Turkish Journal of Mathematics

Volume 43 | Number 5 Article 13

1-1-2019

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YİĞİT, EBRU; AYIK, GONCA; and AYIK, HAYRULLAH (2019) "Relative ranks of some partial transformation semigroups," *Turkish Journal of Mathematics*: Vol. 43: No. 5, Article 13. https://doi.org/10.3906/mat-1902-23
Available at: https://dctubitak.researchcommons.org/math/vol43/iss5/13

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2218 – 2225
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doi:10.3906/mat-1902-23

Relative ranks of some partial transformation semigroups

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Received: 05.02.2019 • **Accepted/Published Online:** 16.07.2019 • **Final Version:** 28.09.2019

Abstract: Let P_n, T_n, I_n, and S_n be the partial transformation semigroup, the (full) transformation semigroup, the symmetric inverse semigroup, and the symmetric group on $X_n = \{1, \ldots, n\}$, respectively. For $1 \leq r \leq n-1$, let $PK_{n,r}$ be the subsemigroup consisting $\alpha \in P_n$ such that $|\mathrm{im}\alpha| \leq r$ and let $SPK_{n,r} = PK_{n,r} \setminus T_n$. In this paper, we first examine the subsemigroup $I_{n,r} = I_n \cup PK_{n,r}$ and we find the necessary and sufficient conditions for any nonempty subset of $PK_{n,r}$ to be a (minimal) relative generating set of the subsemigroup $I_{n,r}$ modulo I_n. Then we examine the subsemigroups $PI_{n,r} = SI_n \cup PK_{n,r}$ and $SI_{n,r} = SI_n \cup SPK_{n,r}$ for $1 \leq r \leq n-1$ where $SI_n = I_n \setminus S_n$ and compute their relative rank.

Key words: (Partial) transformation semigroup, symmetric inverse semigroup, symmetric group, (minimal) generating set, relative rank

1. Introduction

The partial transformation semigroup P_X, the (full) transformation semigroup T_X and the symmetric inverse semigroup I_X on a set X have been extensively studied over the last sixty years, both in the finite and in the infinite cases. Among recent contributions are [1–6, 13, 16]. Here we are concerned solely with the case where $X = X_n = \{1, \ldots, n\}$, and we denote the semigroups P_{X_n}, T_{X_n}, and I_{X_n}, by P_n, T_n, and I_n, respectively. Moreover, we denote the subsemigroup $I_n \setminus S_n$ by SI_n where S_n is the symmetric group on X_n.

It is well known that I_n is an inverse semigroup and every finite inverse semigroup S is embeddable in I_n, the analog of Cayley's theorem for finite groups. Hence, as emphasized in [1], the importance of I_n to inverse semigroup theory is similar to that of the symmetric group S_n to group theory. Moreover, Gomes and Howie remarked in [11] that very little has been written on the symmetric inverse semigroups. Despite the appearance of the books of Lipscomb [18], and Ganyushkin and Mazorchuk [8], as well as a handful of papers (for example, [10]), the study of I_n is still in its infancy compared to that of T_n.

An element α of P_n is called an idempotent if $\alpha^2 = \alpha$. We denote the set of all idempotents in any subset U of any semigroup by $E(U)$. Let S be a semigroup and let A be a nonempty subset of S. Then the subsemigroup generated by A, that is the smallest subsemigroup of S containing A, is denoted by $\langle A \rangle$. If a semigroup S has a finite subset A such that $S = \langle A \rangle$, then S is called a *finitely generated* semigroup. The *rank* of a finitely generated semigroup S is defined by $\mathrm{rank}(S) = \min\{\, |A| : \langle A \rangle = S \,\}$. For a fixed subset G of

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2010 *AMS Mathematics Subject Classification:* 20M20

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a semigroup S, if there exists a subset A of S such that $\langle A \cup G \rangle = S$, then A is called a *relative generating set of* S *modulo* G. Then the *relative rank* of a finitely generated semigroup S modulo G is defined by

$$\operatorname{rerank}(S:G) = \min\{|A| : \langle A \cup G \rangle = S\}.$$

For $1 \leq r \leq n$, let

$$\begin{array}{rclrcl} K_{n,r} & = & \{\alpha \in T_n : |\operatorname{im}(\alpha)| \leq r\}, & T_{n,r} & = & S_n \cup K_{n,r}, \\ PK_{n,r} & = & \{\alpha \in P_n : |\operatorname{im}(\alpha)| \leq r\}, & PT_{n,r} & = & S_n \cup PK_{n,r}, \\ SPK_{n,r} & = & PK_{n,r} \setminus T_n = PK_{n,r} \setminus K_{n,r}, & A_{n,r} & = & A_n \cup K_{n,r}, \\ PA_{n,r} & = & A_n \cup PK_{n,r}, & I_{n,r} & = & I_n \cup PK_{n,r}, \\ SI_{n,r} & = & SI_n \cup SPK_{n,r} \qquad \text{and} & PI_{n,r} & = & SI_n \cup PK_{n,r}, \end{array}$$

where A_n denotes the alternating group on X_n.

Howie and McFadden proved in [15] that the rank of $K_{n,r}$ is $S(n,r)$, the Stirling number of the second kind, for $2 \leq r \leq n-1$. Recall that the Stirling number $S(n,r)$ of the second kind is defined by

$$S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,r) = S(n-1,r-1) + r \cdot S(n-1,r)$$

for $2 \leq r \leq n-1$. Moreover, Garba proved in [9] that the rank of the subsemigroup $PK_{n,r}$ of P_n is $S(n+1,r+1)$ for $2 \leq r \leq n-1$.

For $n, r \in \mathbb{Z}^+$ with $r \leq n$, let $P_r(n)$ be the set of all integer solutions of the equation

$$x_1 + x_2 + \cdots + x_r = n \quad \text{with} \quad x_1 \geq x_2 \geq \cdots \geq x_r \geq 1,$$

and let $p_r(n) = |P_r(n)|$. If an r-tuple $(n_1, n_2, \ldots, n_r)$ is a solution of the equation given above, then it is called a *partition of* n *with* r *terms* (see [12]). Ayık et al. developed a notation for certain primitive elements of T_n, called path-cycle, and described an algorithm to decompose an arbitrary transformation α in T_n into a product of path-cycles in [2]. In addition, they used these techniques to obtain some informations about generators of T_n, and proved that, $\operatorname{rerank}(T_{n,r} : S_n) = p_r(n)$ for $1 \leq r \leq n-1$ (see also [17, Theorem 8]).

In [19], we obtained the necessary and sufficient conditions for any nonempty subset U of $K_{n,r}$ ($PK_{n,r}$) to be a (minimal) relative generating set of $T_{n,r}$ ($PT_{n,r}$) modulo S_n for $1 \leq r \leq n-1$. Then we concluded the same result in [2, 17] that $\operatorname{rerank}(T_{n,r} : S_n) = p_r(n)$ and, we obtained the new result

$$\operatorname{rerank}(PT_{n,r} : S_n) = \sum_{s=0}^{n-r} p_r(n-s)$$

for $1 \leq r \leq n-1$. Moreover, we showed that

$$\operatorname{rerank}(A_{n,r} : A_n) = p_r(n) \text{ and } \operatorname{rerank}(PA_{n,r} : A_n) = \sum_{s=0}^{n-r} p_r(n-s)$$

for each $1 \leq r \leq n-1$.

In this paper, we first find the necessary and sufficient conditions for any nonempty subset U of $PK_{n,r}$ to be a (minimal) relative generating set of $I_{n,r}$ (respectively $PI_{n,r}$) modulo I_n (respectively SI_n)

for $1 \le r \le n-1$. Moreover, we find the necessary and sufficient conditions for any subset U of $SPK_{n,r}$ to be a (minimal) relative generating set of $SI_{n,r}$ modulo SI_n. Then we conclude that

$$\begin{aligned} \operatorname{rerank}(I_{n,r} : I_n) &= p_r(n), \\ \operatorname{rerank}(SI_{n,r} : SI_n) &= p_r(n-1) \text{ and} \\ \operatorname{rerank}(PI_{n,r} : SI_n) &= S(n,r) \end{aligned}$$

for $1 \le r \le n-1$.

2. Preliminaries

The *height* and the *kernel* of any partial transformation $\alpha \in P_n$ are defined by

$$\begin{aligned} \mathrm{h}(\alpha) &= |\mathrm{im}(\alpha)| \text{ and} \\ \ker(\alpha) &= \{(x,y) \in X_n \times X_n : \text{ either } x,y \in \mathrm{dom}(\alpha) \text{ and } x\alpha = y\alpha \\ &\qquad \text{or } x,y \notin \mathrm{dom}(\alpha)\}, \end{aligned}$$

respectively. For any $\alpha, \beta \in P_n$ (also $\alpha, \beta \in T_n$), recall that $\mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha)$, $\mathrm{im}(\alpha\beta) \subseteq \mathrm{im}(\beta)$, $\ker(\alpha) \subseteq \ker(\alpha\beta)$, and that

$$\begin{aligned} (\alpha,\beta) \in \mathcal{D} \quad &\Leftrightarrow \quad \mathrm{h}(\alpha) = \mathrm{h}(\beta) \\ (\alpha,\beta) \in \mathcal{H} \quad &\Leftrightarrow \quad \mathrm{im}(\alpha) = \mathrm{im}(\beta),\ \ker(\alpha) = \ker(\beta) \text{ and } \mathrm{dom}(\alpha) = \mathrm{dom}(\beta) \end{aligned}$$

where the equivalences $\mathcal{D}$ and $\mathcal{H}$ denote Green's relations (see, for examples [14] and [8, Theorem 4.5.1]). For $1 \le r \le n$, we denote that Green's $\mathcal{D}$-class, consists of all elements in T_n (respectively P_n) of height r, by D_r^T (respectively D_r^P). If the implied semigroup is clear from the context, we will use the simpler notation D_r.

It is shown in [7] that Green's $\mathcal{D}$-class D_r^T is generated by its idempotents, and so it follows from [15, Lemma 4] that $K_{n,r} = \langle E(D_r^T) \rangle$ for $2 \le r \le n-1$. It is also shown in [9] that a subset A of Green's $\mathcal{D}$-class D_r^P is a generating set of $PK_{n,r}$ if and only if $E(D_r^P) \subseteq \langle A \rangle$ for $2 \le r \le n-1$. Therefore, to show a subset A of D_r^P is a generating set of $PK_{n,r}$, it is enough to prove $D_r^P \subseteq \langle A \rangle$.

For a given nonempty set X and a positive integer r where $1 \le r \le |X|$, let $A_1, \ldots, A_r$ be a collection of nonempty disjoint subsets of X. Then $\xi = \{A_1, \ldots, A_r\}$ is called a *partition of* X (with r terms) if $X = \bigcup_{i=1}^{r} A_i$. A partition $\{A_1, \ldots, A_r\}$ of X is called an *ordered partition* if $|A_1| \ge \cdots \ge |A_r|$, and is denoted by $(A_1, \ldots, A_r)$. For $1 \le r \le n$, it is clear that $\alpha \in D_r^P$ if and only if there exists a unique partition $\{A_1, \ldots, A_r\}$ of $\mathrm{dom}(\alpha)$ such that $\ker(\alpha) = \bigcup_{i=1}^{r+1} (A_i \times A_i)$ where $A_{r+1} = X_n \setminus \mathrm{dom}(\alpha) = \mathrm{cdom}(\alpha)$; or equivalently, there exists a unique subset $\{a_1, \ldots, a_r\}$ of X_n with cardinality r, such that $\mathrm{im}(\alpha) = \{a_1, \ldots, a_r\}$. Without loss of generality suppose that $A_i\alpha = a_i$ for each $1 \le i \le r$, and so α can be written in the following tabular form:

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & A_{r+1} \\ a_1 & \cdots & a_r & - \end{pmatrix} \quad \left(\text{and } \alpha = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix} \text{ if } \alpha \in D_r^T\right).$$

For $\alpha \in D_r^T$, written in the above tabular form, there clearly exists a permutation $\sigma \in S_r$ such that $(|A_{1\sigma}|, \ \ldots \ , |A_{r\sigma}|)$ is a partition of n with r terms. In this case the *partition* of α is defined by

$$\mathrm{part}(\alpha) = (|A_{1\sigma}|, \ \ldots \ , |A_{r\sigma}|)\,.$$

For $\alpha \in D_r^P$ if $|\mathrm{cdom}(\alpha)| = |A_{r+1}| = s \geq 1$, similarly there exists a permutation $\sigma \in S_r$ such that $(|A_{1\sigma}|, \ \ldots \ , |A_{r\sigma}|)$ is a partition of $n-s$ with r terms. In this case, the *co-partition* of α is defined by

$$\mathrm{copart}(\alpha) = (\, |A_{1\sigma}|, \ \ldots \ , |A_{r\sigma}| : |A_{r+1}|\,)\,.$$

If $|\mathrm{cdom}(\alpha)| = |A_{r+1}| = s = 0$, then for convenience the *copartition* of α is defined by $\mathrm{copart}(\alpha) = \mathrm{part}(\alpha)$.

From now on, we consider the case $1 \leq r \leq n-1$, since $D_n^P = D_n^T = S_n$. We also assume that $\alpha \in D_r^P$ is in the above tabular form unless stated otherwise.

First, recall Proposition 1 and Lemma 2 in [19] :

Proposition 2.1 *For $1 \leq r \leq n-1$, let $\alpha, \beta \in D_r$. Then $\alpha\beta \in D_r$ if and only if $\ker(\alpha\beta) = \ker(\alpha)$. Moreover, $\alpha\beta \in D_r$ implies $\mathrm{cdom}(\alpha\beta) = \mathrm{cdom}(\alpha)$.*

Lemma 2.2 *For $1 \leq r \leq n-1$, let $\alpha, \beta \in D_r$. Then $\mathrm{copart}(\alpha) = \mathrm{copart}(\beta)$ if and only if there exist $\lambda, \mu \in S_n$ such that $\alpha = \lambda\beta\mu$. In particular, for $\alpha, \beta \in T_n$, $\mathrm{part}(\alpha) = \mathrm{part}(\beta)$ if and only if there exist $\lambda, \mu \in S_n$ such that $\alpha = \lambda\beta\mu$.*

Next we state and prove the following similar lemma which will be used throughout this paper:

Lemma 2.3 *For $1 \leq r \leq n-1$, let $\alpha \in D_r^P$ and let $\mathrm{copart}(\alpha) = (n_1, n_2, \ldots, n_r : s)$ with $s \geq 1$.*

(i) For any $\beta \in D_r^T$ with $\mathrm{part}(\beta) = (n_1 + s, n_2, \ldots, n_r)$, there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda\beta\mu$.

(ii) For any $\beta \in D_r^P$ with $\mathrm{copart}(\beta) = (n_1 + s - k, n_2, \ldots, n_r : k)$ where $1 \leq k \leq s$, there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda\beta\mu$.

(iii) For any $\beta \in D_r^P$, $\mathrm{copart}(\alpha) = \mathrm{copart}(\beta)$ if and only if $|\mathrm{dom}(\alpha)| = |\mathrm{dom}(\beta)|$ and there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda\beta\mu$.

Proof Without loss of generality, let

$$\alpha \ = \ \begin{pmatrix} A_1 & \cdots & A_r & A_{r+1} \\ a_1 & \cdots & a_r & - \end{pmatrix}$$

where $|A_i| = n_i$ for $1 \leq i \leq r$ and $|A_{r+1}| = s \geq 1$.

(i)-(ii) For any $\beta \in D_r^P$ without loss of generality let

$$\beta \ = \ \begin{pmatrix} B_1 & \cdots & B_r & B_{r+1} \\ b_1 & \cdots & b_r & - \end{pmatrix}$$

where

$$|B_1| = \begin{cases} n_1 + s & \text{if } |B_{r+1}| = 0 \\ n_1 + s - k & \text{if } 1 \leq |B_{r+1}| = k \leq s \end{cases} \qquad \text{and} \qquad |B_i| = n_i$$

for each $2 \leq i \leq r$. Then it is clear that there exists $\lambda \in SI_n$ such that $A_1\lambda \subseteq B_1$, $A_i\lambda = B_i$ for each $2 \leq i \leq r$ and $\mathrm{cdom}(\lambda) = \mathrm{cdom}(\alpha)$. Moreover, consider the partial injective transformation $\mu : \mathrm{im}(\beta) \longrightarrow \mathrm{im}(\alpha)$ in SI_n defined by $b_i\mu = a_i$ for each $1 \leq i \leq r$. Since $A_i(\lambda\beta\mu) = a_i$ for each $1 \leq i \leq r$ and $\mathrm{cdom}(\lambda\beta\mu) = A_{r+1}$, we have $\alpha = \lambda\beta\mu$, as required.

(iii) $(\Rightarrow)$ Suppose that $\mathrm{copart}(\alpha) = \mathrm{copart}(\beta)$. Then, from (ii), it follows that $k = s$.

$(\Leftarrow)$ Suppose that $|\mathrm{dom}(\alpha)| = |\mathrm{dom}(\beta)|$ and that there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda\beta\mu$. Then $\mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\lambda)$ and $(\mathrm{dom}(\alpha))\lambda \subseteq \mathrm{dom}(\beta)$, and so $(\mathrm{dom}(\alpha))\lambda = \mathrm{dom}(\beta)$. Since $\alpha, \beta \in D_r^P$, it follows that $\mathrm{im}(\beta) \subseteq \mathrm{dom}(\mu)$, and so $(\mathrm{im}(\beta))\mu = \mathrm{im}(\alpha)$. Thus, it is easy to see that $\mathrm{dom}(\beta) = \bigcup_{i=1}^r A_i\lambda$ and that

$$\beta = \begin{pmatrix} A_1\lambda & \cdots & A_r\lambda & B_{r+1} \\ a_1\mu^{-1} & \cdots & a_r\mu^{-1} & - \end{pmatrix},$$

where $B_{r+1} = \mathrm{cdom}(\beta)$. Therefore, since $|A_i| = |A_i\lambda|$ for each $1 \le i \le r$, we have $\mathrm{copart}(\alpha) = \mathrm{copart}(\beta)$, as required. □

For $1 \le r \le n-1$, let $\alpha, \beta \in I_n$ with $\mathrm{h}(\alpha) = \mathrm{h}(\beta) = r$. Then it is clear that $\mathrm{h}(\alpha\beta) = r$ if and only if $\mathrm{im}(\alpha) = \mathrm{dom}(\beta)$.

3. Relative ranks

Notice that, since $I_{n,r} \setminus I_n = PK_{n,r}$ is an ideal of $I_{n,r}$, any generating set of $I_{n,r}$ must contain a generating set of I_n for $1 \le r \le n-1$. Thus, if $W \subseteq I_{n,r}$ is a generating set of $I_{n,r}$, then there exist $U \subseteq D_r^P \cap W$ and $V \subseteq I_n \cap W$ such that $I_n = \langle V \rangle$ and $I_{n,r} = \langle U \cup V \rangle = \langle U \cup I_n \rangle$. Therefore, any minimal relative generating set of $I_{n,r}$ modulo I_n must be a subset of D_r^P for $1 \le r \le n-1$.

Theorem 3.1 *Let $1 \le r \le n-1$ and $U \subseteq PK_{n,r}$. Then $I_{n,r} = \langle U \cup I_n \rangle$ if and only if, for each partition $p = (n_1, \ldots, n_r) \in P_r(n)$, there exists $\beta \in U \cap D_r^T$ such that $\mathrm{part}(\beta) = p$.*

Proof $(\Leftarrow)$ Let $1 \le r \le n-1$. For each partition $p = (n_1, \ldots, n_r) \in P_r(n)$, we fix an arbitrary element $\beta_p \in U \cap D_r^T$ with $\mathrm{part}(\beta_p) = p$. Then we denote the set of all these fixed elements by $V = \{\beta_p \in U : p \in P_r(n)\}$.

For any element $\alpha \in D_r^P$ either $\alpha \in K_{n,r}$ or $\alpha \in SPK_{n,r}$. If $\alpha \in K_{n,r}$ with $\mathrm{copart}(\alpha) = \mathrm{part}(\alpha) = (n_1, \ldots, n_r) = p$, then there exists $\beta_p \in V$ such that $\mathrm{part}(\beta_p) = p$, and so it follows from Lemma 2.2 that there exist $\lambda, \mu \in S_n \subseteq I_n$ such that $\alpha = \lambda\beta_p\mu$. Now suppose that $\alpha \in SPK_{n,r}$ with $\mathrm{copart}(\alpha) = (n_1, n_2, \ldots, n_r : s)$ where $s \ge 1$. Then there exists $\beta_p \in V$ such that $\mathrm{part}(\beta_p) = (n_1 + s, n_2, \ldots, n_r) = p$ and so, it follows from Lemma 2.3 (i) that there exist $\lambda, \mu \in SI_n \subseteq I_n$ such that $\alpha = \lambda\beta_p\mu$. Thus, the set $V \cup I_n$ and so, the set $U \cup I_n$ generates D_r^P. Therefore, it follows from Lemma 2.6 given in [9] that $U \cup I_n$ is a generating set of $I_{n,r}$.

$(\Rightarrow)$ For $1 \le r \le n-1$, let $U \subseteq PK_{n,r}$ be a relative generating set of $I_{n,r}$ modulo I_n, that is $I_{n,r} = \langle U \cup I_n \rangle$. Then, for an arbitrary partition $p \in P_r(n)$, consider an arbitrary element $\beta \in D_r^T$ such that $\mathrm{part}(\beta) = p$. Since $I_{n,r} = \langle U \cup I_n \rangle$, β can be written as a product of finitely many elements of $U \cup I_n$. It follows from $\beta \notin I_n$ that either $\beta = \beta_1\delta$ or $\beta = \alpha_1\beta_1\delta$ for some $\beta_1 \in U$, $\delta \in \langle U \cup I_n \rangle \subseteq I_{n,r}$ and $\alpha_1 \in I_n \setminus \{\varepsilon\}$ where ε is the identity permutation on X_n. Now let

$$\gamma = \begin{cases} \beta_1 & \text{if } \beta = \beta_1\delta \\ \alpha_1\beta_1 & \text{if } \beta = \alpha_1\beta_1\delta, \end{cases}$$

and so $\beta = \gamma\delta$. Since $X_n = \mathrm{dom}(\beta) \subseteq \mathrm{dom}(\gamma)$, it follows that $\gamma \in T_n$ (and $\alpha_1 \in S_n$ in the second case), and so $\beta_1 \in T_n$ in both cases. Since $\mathrm{h}(\gamma) \ge \mathrm{h}(\beta) = r$ and $\beta_1 \in K_{n,r}$, it follows that $\gamma, \beta_1 \in D_r^T$. Thus, since

$\ker(\gamma) \subseteq \ker(\beta)$, we have $\ker(\beta) = \ker(\gamma)$ and so,

$$p = \text{part}(\beta) = \text{part}(\gamma) = \text{part}(\beta_1).$$

Therefore, $\beta_1 \in U \cap D_r^T$ and $\text{part}(\beta_1) = p$, as required. □

As an immediate consequence, we have the following corollary:

Corollary 3.2 *For each* $1 \leq r \leq n-1$,

$$\text{rerank}(I_{n,r} : I_n) = p_r(n)$$

where $p_r(n)$ *is the number of partitions of* n *with* r *terms.*

Notice that, since $SI_{n,r} \setminus SI_n = SPK_{n,r}$ is an ideal of $SI_{n,r}$, any generating set of $SI_{n,r}$ must contain a generating set of SI_n. Similarly, any minimal relative generating set of $SI_{n,r}$ modulo SI_n must be a subset of $D_r^P \cap SPK_{n,r} = D_r^P \setminus D_r^T$ for each $1 \leq r \leq n-1$.

Theorem 3.3 *Let* $1 \leq r \leq n-2$ *and* $U \subseteq SPK_{n,r}$. *Then* $SI_{n,r} = \langle U \cup SI_n \rangle$ *if and only if, for each partition* $(n_1, \ldots, n_r) \in P_r(n-1)$, *there exists* $\beta \in U \cap D_r^P$ *such that* $\text{copart}(\beta) = (n_1, \ldots, n_r : 1)$.

Proof ($\Leftarrow$) Let $1 \leq r \leq n-2$. For each partition $p = (n_1, \ldots, n_r) \in P_r(n-1)$, we fix an arbitrary element $\beta_p \in U \cap D_r^P$ with $\text{copart}(\beta_p) = (n_1, \ldots, n_r : 1)$. Then we denote the set of all these fixed elements by $V = \{\beta_p \in U : p \in P_r(n-1)\}$.

For any element $\alpha \in D_r^P \cap SPK_{n,r}$, let $\text{copart}(\alpha) = (n_1, \ldots, n_r : s)$. Since $s \geq 1$, we have $(n_1 + s - 1, n_2, \ldots, n_r) = p \in P_r(n-1)$. Then there exists $\beta_p \in V$ such that $\text{part}(\beta_p) = p$, so it follows from Lemma 2.3 (*ii*) (when $k = 1$) that there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda \beta_p \mu$. Thus, the set $V \cup SI_n$, and so the set $U \cup SI_n$ generates $D_r^P \setminus D_r^T$. It follows from Lemma 2.6 in [9] that $U \cup SI_n$ is a generating set of $SI_{n,r}$.

($\Rightarrow$) For $1 \leq r \leq n-2$, let $U \subseteq SPK_{n,r}$ be a relative generating set of $SI_{n,r}$ modulo SI_n. Then, for an arbitrary partition $p = (n_1, \ldots, n_r) \in P_r(n-1)$, consider an arbitrary element $\beta \in D_r^P$ such that $\text{copart}(\beta) = (n_1, \ldots, n_r : 1)$. Since $SI_{n,r} = \langle U \cup SI_n \rangle$, β can be written as a product of finitely many elements of $U \cup SI_n$. It follows from the fact $\beta \notin SI_n$ that either $\beta = \beta_1 \delta$ or $\beta = \alpha_1 \beta_1 \delta$ for some $\beta_1 \in U$, $\alpha_1 \in SI_n$ and $\delta \in SI_{n,r} \cup \{\varepsilon\}$ where ε is the identity permutation on X_n. Now let

$$\gamma = \begin{cases} \beta_1 & \text{if } \beta = \beta_1 \delta \\ \alpha_1 \beta_1 & \text{if } \beta = \alpha_1 \beta_1 \delta, \end{cases}$$

and so $\beta = \gamma \delta$. Since $|\text{cdom}(\beta)| = 1$ and $\text{dom}(\beta) \subseteq \text{dom}(\gamma) \neq X_n$, we have $\text{dom}(\beta) = \text{dom}(\gamma)$. Moreover, since $\text{h}(\gamma) \geq \text{h}(\beta) = r$ and $\beta_1 \in SPK_{n,r}$, it follows that $\gamma, \beta_1 \in D_r^P$. Thus, since $\ker(\gamma) \subseteq \ker(\beta)$ and $\text{dom}(\beta) = \text{dom}(\gamma)$, we have $\ker(\beta) = \ker(\gamma)$, so $\text{copart}(\beta) = \text{copart}(\gamma)$.

Now let $\gamma = \beta_1$, then clearly $\text{copart}(\beta) = \text{copart}(\beta_1)$. Otherwise, since $\alpha_1 \in SI_n$, $\text{dom}(\beta) = \text{dom}(\gamma) \subseteq \text{dom}(\alpha_1)$ and $|\text{cdom}(\beta)| = 1$, we have $\text{dom}(\beta) = \text{dom}(\alpha_1)$ and $\text{im}(\alpha_1) = \text{dom}(\beta_1)$, so

$$\text{copart}(\beta) = \text{copart}(\gamma) = \text{copart}(\beta_1),$$

as claimed. □

As an immediate consequence, we have the following corollary:

Corollary 3.4 *For each* $1 \leq r \leq n-2$ *,*

$$\text{rerank}(SI_{n,r} : SI_n) = p_r(n-1)$$

where $p_r(n-1)$ *is the number of partitions of* $n-1$ *with* r *terms.*

Theorem 3.5 *Let* $1 \leq r \leq n-1$ *and* $U \subseteq PK_{n,r}$ *. Then* $PI_{n,r} = \langle U \cup SI_n \rangle$ *if and only if, for each partition* $\{A_1, \ldots, A_r\}$ *of* X_n *, there exists* $\beta \in U \cap D_r^T$ *such that*

$$\ker(\beta) = \bigcup_{i=1}^{r} (A_i \times A_i).$$

Proof $(\Rightarrow)$ For $1 \leq r \leq n-1$, let $U \subseteq PK_{n,r}$ be a relative generating set of $PI_{n,r}$ modulo SI_n. Then, for an arbitrary partition $\{A_1, \ldots, A_r\}$ of X_n, consider an arbitrary element $\beta \in D_r^T \subseteq K_{n,r} \subseteq PI_{n,r}$ with $\ker(\beta) = \bigcup_{i=1}^{r}(A_i \times A_i)$. Since $PI_{n,r} = \langle U \cup SI_n \rangle$ and $\text{dom}(\beta) = X_n$, there exist $\beta_1 \in U \cap T_n$ and $\delta \in PI_{n,r} \cup \{\varepsilon\}$, where ε is the identity permutation on X_n, such that $\beta = \beta_1 \delta$. Then since $\beta_1 \in K_{n,r}$, $\ker(\beta_1) \subseteq \ker(\beta)$ and $\text{h}(\beta_1) \geq \text{h}(\beta) = r$, it follows that $\text{h}(\beta_1) = r$, so $\ker(\beta) = \ker(\beta_1)$. Therefore, $\ker(\beta_1) = \bigcup_{i=1}^{r}(A_i \times A_i)$ and $\beta_1 \in U \cap D_r^T$, as required.

$(\Leftarrow)$ Let $1 \leq r \leq n-1$. Recall that $PK_{n,r}$ is the disjoint union of $SPK_{n,r}$ and $K_{n,r}$. Then consider any $\alpha \in SPK_{n,r}$ with $\text{copart}(\alpha) = (n_1, \ldots, n_r : s)$. Since $s \geq 1$, it follows from Lemma 2.3 (i) that for any $\beta \in D_r^T$ with $\text{part}(\beta) = (n_1 + s, n_2, \ldots, n_r)$ there exist $\lambda, \mu \in SI_n$ such that $\alpha = \lambda \beta \mu$. Therefore, since $K_{n,r} = \langle D_r^T \rangle$, to show $PI_{n,r} = \langle U \cup SI_n \rangle$ it is enough to show that $D_r^T \subseteq \langle U \cup SI_n \rangle$.

For each partition $\mathcal{A} = \{A_1, \ldots, A_r\}$ of X_n into r subsets, we fix an arbitrary element $\beta_{\mathcal{A}} \in U \cap D_r^T$ such that $\ker(\beta_{\mathcal{A}}) = \bigcup_{i=1}^{r}(A_i \times A_i)$. Then we denote the set of all these fixed elements by

$$V = \{\beta_{\mathcal{A}} \in U : \mathcal{A} \text{ is a partition of } X_n \text{ into } r \text{ subsets}\}.$$

For any $\alpha \in D_r^T$, let $\ker(\alpha) = \bigcup_{i=1}^{r}(A_i \times A_i)$ where $\mathcal{A} = \{A_1, \ldots, A_r\}$ is a partition of X_n. Then there exists $\beta_{\mathcal{A}} \in V \subseteq U \cap D_r^T$ such that $\ker(\beta_{\mathcal{A}}) = \bigcup_{i=1}^{r}(A_i \times A_i)$. Let $A_i \beta_{\mathcal{A}} = b_i$ for $1 \leq i \leq r$. Now consider the map $\delta : \text{im}(\beta_{\mathcal{A}}) \to \text{im}(\alpha)$ defined by $b_i \delta = A_i \alpha$ for each $1 \leq i \leq r$. Then it is clear that $\delta \in SI_n$ and that $\alpha = \beta_{\mathcal{A}} \delta \in \langle U \cup SI_n \rangle$, as required. $\square$

As an immediate consequence, we have the following corollary:

Corollary 3.6 *For each* $1 \leq r \leq n-1$ *,*

$$\text{rerank}(PI_{n,r} : SI_n) = S(n, r)$$

where $S(n,r)$ *is the Stirling number of the second kind.*

Proof The proof follows from the fact that the number of partitions of X_n into r subsets is $S(n,r)$. $\square$

References

[1] Al-Kharousi F, Kehinde R, Umar A. Combinatorial results for certain semigroups of partial isometries of a finite chain. Australasian Journal of Combinatorics 2014; 58(3): 365-375.

[2] Ayık G, Ayık H, Howie JM. On factorisations and generators in transformation semigroup. Semigroup Forum 2005; 70: 225-237.

[3] Ayık G, Ayık H, Howie JM, Ünlü Y. Rank properties of the semigroup of singular transformations on a finite set. Communications in Algebra 2008; 36: 2581-2587.

[4] Ayık H, Bugay L. Generating sets of finite transformation semigroups $PK(n,r)$ and $K(n,r)$. Communications in Algebra 2015; 43: 412-422.

[5] Bugay L, Yağcı M, Ayık H. The ranks of certain semigroups of partial isometries. Semigroup Forum 2018; 97: 214-222.

[6] East J. Infinite partition monoids. International Journal of Algebra and Computation 2014; 24: 429-460.

[7] Evseen AE, Podran NE. Semigroups of transformations generated by idempotents of given defect. Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika 1972; 117: 44-50.

[8] Ganyushkin O, Mazorchuk V. Classical Finite Transformation Semigroups. Berlin, Germany: Springer-Verlag, 2009.

[9] Garba GU. Idempotents in partial transformation semigroups. Proceedings of the Royal Society of Edinburgh 1990; 116A: 359-366.

[10] Garba GU. On the nilpotent ranks of certain semigroups of transformations. Glasgow Mathematical Journal 1994; 36: 1-9.

[11] Gomes GMS, Howie JM. Nilpotents in finite symmetric inverse semigroups. Proceedings of the Royal Society of Edinburgh 1987; 30: 383-395.

[12] Hardy GH, Wright EM. An Introduction to the Theory of Numbers. New York, USA: Oxford University Press, 1979.

[13] Higgins PM. The product of the idempotents and an H-class of the finite full transformation semigroup. Semigroup Forum 2012; 84: 216-228.

[14] Howie JM. Fundamentals of Semigroup Theory. New York, NY, USA: Oxford University Press, 1995.

[15] Howie JM, McFadden RB. Idempotent rank in finite full transformation semigroups. Proceedings of the Royal Society of Edinburgh 1990; 114A: 161-167.

[16] Kearnes KA, Szendrei Á, Wood J. Generating singular transformations. Semigroup Forum 2001; 63: 441-448.

[17] Levi I, McFadden RB. S_n-Normal semigroups. Proceedings of the Royal Society of Edinburgh 1994; 37: 471-476.

[18] Lipscomb S. Symmetric Inverse Semigroups, Mathematical Surveys and Monographs. Providence, RI, USA: American Mathematical Society, 1996.

[19] Yiğit E, Ayık G, Ayık H. Minimal relative generating sets of some partial transformation semigroups. Communications in Algebra 2017; 45: 1239-1245.