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A certain subclass of bi-univalent analytic functions introduced by means of the q-analogue of Noor integral operator and Horadam polynomials

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Abstract: In the present study, by using the Horadam Polnomials and $q-$analogue of Noor integral oprerator, we target to construct an interesting connection between the geometric function theory and that of special functions. Also, by defining a new class of bi-univalent analytic functions, we investigate coefficient estimates and famous Fekete-Szegö inequality for functions belonging to this interesting class.

Key words: $q-$analogue of Noor integral operator, Horadam polynomials, bi-univalent functions

1. Introduction and preliminaries

Let $\mathcal{A}$ be the family of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $\mathfrak{D} = \{z : |z| < 1\}$ and normalized under the conditions given by $f(0) = 0 = f'(0) - 1$. Let $S = \{f \in \mathcal{A} : f$ is univalent in $\mathfrak{D}\}$.

According to the Koebe one-quarter theorem [4], every function $f \in S$ has an inverse f^{-1} which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathfrak{D})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f) \ , \ r_0(f) \geq \frac{1}{4}\right), \quad (w \in \mathfrak{D}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

Definition 1 *A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both f and f^{-1} are univalent in $\mathfrak{D}$. Let Σ denote the class of bi-univalent functions in $\mathfrak{D}$ given by (1).*

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Definition 2 *For analytic functions f and g in $\mathfrak{D}$, f is said to be subordinate to g if there exists an analytic function w such that*

$$w(0)=0 \quad , \ |w(z)|<1 \quad \text{and } f(z)=g\left(w(z)\right).$$

This subordination is denoted by

$$f(z)\prec g(z). \tag{3}$$

In particular, when g is univalent in $\mathfrak{D}$,

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0) \quad \text{and} \quad f(\mathfrak{D})\subset g(\mathfrak{D}) \qquad (z\in\mathfrak{D}).$$

Definition 3 *[14] For $q\in(0,1)$, the q-derivative of function $f\in\mathcal{A}$ is defined by*

$$\partial_q f(z)=\frac{f(qz)-f(z)}{(q-1)z}, z\neq 0 \tag{4}$$

and

$$\partial_q f(0)=f'(0).$$

Thus, we have

$$\partial_q f(z)=1+\sum_{k=2}^{\infty}[k,q]\,a_k z^{k-1}, \tag{5}$$

where $[k,q]$ is given by

$$[k,q]=\frac{1-q^k}{1-q}, \qquad [0,q]=0 \tag{6}$$

and the q-fractional is defined by

$$[k,q]!=\begin{cases}\prod\limits_{n=1}^{k}[n,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{7}$$

Also, the $q-$ generalized Pochhammer symbol for $\mathfrak{p}\geq 0$ is given by

$$[\mathfrak{p},q]_k=\begin{cases}\prod\limits_{n=1}^{k}[\mathfrak{p}+n-1,q]\,, & k\in\mathbb{N}\\ 1, & k=0\end{cases}. \tag{8}$$

As $q\to 1^-$, then we get $[k,q]\to k$. Thus, if we choose the function $g(z)=z^k$, while $q\to 1^-$, then we have

$$\partial_q g(z)=\partial_q z^k=[k,q]\,z^{k-1}=g'(z),$$

where g' is the ordinary derivative.

Recently Arif et al. [3] defined the function $F^{-1}_{q,\mu+1}(z)$ given by the following relation

$$F^{-1}_{q,\mu+1}(z)*F_{q,\mu+1}(z)=z\partial_q f(z), \quad (\mu>-1), \tag{9}$$

where

$$F_{q,\mu+1}(z) = z + \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k-1,q]!} z^k, \quad z \in \mathfrak{D}. \tag{10}$$

Due to the fact that the series defined in (10) is convergent absolutely in $z \in \mathfrak{D}$, by making use of the definition of q-derivative through convolution, we now define the integral operator $\zeta_q^{\mu} : \mathfrak{D} \to \mathfrak{D}$ by

$$\zeta_q^{\mu} f(z) = F_{q,\mu+1}^{-1}(z) * f(z) = z + \sum_{k=2}^{\infty} \phi_{k-1} a_k z^k, \quad (z \in \mathfrak{D}), \tag{11}$$

where

$$\phi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}. \tag{12}$$

We note that

$$\zeta_q^0 f(z) = z\partial_q f(z), \ \zeta_q^1 f(z) = f(z) \tag{13}$$

also

$$\lim_{q\to 1^-} \zeta_q^{\mu} f(z) = z + \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k. \tag{14}$$

This shows that, by taking $q \to 1^-$, the operator defined in (11) reduces to the well known Noor integral operator studied in ([12, 13]). For more details on the q-analogue of differential and integral operators, see the work by Aldweby and Darus (see[2]).

Recently, Horzum and Koçer (see[9]) studied the Horadam polynomials $h_n(x)$, which are given by the following recurrence relation (see also [8]):

$$\begin{aligned} h_n(x) &= pxh_{n-1}(x) + \rho h_{n-2}(x) \\ n &\in \mathbb{N} - \{1,2\}, \qquad \mathbb{N} = \{1,2,3,...\} \end{aligned} \tag{15}$$

with

$$\begin{aligned} h_1(x) &= c, \\ h_2(x) &= dx, \\ h_3(x) &= dpx^2 + c\rho \end{aligned} \tag{16}$$

for some real constants c, d, p and ρ.

First of all, we present some special cases of the polynomials $h_n(x)$ (see [9] and [8]):

1. Choosing $c = d = p = \rho = 1$, we obtain Fibonacci polynomials $F_n(x)$;

2. Choosing $c = 2$ and $d = p = \rho = 1$; we obtain Lucas polynomials $L_n(x)$;

3. Choosing $c = \rho = 1$ and $d = p = 2$; we obtain Pell polynomials $P_n(x)$;

4. Choosing $c = d = p = 2$ and $\rho = 1$; we obtain the Pell–Lucas polynomials $Q_n(x)$;

5. Choosing $c = 1$, $d = p = 2$ and $\rho = -1$; we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

The Fibonacci polynomials, the Lucas polynomials, the Chebyshev polynomials, the Pell polynomials, Lucas–Lehmer polynomials, and the families of orthogonal polynomials and other special polynomials also their generalizations are very important in different disciplines in the mathematical, physical, statistical, and engineering sciences. These kind of polynomials have been studied in several papers from a theoretical point of view (see, e.g., [6–8, 10, 11, 15, 15–18]).

Theorem 4 *[9] Let $\Omega(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then,*

$$\Omega(x,z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{c + (d - cp)\, xz}{1 - pxz - \rho z^2}. \tag{17}$$

Remark 5 *[16] Choosing $d = p = 2, \rho = -1$ and $x \to t$, in Theorem 4, the generating function $\Omega(x,z)$ reduces the Chebyshev polynomials $U_k(t)$ of the second kind, which is given by*

$$U_k(t) = (k+1)_2F_1(-k, k+2; \frac{3}{2}; \frac{1-t}{2}) = \frac{\sin(k+1)\varphi}{\sin\varphi}, \quad (t = \sin\varphi)$$

in terms of the celebrated Gauss hypergeometric function ${}_2F_1$.

2. The class $\Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$

Definition 6 *A function $f \in \Sigma$ is said to be in the class $\Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$ if the following conditions hold true:*

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu} f(z)}{z} + \alpha\partial_q(\zeta_q^{\mu} f(z)) - 1\right] \prec \Omega(x,z) + 1 - c \tag{18}$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu} g(w)}{w} + \alpha\partial_q(\zeta_q^{\mu} g(w)) - 1\right] \prec \Omega(x,w) + 1 - c, \tag{19}$$

where $(\mu > -1,\ 0 < q < 1,\ \tau > 0, \alpha \geq 0)$ and the c, d constants are given by (16) , $g = f^{-1}$ is given by (2) .

Remark 7 *By taking $\mu = 1$ in Definition 6, one can easily see that a function $f \in \Sigma$ is in $\Im_{\Sigma}^{1,q}(\alpha,\tau;x)$ if the following conditions hold true:*

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{f(z)}{z} + \alpha\partial_q f(z) - 1\right] \prec \Omega(x,z) + 1 - c$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{g(w)}{w} + \alpha\partial_q g(w) - 1\right] \prec \Omega(x,w) + 1 - c.$$

Remark 8 *By taking* $\mu = 0$ *in Definition 6 one can easily see that a function* $f \in \Sigma$ *is in* $\Im_{\Sigma}^{0,q}(\alpha, \tau; x)$ *if the following conditions hold true:*

$$1 + \frac{1}{\tau}\left[\partial_q f(z) + \alpha z \partial_q^2 f(z) - 1\right] \prec \Omega(x, z) + 1 - c$$

and

$$1 + \frac{1}{\tau}\left[\partial_q g(w) + \alpha z \partial_q^2 g(w) - 1\right] \prec \Omega(x, w) + 1 - c.$$

Also choosing different values for q, μ, τ, and α in Definition 6, we have following new classes:

Remark 9 *By taking* $q \to 1^-$, *one can easily see that a function* $f \in \Sigma$ *is in* $\Im_{\Sigma}^{\mu}(\alpha, \tau; x)$ *if the following conditions hold true:*

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta^{\mu} f(z)}{z} + \alpha(\zeta^{\mu} f(z))' - 1\right] \prec \Omega(x, z) + 1 - c$$

and

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta^{\mu} g(w)}{w} + \alpha(\zeta^{\mu} g(w))' - 1\right] \prec \Omega(x, w) + 1 - c,$$

where $g = f^{-1}$ *is given by (2) .*

Remark 10 *Upon setting* $q \to 1^-$ *and for* $\tau = 1$, *one can easily see that a function* $f \in \Sigma$ *is in*

$$\Im_{\Sigma}^{\mu}(\alpha, 1; x) = \Im_{\Sigma}^{\mu}(\alpha; x)$$

if the following conditions hold true:

$$(1-\alpha)\frac{\zeta^{\mu} f(z)}{z} + \alpha(\zeta^{\mu} f(z))' \prec \Omega(x, z) + 1 - c$$

and

$$(1-\alpha)\frac{\zeta^{\mu} g(w)}{w} + \alpha(\zeta^{\mu} g(w))' \prec \Omega(x, w) + 1 - c,$$

where $g = f^{-1}$ *is given by (2) .*

Remark 11 *Upon setting* $q \to 1^-$, *for* $\tau = 1$ *and* $\alpha = 1$, *one can easily see that a function* $f \in \Sigma$ *is in*

$$\Im_{\Sigma}^{\mu}(1, 1; x) = \Im_{\Sigma}^{\mu}(x)$$

if the following conditions hold true:

$$(\zeta^{\mu} f(z))' \prec \Omega(x, z) + 1 - c$$

and

$$(\zeta^{\mu} g(w))' \prec \Omega(x, w) + 1 - c,$$

where $g = f^{-1}$ *is given by (2) .*

Remark 12 *Upon setting $q \to 1^-$, for $\tau = 1$, $\alpha = 1$ and $\mu = 1$, one can easily see that a function $f \in \Sigma$ is in*

$$\Im_{\Sigma}^{1}(1,1;x) = \Sigma^{1}(x)$$

if the following conditions hold true:

$$f'(z) \prec \Omega(x,z) + 1 - c$$

and

$$g'(w) \prec \Omega(x,w) + 1 - c,$$

where $g = f^{-1}$ is given by (2). The class $\Sigma^{1}(x)$ was investigated and studied by Alamous [1].

Remark 13 *Upon setting $q \to 1^-$, for $\tau = 1$, $\alpha = 1$ and $\mu = 0$, one can easily see that a function $f \in \Sigma$ is in*

$$\Im_{\Sigma}^{0}(1,1;x) = \Im_{\Sigma}(x)$$

if the following conditions hold true:

$$(zf'(z))' \prec \Omega(x,z) + 1 - c$$

and

$$(wg'(w))' \prec \Omega(x,w) + 1 - c,$$

where $g = f^{-1}$ is given by (2) .

We first state and prove the following result.

3. Initial coefficient estimates

Theorem 14 *Let the function $f \in \Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$ be of the form (1). Then,*

$$|a_2| \leq |\tau| \frac{|dx|\sqrt{|dx|}}{\sqrt{\left|\left[\tau(1+\alpha q+\alpha q^2)\phi_2 d - p\,(1+\alpha q)^2\,\phi_1^2\right] dx^2 - (1+\alpha q)^2\,\phi_1^2 c\rho\right|}} \tag{20}$$

and

$$|a_3| \leq |\tau| \frac{|dx|}{(1+\alpha q+\alpha q^2)\phi_2} + \tau^2 \frac{d^2x^2}{(1+\alpha q)^2\,\phi_1^2}, \tag{21}$$

where

$$\mu > -1, 0 < q < 1, \tau > 0, \alpha \geq 0.$$

Proof Let $f \in \Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$, $\mu > -1$, $0 < q < 1$, $\tau > 0, \alpha \geq 0$. Then by Definition 6, for some analytic functions Φ and Λ such that

$$\Phi(0) = \Lambda(0) = 0, \quad |\Phi(z)| < 1 \quad \text{and} \quad |\Lambda(w)| < 1 \quad (z, w \in \mathfrak{D}) \ ,$$

we can write

$$1 + \frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu} f(z)}{z} + \alpha\partial_q(\zeta_q^{\mu} f(z)) - 1\right] \prec \Omega(x,\Phi(z)) + 1 - c \tag{22}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}g(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}g(w))-1\right]\prec\Omega(x,\Lambda(w))+1-c. \tag{23}$$

or, equivalently,

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}f(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}f(z))-1\right]=1+h_1(x)-c+h_2(x)\Phi(z)+h_3(x)\Phi^2(z)+\cdots \tag{24}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}g(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}g(w))-1\right]=1+h_1(x)-c+h_2(x)\Lambda(w)+h_3(x)\Lambda^2(w)+\cdots. \tag{25}$$

From (24) and (25), we get

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}f(z)}{z}+\alpha\partial_q(\zeta_q^{\mu}f(z))-1\right]=1+h_2(x)u_1z+\left[h_2(x)u_2+h_3(x)u_1^2\right]z^2+\cdots \tag{26}$$

and

$$1+\frac{1}{\tau}\left[(1-\alpha)\frac{\zeta_q^{\mu}g(w)}{w}+\alpha\partial_q(\zeta_q^{\mu}g(w))-1\right]=1+h_2(x)v_1w+\left[h_2(x)v_2+h_3(x)v_1^2\right]w^2+\cdots. \tag{27}$$

It is well known that if

$$|\Phi(z)|=\left|u_1z+u_2z^2+u_3z^3+\cdots\right|<1$$

and

$$|\Lambda(w)|=\left|v_1w+v_2w^2+v_3w^3+\cdots\right|<1,$$

for $z,w\in\mathfrak{D}$, then

$$|u_i|\le 1\quad\text{and}\quad|v_i|\le 1\quad i\in\mathbb{N}.$$

If we compare the corresponding coefficients (26) and (27), we can obtain

$$\frac{1}{\tau}\left(1+\alpha q\right)\phi_1a_2=h_2(x)u_1 \tag{28}$$

$$\frac{1}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2a_3\right]=h_2(x)u_2+h_3(x)u_1^2 \tag{29}$$

and

$$-\frac{1}{\tau}\left(1+\alpha q\right)\phi_1a_2=h_2(x)v_1 \tag{30}$$

$$\frac{1}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2\left(2a_2^2-a_3\right)\right]=h_2(x)v_2+h_3(x)v_1^2. \tag{31}$$

From (28) and (30), we find that

$$u_1=-v_1 \tag{32}$$

and

$$2\left[\frac{1}{\tau}\left(1+\alpha q\right)\phi_1\right]^2 a_2^2 = h_2^2(x)\left(u_1^2+v_1^2\right). \tag{33}$$

Also, by using (29) and (31), we obtain

$$\frac{2}{\tau}\left[(1+\alpha q+\alpha q^2)\phi_2\right] a_2^2 = h_2(x)\left(u_2+v_2\right)+h_3(x)\left(u_1^2+v_1^2\right). \tag{34}$$

By substituting (33) in (34), we reduce that

$$2\left[\tau(1+\alpha q+\alpha q^2)\phi_2 h_2^2(x)-(1+\alpha q)^2\phi_1^2 h_3(x)\right]a_2^2=\tau^2 h_2^3(x)\left(u_2+v_2\right) \tag{35}$$

which yields

$$|a_2| \le |\tau|\frac{|dx|\sqrt{|dx|}}{\sqrt{\left|\left[\tau(1+\alpha q+\alpha q^2)\phi_2 d-p(1+\alpha q)^2\phi_1^2\right]dx^2-(1+\alpha q)^2\phi_1^2 c\rho\right|}}. \tag{36}$$

Moreover, if we subtract (31) from (29) and using (32), we have

$$a_3-a_2^2=\tau\frac{h_2(x)\left(u_2-v_2\right)}{2(1+\alpha q+\alpha q^2)\phi_2}. \tag{37}$$

Then, in view of (33) and (37), we have

$$a_3=\tau\frac{h_2(x)\left(u_2-v_2\right)}{2(1+\alpha q+\alpha q^2)\phi_2}+\tau^2\frac{h_2^2(x)\left(u_1^2+v_1^2\right)}{2(1+\alpha q)^2\phi_1^2}. \tag{38}$$

Applying $h_2(x)$ and taking modulus, we deduce that

$$|a_3| \le |\tau|\frac{|dx|}{(1+\alpha q+\alpha q^2)\phi_2}+\tau^2\frac{d^2x^2}{(1+\alpha q)^2\phi_1^2}.$$

So the proof of Theorem 14 is completed. □

In the next section, we introduce the Fekete–Szegö inequalities for functions in the class $\Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$, which is introduced by Definition 6. In Theorem 15, we will give these inequalities.

4. The Fekete-Szegö inequality for the class $\Im_{\Sigma}^{\mu,q}(\alpha,\tau;x)$

The following classical Fekete–Szegö inequality, which is studied via Loewner's chain method, contains the Taylor–Maclaurin coefficients of $f \in S$ given by (1):

$$\left|a_3-\delta a_2^2\right| \leqq 1+2\exp\left(\frac{-2\delta}{1-\delta}\right) \quad (0 \leqq \delta < 1).$$

In its limit as $\delta \to 1^-$, we have a fundamental inequality given by

$$\left|a_3-a_2^2\right| \leqq 1.$$

In fact, for the normalized function $f \in S$, the coefficient functional $\Psi_\delta(f)$ given by the relation

$$\Psi_\delta(f) = a_3 - \delta a_2^2$$

is very important in function theory. The problem of maximizing the modulus of the functional $\Psi_\delta(f)$ is said to be the Fekete–Szegö problem (see [5, 16]).

Theorem 15 *Let the function $f \in \Im_\Sigma^{\mu,q}(\alpha,\tau;x)$ be of the form (1). Then, for some $\delta \in \mathbb{R}$*

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} |\tau| \frac{|dx|}{(1+\alpha q+\alpha q^2)\phi_2} & , \quad |\delta-1| \leq 1-\frac{(1+\alpha q)^2\phi_1^2\left(dpx^2+c\rho\right)}{(1+\alpha q+\alpha q^2)\phi_2|\tau|d^2x^2} \\ \tau^2\frac{|\delta-1||dx|^3}{\left|\tau\left[(1+\alpha q+\alpha q^2)\phi_2 d-(1+\alpha q)^2\phi_1^2 p\right]dx^2-(1+\alpha q)^2\phi_1^2\rho c\right|} & , \quad |\delta-1| \geq 1-\frac{(1+\alpha q)^2\phi_1^2\left(dpx^2+c\rho\right)}{(1+\alpha q+\alpha q^2)\phi_2|\tau|d^2x^2} \end{cases}. \tag{39}$$

Proof From equation (37), for $\delta \in \mathbb{R}$, we write

$$a_3-\delta a_2^2 = \tau\frac{h_2(x)\left(u_2-v_2\right)}{2(1+\alpha q+\alpha q^2)\phi_2} + (1-\delta)a_2^2. \tag{40}$$

By substituting (35) in (40), we have

$$\begin{aligned} a_3-\delta a_2^2 &= \tau\frac{h_2(x)\left(u_2-v_2\right)}{2(1+\alpha q+\alpha q^2)\phi_2} + (1-\delta)\left[\tau^2\frac{h_2^3(x)\left(u_2+v_2\right)}{2\tau(1+\alpha q+\alpha q^2)\phi_2 h_2^2(x)-2(1+\alpha q)^2\phi_1^2 h_3(x)}\right] \\ &= h_2(x)\left[\left(\Phi(\delta,x)+\frac{\tau}{2(1+\alpha q+\alpha q^2)\phi_2}\right)u_2\right. \\ &\quad \left. +\left(\Phi(\delta,x)-\frac{\tau}{2(1+\alpha q+\alpha q^2)\phi_2}\right)v_2\right], \end{aligned} \tag{41}$$

where

$$\Phi(\delta,x) = \tau^2\frac{(1-\delta)h_2^2(x)}{2\left[\tau h_2^2(x)(1+\alpha q+\alpha q^2)\phi_2-h_3(x)(1+\alpha q)^2\phi_1^2\right]}.$$

Then, in view of (16) ,we conclude that

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} |\tau|\frac{|h_2(x)|}{(1+\alpha q+\alpha q^2)\phi_2} & , \quad 0 \leq |\Phi(\delta,x)| \leq \frac{|\tau|}{2(1+\alpha q+\alpha q^2)\phi_2} \\ 2\left|h_2(x)\right|\left|\Phi(\delta,x)\right| & , \quad |\Phi(\delta,x)| \geq \frac{|\tau|}{2(1+\alpha q+\alpha q^2)\phi_2} \end{cases}$$

which evidently complete the proof of the theorem. □

An urgent and important corollary of Theorems 14 and 15 for $q \to 1^-$ is asserted by Corollary 16.

Corollary 16 *Let the function $f \in \Im_\Sigma^{\mu,1}(\alpha,\tau;x)$ be of the form (1). Also assume that $\delta \in \mathbb{R}$. Then,*

$$|a_2| \leq |\tau|\frac{|dx|\sqrt{|dx|}}{\sqrt{\left|\left[\tau(1+2\alpha)\phi_2 d-p\left(1+\alpha\right)^2\phi_1^2\right]dx^2-\left(1+\alpha\right)^2\phi_1^2 c\rho\right|}}, \tag{42}$$

$$|a_3| \leq |\tau| \frac{|dx|}{(1+2\alpha)\phi_2} + \tau^2 \frac{d^2x^2}{(1+\alpha)^2 \phi_1^2} \tag{43}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} |\tau| \frac{|dx|}{(1+2\alpha)\phi_2} & , \quad |\delta - 1| \leq 1 - \frac{(1+\alpha)^2 \phi_1^2 \left(dpx^2 + \rho c\right)}{(1+2\alpha)\phi_2 |\tau| d^2 x^2} \\ \tau^2 \frac{|\delta-1||dx|^3}{\left|\tau\left[(1+2\alpha)\phi_2 d - (1+\alpha)^2 \phi_1^2 p\right] dx^2 - (1+\alpha)^2 \phi_1^2 \rho c\right|} & , \quad |\delta - 1| \geq 1 - \frac{(1+\alpha)^2 \phi_1^2 \left(dpx^2 + \rho c\right)}{(1+2\alpha^2)\phi_2 |\tau| d^2 x^2} \end{cases} . \tag{44}$$

Choosing $\tau = 1$ and $q \to 1^-$ in Theorems 14 and 15, we get the Corollary 17.

Corollary 17 *Let the function* $f \in \Im_{\Sigma}^{\mu,1}(\alpha, 1; x)$ *be of the form (1). Also assume that* $\delta \in \mathbb{R}$*. Then,*

$$|a_2| \leq \frac{|dx| \sqrt{|dx|}}{\sqrt{\left|\left[(1+2\alpha)\phi_2 d - p\,(1+\alpha)^2 \phi_1^2\right] dx^2 - (1+\alpha)^2 \phi_1^2 \rho c\right|}},$$

$$|a_3| \leq \frac{|dx|}{(1+2\alpha)\phi_2} + \frac{d^2x^2}{(1+\alpha)^2 \phi_1^2}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|dx|}{(1+2\alpha)\phi_2} & , \quad |\delta - 1| \leq \left(1 - \frac{(1+\alpha)^2 \phi_1^2 \left(dpx^2 + \rho c\right)}{(1+\alpha q + \alpha q^2)\phi_2 d^2 x^2}\right) \\ \frac{|\delta-1||dx|^3}{\left|\left[(1+2\alpha)\phi_2 d - (1+\alpha)^2 \phi_1^2 p\right] dx^2 - (1+\alpha)^2 \phi_1^2 \rho c\right|} & , \quad |\delta - 1| \geq \left(1 - \frac{(1+\alpha)^2 \phi_1^2 \left(dpx^2 + \rho c\right)}{(1+2\alpha)\phi_2 d^2 x^2}\right) \end{cases} .$$

Choosing $\alpha = 1,\ \tau = 1$ and $q \to 1^-$ in Theorems 14 and 15, we get Corollary 18.

Corollary 18 *Let the function* $f \in \Im_{\Sigma}^{\mu,1}(1, 1; x)$ *be of the form (1). Also assume that* $\delta \in \mathbb{R}$*. Then,*

$$|a_2| \leq \frac{|dx| \sqrt{|dx|}}{\sqrt{\left|\left[3\phi_2 d - 4p\phi_1^2\right] dx^2 - 4\phi_1^2 \rho c\right|}},$$

$$|a_3| \leq \frac{|dx|}{3\phi_2} + \frac{d^2x^2}{4\phi_1^2}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|dx|}{3\phi_2} & , \quad |\delta - 1| \leq \left(1 - \frac{4\phi_1^2 \left(dpx^2 + \rho c\right)}{3\phi_2 d^2 x^2}\right) \\ \frac{|\delta-1||dx|^3}{\left|\left[(3\phi_2 d - 4\phi_1^2 p\right] dx^2 - 4\phi_1^2 \rho c\right|} & , \quad |\delta - 1| \geq \left(1 - \frac{4\phi_1^2 \left(dpx^2 + \rho c\right)}{3\phi_2 d^2 x^2}\right) \end{cases} .$$

Choosing $\alpha = 1,\ \tau = 1,\ q \to 1^-$ and $\phi_1 = \phi_2 = 1$ in Theorem 14 and Theorem 15, we have corresponding result of the Alamoush (Thm2.2 in [1]) which we recall here as Corollary 19:

Corollary 19 *Let the function* $f \in \Im_{\Sigma}^{\mu,1}(1,1;x)$ *be of the form (1). Also assume that* $\delta \in \mathbb{R}$*. If* $\phi_1 = \phi_2 = 1,$ *then*

$$|a_2| \leq \frac{|dx|\sqrt{|dx|}}{\sqrt{|[3d-4p]\,dx^2 - 4c\rho|}},$$

$$|a_3| \leq \frac{|dx|}{3} + \frac{d^2x^2}{4}$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{|dx|}{3} & , \quad |\delta - 1| \leq \left(1 - \frac{4\left(dpx^2+\rho c\right)}{3d^2x^2}\right) \\ \frac{|\delta-1||dx|^3}{|(3d-4p)dx^2-4\rho c|} & , \quad |\delta - 1| \geq \left(1 - \frac{4\left(dpx^2+a\right)}{3d^2x^2}\right) \end{cases},$$

where the coefficient ϕ_{k-1} given by (12).

In light of Remark 5, Theorems 14 and 15 give us the following result.

Corollary 20 *For* $t \in (\frac{1}{2}, 1)$*, let the function* $f \in \Im_{\Sigma}^{\mu,1}(1,1;t)$ *be of the form (1) . Then,*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|1-t^2|}},$$

$$|a_3| \leq \frac{2t}{3} + t^2$$

and

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{4t}{3} & , \quad |\delta - 1| \leq \frac{1-t^2}{3t^2} \\ \frac{2|\delta-1|}{|1-t^2|} & , \quad |\delta - 1| \geq \frac{1-t^2}{3t^2} \end{cases}.$$

Choosing $\delta = 1$ in Corollary 20, we obtain the following result.

Corollary 21 *For* $t \in (\frac{1}{2}, 1)$*, let the function* $f \in \Im_{\Sigma}^{\mu,1}(1,1;t)$ *be of the form (1). Then,*

$$\left|a_3 - \delta a_2^2\right| \leq \frac{4t}{3}.$$

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