

1-1-2019

## Approximation by generalized complex Szasz–Mirakyan operators in compact disks

DÖNE KARAHAN

SEVİLAY KIRCI SERENBAY

AYDIN İZGİ

Follow this and additional works at: <https://dctubitak.researchcommons.org/math>



Part of the [Mathematics Commons](#)

---




### Recommended Citation

KARAHAN, DÖNE; SERENBAY, SEVİLAY KIRCI; and İZGİ, AYDIN (2019) "Approximation by generalized complex Szasz–Mirakyan operators in compact disks," *Turkish Journal of Mathematics*: Vol. 43: No. 5, Article 11. <https://doi.org/10.3906/mat-1903-52>

Available at: <https://dctubitak.researchcommons.org/math/vol43/iss5/11>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals.

## Approximation by generalized complex Szász–Mirakyan operators in compact disks

Döne KARAHAN\* , Sevilay K. SERENBAY , Aydın IZGİ 

Department of Mathematics, Faculty of Science and Letters, Harran University, Şanlıurfa, Turkey

Received: 15.03.2019

Accepted/Published Online: 10.07.2019

Final Version: 28.09.2019

**Abstract:** In this study, the generalized complex Szász–Mirakyan operators are introduced and the approximation properties of these operators are studied. Voronovskaya’s theorem with a quantitative estimate for these operators attached to an analytic function is achieved on compact disks.

**Key words:** Complex Szász–Mirakyan operators, quantitative exact approximation, Voronovskaya’s theorem

### 1. Introduction

Concerning the convergence of complex Szász–Mirakyan operators in the complex plane, the first result was established by Gergen et al. [4]. In 1986, some approximation properties of complex Bernstein operators defined

$$B_n(f)(z) = \sum_{k=0}^n \binom{n}{k} z^k (1-z)^{n-k} f\left(\frac{k}{n}\right)$$

in compact disks were studied by Lorentz [7]. Then Gal [3] achieved quantitative estimates for the convergence and Voronovskaya’s theorem of complex Favard–Szász–Mirakyan operators attached to analytic function without exponential growth condition. We may also mention that similar results for the well-known complex approximating operators were achieved by Gal in his book [2]. After that, similar results for different operators, e.g., complex Favard–Szász–Mirakjan–Stancu, complex modified Szász–Mirakjan–Stancu, complex Durrmeyer–Stancu, Szász–Mirakyan, and Durrmeyer–Chlodowsky operators, were discussed in the literature (see [1, 5, 6, 8]). In [9], Serenbay and Dalmanoglu introduced the following generalized Szász–Mirakyan operators in exponential weighted space of functions of one variable and some theorems on the degree of approximation investigated using a method given by Rempulska and Walczak:

$$S_n(f; r; x) = \begin{cases} \sum_{j=0}^{\infty} p_j(a_n x) f\left(\frac{j}{r+b_n}\right) & , \quad x > 0 \\ f(0) & , \quad x = 0 \end{cases} \quad (1.1)$$

where

$$p_j(a_n x) = e^{-a_n x} \frac{(a_n x)^j}{j!}, \quad r > 0.$$

\*Correspondence: [dkarahan@harran.edu.tr](mailto:dkarahan@harran.edu.tr)

2010 AMS Mathematics Subject Classification: 30E10, 41A25

$(a_n)$  and  $(b_n)$  are increasing and unbounded sequences of positive numbers such that

$$(a_n) \leq (b_n) \quad \lim_{n \rightarrow \infty} \frac{1}{r + b_n} = 0, \quad \frac{a_n}{r + b_n} = 1 + O\left(\frac{1}{r + b_n}\right), \tag{1.2}$$

for all  $n \in \mathbb{N}$ .

In this paper, we study the approximation properties of the generalized complex Szász–Mirakyan operators to a complex domain. The generalized complex Szász–Mirakyan operators are achieved from the real version, simply by replacing the real variable  $x$  by complex variable  $z$  in the operators defined by (1.1), which is given below:

$$D_n^r(f; z) = \begin{cases} \sum_{j=0}^{\infty} p_j(a_n z) f\left(\frac{j}{r+b_n}\right) & , \quad z \in \mathbb{C} \quad (z \neq 0) \\ f(0) & , \quad z = 0 \end{cases} \tag{1.3}$$

where

$$p_j(a_n z) = e^{-a_n z} \frac{(a_n z)^j}{j!}, \quad r > 0, \tag{1.4}$$

and  $(a_n)$  and  $(b_n)$  are satisfying the conditions (1.2). Let  $\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}$  be with  $1 < R < \infty$  and suppose that  $f : [R, \infty) \cup \overline{\mathbb{D}}_R \rightarrow \mathbb{C}$  is continuous in  $[R, \infty) \cup \overline{\mathbb{D}}_R$ , analytic in  $\mathbb{D}_R$ , i.e.  $f(z) = \sum_{k=0}^{\infty} c_k z^k$ , for all  $z \in \mathbb{D}_R$ .

### 2. Auxiliary results

**Lemma 2.1** ([9], Lemma 2.1) For all  $n \in \mathbb{N}$ ,  $z \in \mathbb{C}$ , and  $r > 0$ , we have

$$\begin{aligned} D_n^r(1; z) &= 1, \\ D_n^r(\omega; z) &= \frac{a_n z}{r + b_n}, \\ D_n^r(\omega^2; z) &= \frac{a_n z + (a_n z)^2}{(r + b_n)^2}, \\ D_n^r((\omega - z); z) &= z \left( \frac{a_n}{r + b_n} - 1 \right), \\ D_n^r((\omega - z)^2; z) &= \frac{z^2 (a_n - (r + b_n))^2 + a_n z}{(r + b_n)^2}. \end{aligned} \tag{2.1}$$

**Lemma 2.2** For all  $n \in \mathbb{N}, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , and  $z \in \mathbb{C}$ , we have

$$D_n^r(\eta_{k+1}; z) = \frac{z}{r + b_n} (D_n^r(\eta_k; z))' + \frac{a_n z}{r + b_n} D_n^r(\eta_k; z), \tag{2.2}$$

where  $\eta_k = z^k$ .

**Proof** From formula (1.3), we can write

$$D_n^r(\eta_k; z) = e^{-a_n z} \sum_{j=0}^{\infty} \frac{(a_n z)^j}{j!} \left( \frac{j}{r + b_n} \right)^k.$$

Also, if we differentiate  $D_n^r(\eta_k; z)$  with respect to  $z \neq 0$ , then we have

$$\begin{aligned} (D_n^r(\eta_k; z))' &= -a_n e^{-a_n z} \sum_{j=0}^{\infty} \frac{(a_n z)^j}{j!} \left(\frac{j}{r+b_n}\right)^k \\ &\quad + e^{-a_n z} \sum_{j=0}^{\infty} j a_n \frac{(a_n z)^{j-1}}{j!} \left(\frac{j}{r+b_n}\right)^k \\ &= -a_n D_n^r(\eta_k; z) + \frac{r+b_n}{z} e^{-a_n z} \sum_{j=0}^{\infty} j a_n \frac{(a_n z)^j}{j!} \left(\frac{j}{r+b_n}\right)^{k+1} \\ &= -a_n D_n^r(\eta_k; z) + \frac{r+b_n}{z} D_n^r(\eta_{k+1}; z), \end{aligned}$$

which implies

$$D_n^r(\eta_{k+1}; z) = \frac{z}{r+b_n} (D_n^r(\eta_k; z))' + \frac{a_n z}{r+b_n} D_n^r(\eta_k; z),$$

$\forall n \in \mathbb{N}, k \in \mathbb{N}_0$ , and  $z \in \mathbb{C}$ . □

### 3. Approximation results

**Theorem 3.1** *Let the hypothesis on  $\mathbb{D}_R$  and  $f$  function hold and let there exist  $M, C, B, L > 0$  and  $A \in (\frac{1}{R}, 1)$ , with the property  $|c|_k \leq M \frac{A^k}{k!}$ ,  $\forall k = 0, 1, \dots$  (which implies  $|f(z)| \leq M e^{A|z|}$ , for all  $z \in \mathbb{D}_R$  and  $|f(x)| \leq L e^{B|x|}$ , for all  $x \in [R, +\infty)$ ).*

(i) *Let  $1 \leq \mu \leq \frac{1}{A}$  be permissive fixed.  $\forall |z| \leq \mu$  and  $n \in \mathbb{N}$ , and we get*

$$|D_n^r(f; z) - f(z)| \leq \frac{N_{A,C}}{r+b_n}, \tag{3.1}$$

where  $N_{A,C} = M \sum_{k=1}^{\infty} (k+1)(k+C)(A\mu)^k < \infty$ .

(ii) *If  $1 \leq \mu \leq \mu_1 \leq \frac{1}{A}$  are permissive fixed, then  $\forall |z| \leq \mu$  and  $n, p \in \mathbb{N}$ , and*

$$|(D_n^r(\eta_{k-2}; z))^{(p)} - f^{(p)}(z)| \leq \frac{p! \mu_1 N_{A,C}}{(r+b_n)(\mu_1 - \mu)^{p+1}}, \tag{3.2}$$

where  $N_{A,C}$  is given as in (i).

### Proof

(i) From (2.2), it is a recurrence formula:

$$\begin{aligned} D_n^r(\eta_k; z) - \eta_k &= \frac{z}{r+b_n} (D_n^r(\eta_{k-1}; z) - \eta_{k-1})' + \frac{a_n z}{r+b_n} (D_n^r(\eta_{k-1}; z) - \eta_{k-1}) \\ &\quad + \frac{k-1}{r+b_n} \eta_{k-1} + \left(\frac{a_n}{r+b_n} - 1\right) \eta_k, \end{aligned} \tag{3.3}$$

$\forall z \in \mathbb{C}, k, n \in \mathbb{N}$ .

Now let  $1 \leq \mu \leq R$ . Denoting with  $\|\cdot\|_\mu$  the norm in  $C(\overline{\mathbb{D}_\mu})$ , where  $\mathbb{D}_\mu = \{z \in \mathbb{C} : |z| \leq \mu, \}$ , by a linear transformation, Bernstein's inequality in the closed unit disk becomes  $|P'_k(z)| \leq \frac{k}{n} \|P_k\|_\mu$ , for all  $|z| \leq \mu$ , where  $P_k(z)$  is a polynomial of degree  $\leq k$ . Therefore, from equation (3.3) we have

$$\begin{aligned} \|D_n^r(\eta_k; z) - \eta_k\|_\mu &\leq \left(\mu + \frac{k-1}{r+b_n}\right) \|D_n^r(\eta_{k-1}; z) - \eta_{k-1}\|_\mu \\ &\quad + \frac{k-1}{r+b_n} \mu^{k-1} + \left|\frac{a_n}{r+b_n} - 1\right| \mu^k. \end{aligned} \tag{3.4}$$

From (1.2), we have

$$\left|\frac{a_n}{r+b_n} - 1\right| \leq \frac{C}{r+b_n}, \tag{3.5}$$

where  $C > 0$  is a constant number.

From (3.5) into (3.4), we get

$$\|D_n^r(\eta_k; z) - \eta_k\|_\mu \leq \left(\mu + \frac{k-1}{r+b_n}\right) \|D_n^r(\eta_{k-1}; z) - \eta_{k-1}\|_\mu + \frac{k+C}{r+b_n} \mu^k.$$

By writing the last inequality, for  $k = 2, 3, \dots$ , we can achieve the following inequalities step by step:

$$\begin{aligned} \|D_n^r(\eta_k; z) - \eta_k\|_\mu &\leq \left(\mu + \frac{k-1}{r+b_n}\right) \left(\left(\mu + \frac{k-2}{r+b_n}\right) \|D_n^r(\eta_{k-2}; z) - \eta_{k-2}\|_\mu \right. \\ &\quad \left. + \frac{k-1+C}{r+b_n} \mu^{k-1}\right) + \frac{k+C}{r+b_n} \mu^k \\ &\leq \dots \leq \frac{(k+1)!}{r+b_n} (k+C) \mu^k. \end{aligned}$$

In conclusion, for any  $n \in \mathbb{N}$ ,  $|z| \leq \mu$ , we have

$$\|D_n^r(\eta_k; z) - \eta_k\|_\mu \leq \frac{(k+1)!}{r+b_n} (k+C) \mu^k. \tag{3.6}$$

Now, from the hypothesis on  $f$ , it follows that  $D_n^r(f; z)$  is analytic in  $\mathbb{D}_\mu$ . For this reason, we can write

$$D_n^r(f; z) = \sum_{k=0}^{\infty} c_k D_n^r(\eta_k; z), \quad \text{for all } z \in \mathbb{D}_\mu,$$

which from the hypothesis on  $c_k$  readily implies  $\forall |z| \leq \mu$ ,

$$\begin{aligned} |D_n^r(f; z) - f(z)| &\leq \sum_{k=2}^{\infty} c_k |D_n^r(\eta_k; z) - \eta_k| \leq \sum_{k=2}^{\infty} M \frac{A^k}{k!} \frac{(k+1)!}{r+b_n} (k+C) \mu^k \\ &= \frac{M}{r+b_n} \sum_{k=2}^{\infty} (k+1)(k+C) (A\mu)^k = \frac{N_{A,C}}{r+b_n}. \end{aligned}$$

Since  $f(z)$  is analytic in  $\mathbb{D}_\mu$ , we get  $f''(z) = \sum_{k=2}^\infty c_k k(k-1)\mu^{k-2}$  and the series is absolutely convergent in  $|z| \leq \mu$ . Thus, we get  $\sum_{k=2}^\infty |c_k| k(k-1)\mu^{k-2} < \infty$ , which implies  $N_{A,C} = M \sum_{k=2}^\infty (k+1)(k+C)(A\mu)^k < \infty$ .

(ii) Denoting by  $\gamma$  the circle of radius  $\mu_1 > \mu$  and center 0, since for any  $|z| \leq \mu$  and  $\nu \in \gamma$ , we have  $|\nu - z| \geq \mu_1 - \mu$ , by Cauchy's formulas it follows that  $\forall |z| \leq \mu$  and  $n \in \mathbb{N}$ , and we get

$$\begin{aligned} |(D_n^r(\eta_{k-2}; z))^{(p)} - f^{(p)}(z)| &= \frac{p!}{2\pi} \left| \int_\gamma \frac{D_n^r(f; \nu) - f(\nu)}{(\nu - z)^{p+1}} d\nu \right| \leq \frac{N_{A,C}}{r + b_n} \frac{p!}{2\pi} \frac{2\pi\mu_1}{(\mu_1 - \mu)^{p+1}} \\ &= \frac{N_{A,C}}{r + b_n} \frac{p!\mu_1}{(\mu_1 - \mu)^{p+1}}. \end{aligned}$$

□

#### 4. Voronovskaya's theorem

Now we achieve the Voronovskaya type formula with a quantitative estimate for the complex Szász–Mirakyan operator.

**Theorem 4.1** *Suppose that the hypothesis on the function  $f$  and the constants  $R, M, C, B, L$  in the statement of Theorem 3.1 hold and let  $1 \leq \mu < \frac{1}{A}$  be permissive fixed.  $\forall |z| \leq \mu$  and  $n \in \mathbb{N}$ , and we have*

$$\begin{aligned} &\left| D_n^r(f; z) - f(z) - \frac{Cz}{r + b_n} f'(z) - \frac{z}{2(r + b_n)} f''(z) \right| \\ &\leq \frac{3a_n M}{(r + b_n)^2} \sum_{k=2}^\infty (k + 2)(k + 1)(k - 1 + C)(A\mu)^k, \end{aligned} \tag{4.1}$$

where  $\sum_{k=2}^\infty (k + 2)(k + 1)(k - 1 + C)(A\mu)^k < \infty$ .

**Proof** Denoting  $\eta_k = z^k, k = 0, 1, \dots$ , by Theorem 3.1, (i), we can write  $D_n^r(f; z) = \sum_{k=0}^\infty c_k D_n^r(\eta_k; z)$ , which readily implies

$$\begin{aligned} &\left| D_n^r(f; z) - f(z) - \frac{Cz}{r + b_n} f'(z) - \frac{z}{2(r + b_n)} f''(z) \right| \\ &\leq \sum_{k=2}^\infty |c_k| \left| D_n^r(\eta_k; z) - \eta_k - \frac{kC}{r + b_n} z^k - \frac{k(k-1)}{2(r + b_n)} z^{k-1} \right|, \end{aligned}$$

$\forall z \in \mathbb{D}_R$  and  $n \in \mathbb{N}$ .

By recurrence relationship (3.3) satisfied by  $D_n^r(\eta_k; z)$ , denoting

$D_n^{r*}(\eta_k; z) = D_n^r(\eta_k; z) - \eta_k - \frac{kC}{r+b_n}z^k - \frac{k(k-1)}{2(r+b_n)}z^{k-1}$ , we instantly achieve the new recurrence

$$\begin{aligned} D_n^{r*}(\eta_k; z) &= \frac{z}{r+b_n} (D_n^{r*}(\eta_{k-1}; z))' + \frac{a_n z}{r+b_n} D_n^{r*}(\eta_{k-1}; z) \\ &+ \left( \frac{k-1}{r+b_n} + \frac{a_n}{r+b_n} + \frac{(k-1)a_n C}{(r+b_n)^2} - \frac{kC}{r+b_n} - 1 \right) z^k \\ &+ \left( \frac{(k-1)^2 C}{(r+b_n)^2} + \frac{(k-1)(k-2)a_n}{2(r+b_n)^2} - \frac{k(k-1)}{2(r+b_n)} \right) z^{k-1} \\ &+ \frac{(k-1)(k-2)^2}{2(r+b_n)^2} z^{k-2}, \end{aligned}$$

$\forall k \geq 2, n \in \mathbb{N}$ , and  $z \in \mathbb{D}_R$ . For  $k \geq 2, n \in \mathbb{N}$ , and  $|z| \leq \mu$ , we have

$$\begin{aligned} |D_n^{r*}(\eta_k; z)| &= \frac{|z|}{r+b_n} |(D_n^{r*}(\eta_{k-1}; z))'| + \frac{a_n |z|}{r+b_n} |D_n^{r*}(\eta_{k-1}; z)| \\ &+ \frac{\mu^k}{r+b_n} |k(1-C+Ca_n) + a_n(1-C) - 2| \\ &+ \frac{\mu^{k-1}}{2(r+b_n)} |(k-1)(2(k-1)C + a_n(k-2) - k)| \\ &+ \frac{(k-1)(k-2)^2}{2(r+b_n)^2} \mu^{k-2} \\ &\leq \frac{a_n |z|}{r+b_n} |D_n^{r*}(\eta_{k-1}; z)| + \frac{|z|}{2(r+b_n)} \left[ \frac{2(k-1)}{\mu} \|D_n^{r*}(\eta_{k-1}; z)\|_\mu \right. \\ &+ 2\mu^{k-1} |k(1-C+Ca_n)a_n(1-C) - 2| \\ &+ \mu^{k-2} |(k-1)(2(k-1)C + a_n(k-2) - k)| \\ &+ \left. \mu^{k-3} (k-1)(k-2)^2 \right] \\ &\leq \frac{a_n |z|}{r+b_n} |D_n^{r*}(\eta_{k-1}; z)| + \frac{|z|}{2(r+b_n)} \left[ \frac{2(k-1)}{\mu} \|D_n^r(\eta_{k-1}; z) - \eta_{k-1}\|_\mu \right. \\ &+ \frac{2(k-1)}{\mu} \frac{kC}{r+b_n} \mu^k + \frac{2(k-1)}{\mu} \frac{k(k-1)}{2(r+b_n)} \mu^{k-1} \\ &+ 2\mu^{k-1} |k(1-C+Ca_n)a_n(1-C) - 2| \\ &+ \mu^{k-2} |(k-1)(2(k-1)C + a_n(k-2) - k)| \\ &+ \left. \mu^{k-3} (k-1)(k-2)^2 \right]. \end{aligned}$$

From (3.6), we get

$$\begin{aligned}
 |D_n^{r*}(\eta_k; z)| &\leq \frac{a_n|z|}{r+b_n}|D_n^{r*}(\eta_{k-1}; z)| + \frac{|z|}{2(r+b_n)} \left[ \frac{2(k-1)}{\mu} \frac{k!}{r+b_n} (k-1+C)\mu^{k-1} \right. \\
 &\quad + \frac{2(k-1)}{\mu} \frac{kC}{r+b_n} \mu^k + \frac{2(k-1)}{\mu} \frac{k(k-1)}{2(r+b_n)} \mu^{k-1} \\
 &\quad + 2\mu^{k-1}|k(1-C+Ca_n)a_n(1-C)-2| \\
 &\quad + \mu^{k-2}|(k-1)(2(k-1)C+a_n(k-2)-k)| \\
 &\quad \left. + \mu^{k-3}(k-1)(k-2)^2 \right] \\
 &\leq \frac{a_n|z|}{r+b_n}|D_n^{r*}(\eta_{k-1}; z)| + \frac{6|z|(k+1)!(k-1+C)}{2(r+b_n)} \mu^{k-1};
 \end{aligned}$$

that is,

$$|D_n^{r*}(\eta_k; z)| \leq \frac{a_n|z|}{r+b_n}|D_n^{r*}(\eta_{k-1}; z)| + \frac{3|z|(k+1)!(k-1+C)}{r+b_n} \mu^{k-1}, \tag{4.2}$$

for all  $|z| \leq \mu$ .

In inequality (4.2), for  $k = 2, 3, \dots$ , we readily achieve the following inequalities step by step:

$$\begin{aligned}
 |D_n^{r*}(\eta_k; z)| &\leq \left( \frac{a_n|z|}{r+b_n} \right)^2 |D_n^{r*}(\eta_{k-2}; z)| + \frac{a_n|z|}{r+b_n} \frac{3|z|(k)!(k-2+C)}{r+b_n} \mu^{k-2} \\
 &\quad + \frac{3|z|(k+1)!(k-1+C)}{r+b_n} \mu^{k-1} \\
 &\leq \dots \leq \frac{3a_n|z|(k+2)!(k-1+C)}{(r+b_n)^2} \mu^{k-1}.
 \end{aligned}$$

Thus, we have achieved that  $\forall n \in \mathbb{N}$  and  $k = 2, 3, \dots$ ,

$$|D_n^{r*}(\eta_k; z)| \leq \frac{3a_n|z|(k+2)!(k-1+C)}{(r+b_n)^2} \mu^{k-1}. \tag{4.3}$$

This implies

$$\begin{aligned}
 &\left| D_n^r(f; z) - f(z) - \frac{Cz}{r+b_n} f'(z) - \frac{z}{2(r+b_n)} f''(z) \right| \\
 &\leq \sum_{k=2}^{\infty} |c_k| |D_n^{r*}(\eta_k; z)| \leq \frac{3a_n M}{(r+b_n)^2} \sum_{k=2}^{\infty} (k+2)(k+1)(k-1+C)(A\mu)^k
 \end{aligned}$$

where  $\sum_{k=2}^{\infty} (k+2)(k+1)(k-1+C)(A\mu)^k < \infty$ . □

### 5. Exact estimates

**Theorem 5.1** *Let the hypothesis on  $\mathbb{D}_R$  and  $f$  function hold. If  $1 \leq \mu < R$  is permissive fixed and if it is not a polynomial of degree  $\leq 1$ , then we get*

$$\|D_n^r(f; z) - f(z)\|_{\mu} \geq \frac{N_{\mu}(f)}{r+b_n}, \tag{5.1}$$

where  $\|f\|_{\mu} = \max_{|z| \leq \mu} |f(z)|$  and the constant  $N_{\mu}(f) > 0$  depends only on  $f$  and  $\mu$ .



**Proof** Define

$$P_n^r(f; z) = D_n^r(f; z) - f(z) - \frac{Cz}{r + b_n} f'(z) - \frac{z}{2(r + b_n)} f''(z).$$

$\forall z \in \mathbb{D}_R$  and  $n \in \mathbb{N}$ , and we have

$$D_n^r(f; z) - f(z) = \frac{1}{r + b_n} \left[ Cz f'(z) + \frac{z}{2} f''(z) + \frac{1}{r + b_n} ((r + b_n)^2 P_n^r(f; z)) \right].$$

Due to the property  $\|F + G\|_\mu \geq \|F\|_\mu - \|G\|_\mu \geq \|F\|_\mu - \|G\|$ , it follows that

$$\|D_n^r(f; z) - f(z)\|_\mu \geq \frac{1}{r + b_n} \left[ \|Cz f'(z) + \frac{z}{2} f''(z)\|_\mu - \frac{1}{r + b_n} ((r + b_n)^2 \|P_n^r(f; z)\|_\mu) \right].$$

Considering the hypothesis that  $f$  is not a polynomial of degree 0 in  $\mathbb{D}_R$ , we have

$$\|Cz f'(z) + \frac{z}{2} f''(z)\|_\mu > 0.$$

Indeed, if we suppose the contrary, then we have

$$Cz f'(z) + \frac{z}{2} f''(z) = 0, \quad \forall z \in \overline{\mathbb{D}}_R.$$

Defining  $y(z) = f'(z)$  and looking for the analytic function  $y(z)$  under the form  $\sum_{k=0}^\infty a_k z^k$ , after replacement in the differential equation, the coefficient identification method readily leads to  $a_k = 0$  for all  $k \in \mathbb{N} \cup \{0\}$ . This implies that  $y(z) = 0$  for all  $z \in \overline{\mathbb{D}}_R$  and for this reason  $f$  is constant on  $\overline{\mathbb{D}}_R$ , a contradiction with the hypothesis. Using inequality (4.1), we have

$$(r + b_n)^2 \|P_n^r(f; z)\|_\mu \leq 3a_n M \sum_{k=2}^\infty (k + 2)(k + 1)(k - 1 + C)(A\mu)^k. \tag{5.2}$$

Thus, there exists an index  $n_1$ , depending only on  $f$  and  $\mu$  such that  $\forall n \geq n_1$ , and we have

$$\|Cz f'(z) + \frac{z}{2} f''(z)\|_\mu - \frac{1}{r + b_n} ((r + b_n)^2 \|P_n^r(f; z)\|_\mu) \geq \frac{1}{2} \|Cz f'(z) + \frac{z}{2} f''(z)\|_\mu.$$

For  $n \in \{1, \dots, n_1 - 1\}$  we strictly get

$$\|D_n^r(f; z) - f(z)\|_\mu \geq \frac{M_{\mu, n}(f)}{r + b_n},$$

with  $M_{\mu, n}(f) = (r + b_n) \|D_n^r(f; z) - f(z)\|_\mu > 0$ , which finally implies

$$\|D_n^r(f; z) - f(z)\|_\mu \geq \frac{N_\mu(f)}{r + b_n}$$

for all  $n$ , where

$$N_\mu(f) = \min\{M_{\mu, 1}(f), \dots, M_{\mu, n_1 - 1}(f), \frac{1}{2} \|Cz f'(z) + \frac{z}{2} f''(z)\|_\mu\}.$$

□

Combining now Theorem 3.1, (i), and Theorem 4.2, we instantly have the following.

**Corollary 5.2** *Let the hypothesis on  $\mathbb{D}_R$  and  $f$  function hold. If  $1 \leq \mu < R$  is permissive fixed and if is not a polynomial of degree  $\leq 1$ , then we get*

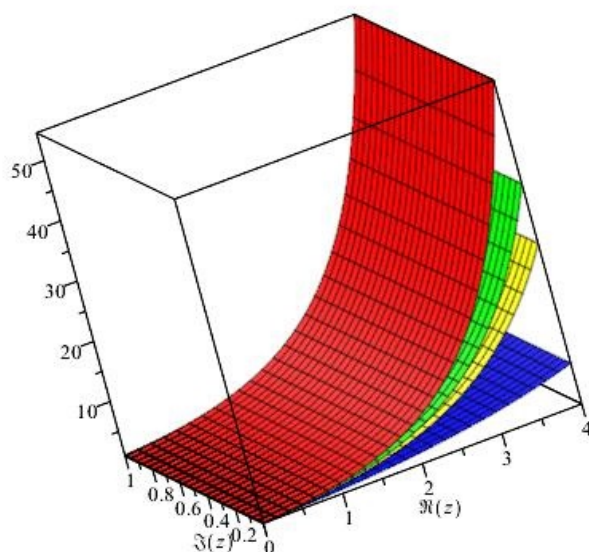
$$\|D_n^r(f; z) - f(z)\|_\mu \sim \frac{1}{r + b_n}, \quad n \in \mathbb{N},$$

where the constants in the equivalence depend on  $f$  and  $\mu$ .

**6. Example**

**Example 6.1** *Choosing  $f(z) = e^z$  we compute the error estimations of the complex Szász–Mirakyan operators  $D_n^r(f; z)$  given in (1.3) and  $f(z)$ . Here we take  $r = 8$ ,  $a_n = n + 1$  and  $b_n = n + 4$ ,  $n = 10, 50, 150$ .*

*In the Figure and the Table, we show a comparison for the error estimates of the function  $f(x)$  and the operator  $D_n^r(f; z)$  by using the software MAPLE 17.*



**Figure.** Curves for the error estimates of  $D_n^r(f; z)$  for  $n = 10$  (blue),  $n = 50$  (yellow),  $n = 150$  (green), and  $f(z)$  (red).

**Table.** Error estimates of  $D_n^r(f; z)$  and  $f(z)$ .

z	n=50	n=250	n=500
1+i	1.470979333	0.1512937837	0.07810854062
2+i	2.387053893	0.6374605565	0.3322984258
1+2i	0.9494674863	0.2391680305	0.1234939398
2+2i	3.008477543	0.8061682558	0.4203058169

### References

- [1] Cetin N, Ispir N. Approximation by complex modified Szász-Mirakjan operators. *Studia Scientiarum Mathematicarum Hungarica* 2013; 50 (3): 355-372.
- [2] Gal GS. *Approximation by Complex Bernstein and Convolution Type Operators*. Singapore: World Scientific, 2009.
- [3] Gal SG. Approximation of analytic functions without exponential growth conditions by complex Favard-Szász-Mirakjan operators. *Rendiconti del Circolo Metematico di Palermo* 2010; 59: 367-376.
- [4] Gergen JJ, Dressel FG, Purcell WH. Convergence of extended Bernstein polynomials in the complex domain. *Pacific Journal of Mathematics* 1963; 13 (4): 1171-1180.
- [5] Gupta V, Verma KD. Approximation by complex Favard-Szász-Mirakjan-Stancu operators in compact disks. *Mathematical Sciences* 2012; 25: 6.
- [6] Izgi A. Approximation by composition of Szász-Mirakyan and Durrmeyer-Chlodowsky operators. *Eurasian Mathematical Journal* 2012; 3 (1): 63-71.
- [7] Lorentz GG. *Bernstein Polynomials*. 2nd ed. New York, NY, USA: Chelsea, 1986.
- [8] Ren MY, Zeng XM, Zeng L. Approximation by complex Durrmeyer-Stancu type operators in compact disks. *Journal of Inequalities and Applications* 2013; 2013: 442.
- [9] Serenbay SK, Dalmanoglu O. Rate of convergence for generalized Szász-Mirakyan operators in exponential weighted space. *Applications and Applied Mathematics* 2017; 12 (2): 884-897.