\colon (\Lambda V, d) \to (B, \delta)$ a model of a given map $f\colon X \to Y$. The morphism φ induces a natural augmentation denoted also by $\varphi\colon \big(\Lambda(V \otimes B^\sharp), \widetilde{d}\big) \to \mathbb{Q}$ which can be thought as the model of the 0-simplex of the mapping space representing f. Applying the process above we obtain the Sullivan algebra

$$\big(\Lambda\big(\overline{V \otimes B^\sharp}^{1} \otimes (V \otimes B^\sharp)^{\geq 2}\big), \widetilde{d}_\varphi\big)$$

which is a Sullivan model of the component $\operatorname{map}_f(X,Y)$ of the free mapping space containing f [3]. In the same way,

$$\big(\Lambda\big(\overline{V \otimes B^\sharp_+}^{1} \otimes (V \otimes B^\sharp_+)^{\geq 2}\big), \widetilde{d}_\varphi\big)$$

is a Sullivan model of $\operatorname{map}^*_f(X,Y)$.

The next result will be used in next sections. It may be considered a rational reformulation of the classical decomposition of Thom [14], see also [9].

Proposition 1 *Let X be a formal finite nilpotent complex and let Y be of the rational homotopy type of a finite type H-space. For $j \geq 0$, let*

$$N_j = \sum_{r-s=j} \dim \Pi_r(Y) \otimes \mathbb{Q} \cdot \dim H^s(X;\mathbb{Q}),$$

$$N'_j = \sum_{r-s=j,\, s\neq 0} \dim \Pi_r(Y) \otimes \mathbb{Q} \cdot \dim H^s(X;\mathbb{Q}).$$

Then,

$$\operatorname{map}(X,Y) \simeq_{\mathbb{Q}} \begin{cases} \prod_{j\geq 1} K(\mathbb{Q},j)^{N_j} & \text{if } N_0 = 0, \\ \bigsqcup_{\mathbb{N}} \Big(\prod_{j\geq 1} K(\mathbb{Q},j)^{N_j}\Big) & \text{if } N_0 \neq 0. \end{cases}$$

$$\operatorname{map}^*(X,Y) \simeq_{\mathbb{Q}} \begin{cases} \prod_{j\geq 1} K(\mathbb{Q},j)^{N'_j} & \text{if } N'_0 = 0, \\ \bigsqcup_{\mathbb{N}} \Big(\prod_{j\geq 1} K(\mathbb{Q},j)^{N'_j}\Big) & \text{if } N'_0 \neq 0. \end{cases}$$

Proof As X is a formal space, $B = (H^*(X;\mathbb{Q}), 0)$ is a model of X. On the other hand the minimal model of the H-space Y is of the form $A = (\Lambda V, 0)$. Then, $(\Lambda(V \otimes B^\sharp), 0)$ is a model of $\operatorname{map}(X,Y)$.

Observe that for any j,

$$\dim(V \otimes B^\sharp)^j = \sum_{r+s=j} = \dim V^r \cdot \dim(B^\sharp)^s$$
$$= \sum_{r-s=j} \dim \Pi_r(Y) \otimes \mathbb{Q} \cdot \dim H^s(X;\mathbb{Q}) = N_j,$$

$$\dim(V \otimes B^\sharp_+)^j = \sum_{r+s=j,\, s\neq 0} \dim V^r \cdot \dim(B^\sharp_+)^s$$
$$= \sum_{r-s=j,\, s\neq 0} \dim \Pi_r(Y) \otimes \mathbb{Q} \cdot \dim H^s(X;\mathbb{Q}) = N'_j.$$

Now, both in the free or pointed case, there is only one component as long as $(V \otimes B^\sharp)^0 = 0$ or $(V \otimes B^\sharp_+)^0 = 0$, that is, whenever $N_0 = 0$ or $N'_0 = 0$. Otherwise, as the differential is trivial, there are a countable number of components, as nonhomotopic augmentations in $(V \otimes B^\sharp)^0$ or $(V \otimes B^\sharp_+)^0$. On the other hand, again by the triviality of the differential, it is clear that each component is of the homotopy type of $\prod_{j\geq 1} K(\mathbb{Q}, j)^{N_j}$ in the free case and $\prod_{j\geq 1} K(\mathbb{Q}, j)^{N'_j}$ in the pointed case.

□

3. The proofs

We first prove Theorem 1 by applying the procedure in Section 2 to obtain a model of $\mathrm{map}(F(\mathbb{R}^m, 3), \mathbb{S}^n)$. Then, we identify from this model the rational homotopy type of its components.

A CDGA model of the configuration space $F(\mathbb{R}^m, k)$ is given by its rational cohomology algebra as these spaces are formal [12]. It is well known [5] that $H^*(F(\mathbb{R}^m, k);\mathbb{Q})$, is given by

$$H^*(F(\mathbb{R}^m, k);\mathbb{Q}) = \Lambda(a_{ij})/I, \qquad i \neq j, \quad i, j = 1, \dots, k, \tag{1}$$

where $\mid a_{ij} \mid = m - 1$, and I is the ideal generated as follows:

$$I = \langle a_{ij} - (-1)^m a_{ji}, \quad a_{ij}^2, \quad a_{ij}a_{jr} + a_{jr}a_{ri} + a_{ri}a_{ij}\rangle.$$

For $k = 3$, we have

$$H^*(F(\mathbb{R}^m, 3)) = \frac{\Lambda(a_{12}, a_{13}, a_{21}, a_{23}, a_{31}, a_{32})}{I},$$

with $\mid \overline{a}_{ij} \mid = m - 1$, $\overline{a_{ji}} = (-1)^m \overline{a_{ij}}$, $\overline{a_{ij}}^2 = 0$ and

$$\overline{a_{12}}\,\overline{a_{23}} + \overline{a_{23}}\,\overline{a_{31}} + \overline{a_{31}}\,\overline{a_{12}} = 0.$$

Then, as a graded vector space $B = H^*(F(\mathbb{R}^m, 3))$ is concentrated in degrees 0, $m-1$ and $2(m-1)$,

$$B = \mathbb{Q} \oplus \langle \overline{a_{12}}, \overline{a_{13}}, \overline{a_{23}}\rangle \oplus \langle \overline{a_{12}}\,\overline{a_{23}}, \overline{a_{13}}\,\overline{a_{23}}\rangle.$$

Hence, its dual vector space is

$$B^\sharp = H_*(F(\mathbb{R}^m, 3);\mathbb{Q}) = H_0 \oplus H_{m-1} \oplus H_{2(m-1)},$$

where $H_0 = \mathbb{Q} = \langle 1 \rangle$, $H_{m-1} = \langle \alpha_{12}, \alpha_{13}, \alpha_{23} \rangle$, and $H_{2(m-1)} = \langle \alpha_{12,23}, \alpha_{13,23} \rangle$. Here $1, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{12,23}, \alpha_{13,23}$ simply denotes the dual basis of $1, \overline{a_{12}}, \overline{a_{13}}$, $\overline{a_{23}}, \overline{a_{12}}\,\overline{a_{23}}, \overline{a_{13}}\,\overline{a_{23}}$.

Now, if n is odd, we may apply Proposition 1 and a straightforward computation proves the assertion (i) of Theorem 1.

From now on, we assume that n is an even integer and fix the minimal model of $\mathbb{S}^n$ given by $(\Lambda(x,y), d)$ with $\mid x \mid = n$, $\mid y \mid = 2n-1$, $dx = 0$ and $dy = x^2$.

We will also need the ring structure of $B = H^*(F(\mathbb{R}^m, 3))$ which is given by the following table:

	1	$\overline{a_{12}}$	$\overline{a_{13}}$	$\overline{a_{23}}$	$\overline{a_{12}}\,\overline{a_{23}}$	$\overline{a_{13}}\,\overline{a_{23}}$
1	1	$\overline{a_{12}}$	$\overline{a_{13}}$	$\overline{a_{23}}$	$\overline{a_{12}}\,\overline{a_{23}}$	$\overline{a_{13}}\,\overline{a_{23}}$
$\overline{a_{12}}$	$\overline{a_{12}}$	0	$\overline{a_{12}}\,\overline{a_{23}} - \overline{a_{13}}\,\overline{a_{23}}$	$\overline{a_{12}}\,\overline{a_{23}}$	0	0
$\overline{a_{13}}$	$\overline{a_{13}}$	$(-1)^{m-1}\overline{a_{12}}\,\overline{a_{23}} + (-1)^m\overline{a_{13}}\,\overline{a_{23}}$	0	$\overline{a_{13}}\,\overline{a_{23}}$	0	0
$\overline{a_{23}}$	$\overline{a_{23}}$	$(-1)^{m-1}\overline{a_{12}}\,\overline{a_{23}}$	$(-1)^{m-1}\overline{a_{13}}\,\overline{a_{23}}$	0	0	0
$\overline{a_{12}}\,\overline{a_{23}}$	$\overline{a_{12}}\,\overline{a_{23}}$	0	0	0	0	0
$\overline{a_{13}}\,\overline{a_{23}}$	$\overline{a_{13}}\,\overline{a_{23}}$	0	0	0	0	0

From it, one explicitly determines the coproduct Δ on $B^\sharp$:

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(\alpha_{12}) = 1 \otimes \alpha_{12} + \alpha_{12} \otimes 1,$$

$$\Delta(\alpha_{13}) = 1 \otimes \alpha_{13} + \alpha_{13} \otimes 1,$$

$$\Delta(\alpha_{23}) = 1 \otimes \alpha_{23} + \alpha_{23} \otimes 1,$$

$$\Delta(\alpha_{12,23}) = 1 \otimes \alpha_{12,23} + \alpha_{12,23} \otimes 1 + (-1)^{m+1}\alpha_{12} \otimes \alpha_{23} + \alpha_{23} \otimes \alpha_{12} + (-1)^{m+1}\alpha_{12} \otimes \alpha_{13} + \alpha_{13} \otimes \alpha_{12},$$

$$\Delta(\alpha_{13,23}) = 1 \otimes \alpha_{13,23} + \alpha_{13,23} \otimes 1 + (-1)^{m+1}\alpha_{13} \otimes \alpha_{23} + \alpha_{23} \otimes \alpha_{13} + (-1)^{m}\alpha_{12} \otimes \alpha_{13} - \alpha_{13} \otimes \alpha_{12}.$$

Hence, following the procedure in Section 1, one obtain a model of $\mathrm{map}(F(\mathbb{R}^m, 3), \mathbb{S}^n)$ of the form

$$(\Lambda(V \otimes B^\sharp), \widetilde{d}),$$

where

$$\begin{aligned} V \otimes B^\sharp = &\langle x \otimes 1, x \otimes \alpha_{12}, x \otimes \alpha_{13}, x \otimes \alpha_{23}, x \otimes \alpha_{12,23}, x \otimes \alpha_{13,23}, \\ &y \otimes 1, y \otimes \alpha_{12}, y \otimes \alpha_{13}, y \otimes \alpha_{23}, y \otimes \alpha_{12,23}, y \otimes \alpha_{13,23} \rangle, \end{aligned}$$

in which $x \otimes 1, x \otimes \alpha_{12}, x \otimes \alpha_{13}, x \otimes \alpha_{23}, x \otimes \alpha_{12,23}, x \otimes \alpha_{13,23}$ are cycles and

$$
\begin{aligned}
\tilde{d}(y \otimes 1) =& (x \otimes 1)^2, \\
\tilde{d}(y \otimes \alpha_{12}) =& 2(x \otimes 1)(x \otimes \alpha_{12}), \\
\tilde{d}(y \otimes \alpha_{13}) =& 2(x \otimes 1)(x \otimes \alpha_{13}), \\
\tilde{d}(y \otimes \alpha_{23}) =& 2(x \otimes 1)(x \otimes \alpha_{23}), \\
\tilde{d}(y \otimes \alpha_{12,23}) =& 2\big((x \otimes 1)(x \otimes \alpha_{12,23}), \\
& + (-1)^{m+1}(x \otimes \alpha_{12})(x \otimes \alpha_{23}) + (-1)^{m+1}(x \otimes \alpha_{13})(x \otimes \alpha_{23})\big), \\
\tilde{d}(y \otimes \alpha_{13,23}) =& 2\big((x \otimes 1)(x \otimes \alpha_{13,23}), \\
& + (-1)^{m+1}(x \otimes \alpha_{13})(x \otimes \alpha_{23}) + (-1)^{m}(x \otimes \alpha_{12})(x \otimes \alpha_{13})\big).
\end{aligned}
$$

To simplify the notation, write $V \otimes B^\sharp = W$, $\tilde{d} = d$,

$$
\begin{aligned}
x = x \otimes 1, &\quad y = y \otimes 1, \\
p_{i+j-2} = x \otimes \alpha_{ij}, &\quad q_{i+j-2} = y \otimes \alpha_{ij}, \\
r_{i+j-2} = x \otimes \alpha_{ij,rs}, &\quad s_{i+j-2} = y \otimes \alpha_{ij,rs}.
\end{aligned}
$$

Then,

$$
(\Lambda W, d) = (\Lambda(x, y, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, s_1, s_2), d),
$$

where x, p_i, r_j are cycles ($i = 1,2,3$ and $j = 1,2$) and

$$
\begin{aligned}
& d(y) = x^2, \\
& d(q_i) = 2xp_i, \quad i = 1,2,3, \\
& d(s_1) = 2(xr_1 + (-1)^{m+1}p_1p_3 + (-1)^{m+1}p_2p_3), \\
& d(s_2) = 2(xr_2 + (-1)^{m+1}p_1p_2 + (-1)^{m}p_2p_3).
\end{aligned}
$$

The degrees of the generators are:

$$
\begin{aligned}
& | x |= n, \\
& | y |= 2n - 1, \\
& | p_i |= n - m + 1, \quad i = 1,2,3, \\
& | q_i |= 2n - m, \quad i = 1,2,3, \\
& | r_i |= n - 2m + 2, \quad i = 1,2, \\
& | s_i |= 2n - 2m + 1, \quad i = 1,2.
\end{aligned}
$$

For the pointed maps, the procedure given in Section 2 produces the following model of $\operatorname{map}^*(F(\mathbb{R}^m, 3), \mathbb{S}^n)$. Writing $W_+ = V \otimes B_+^\sharp$ and with the same notation for the generators, this model is

$$
(\Lambda W_+, d) = (\Lambda(p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, s_1, s_2), d),
$$

where

$$
\begin{aligned}
& d(s_1) = (-1)^{m+1}2(p_1p_3 + p_2p_3), \\
& d(s_2) = (-1)^{m+1}2(p_1p_2 - p_2p_3),
\end{aligned}
$$

and the rest of generators are cycles.

We first deal with the case $n = 2$ and analyze each component in the cases $m > 4$, $m = 4$, $m = 3$ and $m = 2$.

For $m > 4$, we have:

degree	W
3	y
2	x
1	
0	
$\cdots$	
$4-m$	q_1, q_2, q_3
$3-m$	p_1, p_2, p_3
$\cdots$	
$5-2m$	s_1, s_2
$4-2m$	r_1, r_2

In this case, there are no generators in degree 0, so the only possible augmentation $\Lambda W \to \mathbb{Q}$ is the trivial one, that is, there is only one component. Also, there are no generators in degree 1, so projecting over the the generators of negative degree we obtain the Sullivan model of the $\mathrm{map}(F(\mathbb{R}^m, 3), \mathbb{S}^2)$ which turns out to be the minimal model of $\mathbb{S}^2$. For the pointed mapping space, observe that W_+ is concentrated in negative degrees and therefore $\mathrm{map}^*(F(\mathbb{R}^m, 3), \mathbb{S}^2) \simeq_{\mathbb{Q}} *$.

For $m = 4$, we have:

degree	W
3	y
2	x
1	
0	q_1, q_2, q_3
-1	p_1, p_2, p_3
-2	
-3	s_1, s_2
-4	r_1, r_2

The existence of generators q_1, q_2, q_3 in degree zero provides an augmentation $\varphi\colon (\Lambda W, d) \to \mathbb{Q}$ for each triad $\lambda_1, \lambda_2, \lambda_3$ of rational numbers given by

$$\varphi(q_1) = \lambda_1, \quad \varphi(q_2) = \lambda_2, \quad \varphi(q_3) = \lambda_3.$$

Note that different triads produces nonhomotopic augmentations as they induce different cohomology morphisms. Therefore, there are a countable number of components in the rationalization of $\mathrm{map}(F(\mathbb{R}^4, 3), \mathbb{S}^2)$. It is straightforward to check that any of them has the same model as $\mathbb{S}^2$, i.e. $\mathrm{map}(F(\mathbb{R}^4, 3), \mathbb{S}^2) \simeq_{\mathbb{Q}} \bigsqcup_{\mathbb{N}} \mathbb{S}^2$. As before, W_+ is concentrated in nonpositive degree so that each component of the pointed mapping space is rationally contractible: $\mathrm{map}^*(F(\mathbb{R}^4, 3), \mathbb{S}^2) \simeq_{\mathbb{Q}} \bigsqcup_{\mathbb{N}} *$.

For $m = 3$, we have:

degree	W
3	y
2	x
1	q_1, q_2, q_3
0	p_1, p_2, p_3
-1	s_1, s_2
-2	r_1, r_2

Observe that in this case, an augmentation $\varphi\colon (\Lambda W, d) \to \mathbb{Q}$ is determined also by a triad of rational numbers $\lambda_i = \varphi(p_i)$, $i = 1, 2, 3$, which satisfy the equations $\varphi(ds_j) = 0$, $j = 1, 2$. In other words, each augmentation corresponds to a solution of the system,

$$\begin{cases} \lambda_2(\lambda_1 - \lambda_3) = 0, \\ \lambda_3(\lambda_1 + \lambda_2) = 0. \end{cases}$$

These are $\{(\lambda, 0, 0), (0, \lambda, 0), (0, 0, \lambda), (\lambda, -\lambda, \lambda)\}_{\lambda \in \mathbb{Q}}$. Note also that different solutions correspond to nonhomotopic augmentations and hence, the mapping space has a countable number of components.

The model of the component corresponding to $\lambda = 0$ is $(\Lambda(q_1, q_2, q_3), 0) \otimes (\lambda(x, y), d)$, that is, the model of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2$. For the rest of the cases, straightforward computations provide models of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^3$.

Thus, $\operatorname{map}(F(\mathbb{R}^3, 3), \mathbb{S}^2) \simeq_{\mathbb{Q}} (\mathbb{S}^1)^3 \times \mathbb{S}^2 \sqcup \bigsqcup_{\mathbb{N}} (\mathbb{S}^1)^2 \times \mathbb{S}^3$.

In the pointed case, we obtain that the model of each component is $(\Lambda(q_1, q_2, q_3), 0)$ and therefore $\operatorname{map}^*(F(\mathbb{R}^3, 3), \mathbb{S}^2) \simeq_{\mathbb{Q}} \bigsqcup_{\mathbb{N}} (\mathbb{S}^1)^3$.

For $m = 2$, we have:

degree	W
3	y
2	x, q_1, q_2, q_3
1	p_1, p_2, p_3, s_1, s_2
0	r_1, r_2

and each augmentation $\varphi\colon (\Lambda W, d) \to \mathbb{Q}$ is determined by a pair of rational numbers $\varphi(r_1) = \lambda_1$ and $\varphi(r_2) = \lambda_2$. According to the procedure in Section 1, the model of the corresponding component is:

$$(\Lambda W^{\geq 1}, d) = (\Lambda(x, y, p_1, p_2, p_3, q_1, q_2, q_3, s_1, s_2), d),$$

in which x, p_1, p_2, p_3 are cycles and

$$dy = x^2,$$
$$dq_i = 2xp_i, \ i = 1, 2, 3,$$
$$ds_1 = 2(\lambda_1 x - p_1 p_3 - p_2 p_3),$$
$$ds_2 = 2(\lambda_2 x - p_1 p_2 + p_2 p_3).$$

For $\lambda_1 = \lambda_2 = 0$, this is the model of a rational space X that is the total space of a rational fibration of the form

$$(\mathbb{S}^1)^2 \times K(\mathbb{Q}, 2)^3 \to X \to (\mathbb{S}^1)^3 \times \mathbb{S}^2.$$

In the rest of the cases, that is $\lambda_i \neq 0$ for some $i = 1, 2$, changing basis and discarding the contractible part, we obtain the model

$$(\Lambda(x_1, y_1, z_1, t_1, u_2, v_2, w_2, u_3), d),$$

where subscripts indicates degree, and all generators are cycles except $dt_1 = x_1 y_1$. The realization is

$$\mathbb{S}^1 \times H_e \times K(\mathbb{Q}, 2)^3 \times \mathbb{S}^3,$$

where H_e is the Heisenberg manifold whose rational model is precisely

$$(\Lambda(x_1, y_1, z_1, t_1), d), \quad dt_1 = x_1 y_1.$$

Adding up,

$$\operatorname{map}(F(\mathbb{R}^2,3),\mathbb{S}^2)\simeq_{\mathbb{Q}} X\sqcup\bigsqcup_{\mathbb{N}}\mathbb{S}^1\times H\times K(\mathbb{Q},2)^3\times\mathbb{S}^3.$$

In the based case, all the components have the same model,

$$(\Lambda(p_1,p_2,p_3,s_1,s_2,q_1,q_2,q_3),d),$$

in which the p_i's are cycles and

$$ds_1=\pm 2(p_1p_3+p_2p_3)\,,\quad ds_2=\pm 2(p_1p_2-p_2p_3).$$

It is not difficult to modify this model to get

$$(\Lambda(a_1,b_1,c_1,x_1,y_1),d)\otimes(\Lambda(u_2,v_2,w_2),0),$$

where $dx_1=a_1b_1$ and $dy_1=b_1c_1$, here subscripts indicates degree. Let Y be the realization of the first factor which has the homotopy type of a nilmanifold [10]. It is clear that the second one realizes as $K(\mathbb{Q},2)^3$ and thus,

$$\operatorname{map}^*(F(\mathbb{R}^2,3),\mathbb{S}^2)\simeq_{\mathbb{Q}}\bigsqcup_{\mathbb{N}}Y\times K(\mathbb{Q},2)^3.$$

We now tackle the case n even greater or equal than 4. For it, as before, we fix $(\Lambda(x,y),d)$ the minimal model of $\mathbb{S}^n$. We also distinguish different cases.

For $m>2n$, and arguing as before, we obtain only one component with the same model as $\mathbb{S}^n$ in the free case and $\mathbb{Q}$ in the pointed case. Hence,

$$\operatorname{map}(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}}\mathbb{S}^n \text{ and } \operatorname{map}^*(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}} *.$$

For $m=2n$ the model is of the form $(\Lambda(x,y),d)\otimes(\Lambda z,0)$ in the free case and $(\Lambda z,0)$ in the based case, with z of degree 0. There is trivially a countable number of augmentations and the model of the corresponding component is the model of the $\mathbb{S}^n$ in the free case and contractible in the pointed case. Thus,

$$\operatorname{map}(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}}\bigsqcup_{\mathbb{N}}\mathbb{S}^n \text{ and } \operatorname{map}^*(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}}\bigsqcup_{\mathbb{N}} *.$$

For $n+2\le m\le 2n-1$ the model is of the form $(\Lambda(x,y),d)\otimes(\Lambda z_{2n-m},0)$ in the free case and $(\Lambda z_{2n-m},0)$ in the pointed one. From here we deduce that

$$\operatorname{map}(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}} K(\mathbb{Q},2n-m)\times\mathbb{S}^n,$$
$$\operatorname{map}^*(F(\mathbb{R}^m,3),\mathbb{S}^n)\simeq_{\mathbb{Q}} K(\mathbb{Q},2n-m).$$

This completes the proof of Theorem 1.

We finish with the proof of Theorem 2. Recall the explicit description of $H^*(F(\mathbb{R}^m,k);\mathbb{Q})$ given in (1). Hence, this cohomology is concentrated in degrees 0, $m-1$, $2(m-1)$, ..., $(k-1)(m-1)$. Now, from the work of Fadell and Neurwith [6], the Poincaré series of $F(\mathbb{R}^m,k)$ is

$$(1+t^{m-1})(1+2t^{m-1})\cdots(1+(k-1)t^{m-1}).$$

From this and the fact [8] that

$$(1+t)(1+2t)\cdots(1+(k-1)t)=\sum_{j=1}^{k}\begin{bmatrix}k\\j\end{bmatrix}t^j$$

we obtain the dimensions of the nontrivial cohomology of $F(\mathbb{R}^m, k)$:

$$\dim H^{j(m-1)}(F(\mathbb{R}^m,k);\mathbb{Q})=\begin{bmatrix}k\\k-j\end{bmatrix}\quad \text{for } j=0,\,1,\,2,\,\cdots,\,k-1\,.$$

The proof finishes with a direct computation using Proposition 1.

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