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## Fibonacci and Lucas numbers as products of two repdigits

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**Abstract:** In this study, it is shown that the largest Fibonacci number that is the product of two repdigits is  $F_{10} = 55 = 5 \cdot 11 = 55 \cdot 1$  and the largest Lucas number that is the product of two repdigits is  $L_6 = 18 = 2 \cdot 9 = 3 \cdot 6$ .

**Key words:** Fibonacci number, Lucas number, repdigit, Diophantine equations, linear forms in logarithms

### 1. Introduction

Let  $(F_n)$  and  $(L_n)$  be the sequences of Fibonacci and Lucas numbers given by  $F_0 = 0$ ,  $F_1 = 1$ ,  $L_0 = 2$ ,  $L_1 = 1$ ,  $F_n = F_{n-1} + F_{n-2}$ , and  $L_n = L_{n-1} + L_{n-2}$  for  $n \geq 2$ , respectively. Binet formulas for Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n,$$

where  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ , which are the roots of the characteristic equation  $x^2 - x - 1 = 0$ . It can be seen that  $1 < \alpha < 2$ ,  $-1 < \beta < 0$ , and  $\alpha\beta = -1$ . For more about Fibonacci and Lucas sequences with their history, one can see [7]. The relation between the  $n$ th Fibonacci number  $F_n$  and  $\alpha$  is given by

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{1}$$

for  $n \geq 1$ . Also, the relation between the  $n$ th Lucas number  $L_n$  and  $\alpha$  is given by

$$\alpha^{n-1} \leq L_n \leq 2\alpha^n. \tag{2}$$

Inequalities (1) and (2) can be proved by induction. A repdigit is a positive integer whose digits are all equal. Investigation of the repdigits in the second-order linear recurrence sequences has been of interest to mathematicians. In [10], Luca found all Fibonacci and Lucas numbers that are repdigits. The largest repdigits in Fibonacci and Lucas sequences are  $F_5 = 55$  and  $L_5 = 11$ . After that, in [2], the authors showed that the largest Fibonacci number that is a sum of two repdigits is  $F_{20} = 6765 = 6666 + 99$ . In [9], the authors found all repdigits in the Pell and Pell–Lucas sequences. The largest repdigits in Pell and Pell–Lucas sequences are  $P_3 = 5$  and  $Q_2 = 6$ . Later, the authors [1] found all Pell and Pell–Lucas numbers that are sums of two repdigits. It was shown that the largest Pell number that is a sum of two repdigits is  $P_6 = 70 = 4 + 66$  and the largest

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Pell–Lucas number that is a sum of two repdigits is  $Q_6 = 198 = 99 + 99$ . In this paper, we investigate Fibonacci and Lucas numbers that are the product of two repdigits. In other word, we solve the Diophantine equations

$$F_k = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9} \tag{3}$$

and

$$L_k = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9}. \tag{4}$$

It is shown that the largest Fibonacci number that is a product of two repdigits is  $F_{10} = 55 = 5 \cdot 11 = 55 \cdot 1$  and the largest Lucas number that is a product of two repdigits is  $L_6 = 18 = 2 \cdot 9 = 3 \cdot 6$ .

In Section 2, we introduce necessary lemmas and theorems. Then we prove our main theorems in Section 3.

### 2. Auxiliary results

In [5], in order to solve Diophantine equations of a similar form, the authors used Baker’s theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are of crucial importance in effectively solving Diophantine equations (3) and (4), we start by recalling some basic notions from algebraic number theory.

Let  $\eta$  be an algebraic number of degree  $d$  with minimal polynomial

$$a_0x^d + a_1x^{d-1} + \dots + a_d = a_0 \prod_{i=1}^d (X - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the  $a_i$ s are relatively prime integers with  $a_0 > 0$  and the  $\eta^{(i)}$ s are conjugates of  $\eta$ . Then

$$h(\eta) = \frac{1}{d} \left( \log a_0 + \sum_{i=1}^d \log \left( \max \{ |\eta^{(i)}|, 1 \} \right) \right) \tag{5}$$

is called the logarithmic height of  $\eta$ . In particular, if  $\eta = a/b$  is a rational number with  $\gcd(a, b) = 1$  and  $b > 1$ , then  $h(\eta) = \log(\max \{ |a|, b \})$ .

Some known properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{6}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{7}$$

$$h(\eta^m) = |m|h(\eta). \tag{8}$$

The following lemma, Lemma 1, is deduced from Corollary 2.3 of Matveev [11] and provides a large upper bound for the subscript  $n$  in equations (3) and (4)(also see Theorem 9.4 in [4]).

**Lemma 1** *Assume that  $\gamma_1, \gamma_2, \dots, \gamma_t$  are positive real algebraic numbers in a real algebraic number field  $\mathbb{K}$  of degree  $D$ ,  $b_1, b_2, \dots, b_t$  are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \dots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 A_2 \dots A_t),$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and  $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$  for all  $i = 1, \dots, t$ .

Now we give a lemma that was proved by Dujella and Pethő [8] and is a variation of a lemma of Baker and Davenport [3]. This lemma will be used to reduce the upper bound for the subscript  $n$  in equations (3) and (4). Let the function  $\|\cdot\|$  denote the distance from  $x$  to the nearest integer. That is,  $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$  for any real number  $x$ . Then we have:

**Lemma 2** *Let  $M$  be a positive integer, let  $p/q$  be a convergent of the continued fraction of the irrational number  $\gamma$  such that  $q > 6M$ , and let  $A, B, \mu$  be some real numbers with  $A > 0$  and  $B > 1$ . Let  $\epsilon := \|\mu q\| - M\|\gamma q\|$ . If  $\epsilon > 0$ , then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers  $u, v$ , and  $w$  with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

The following lemma was given in [6].

**Lemma 3** *Let  $a, x \in \mathbb{R}$  and  $0 < a < 1$ . If  $|x| < a$ , then*

$$|\log(1 + x)| < \frac{-\log(1 - a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1 - e^{-a}} \cdot |e^x - 1|.$$

### 3. Main Theorems

**Theorem 4** *Let  $m, n, d_1, d_2$  be positive integers with  $m \leq n$  and  $d_1, d_2 \leq 9$ . Let*

$$M = \frac{d_1(10^m - 1)}{9} \text{ and } N = \frac{d_2(10^n - 1)}{9}.$$

If  $F_k = M \cdot N$ , then

$$(k, F_k, M, N) = (1, 1, 1, 1), (2, 1, 1, 1), (3, 2, 1, 2), (4, 3, 1, 3), (5, 5, 1, 5), (6, 8, 2, 4)$$

and

$$(k, F_k, M, N) = (6, 8, 1, 8), (8, 21, 3, 7), (10, 55, 5, 11), (10, 55, 1, 55).$$

**Proof** Assume that  $F_k = M \cdot N$ . Now assume that  $1 \leq m \leq n \leq 22$ . Then by using the Mathematica program, we see that  $k \leq 212$ . In this case, we obtain only the solutions  $F_k \in \{1, 2, 3, 5, 8, 21, 55\}$ . From now on, assume that  $n \geq 23$ . Then the inequality

$$\alpha^{4n-4} < 10^{n-1} \leq \frac{d_2(10^n - 1)}{9} \leq \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9} = F_k \leq \alpha^{k-1}$$

implies that  $k > 4n - 3$ . That is,  $k > 89$  for  $n \geq 23$ . Combining the left side of (1) with (3), we obtain

$$\alpha^{k-2} \leq F_k = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9} \leq (10^n - 1)^2 < 10^{2n} \leq \alpha^{10n}.$$

From this, we get  $k < 10n + 2$ . On the other hand, we rewrite equation (3) as

$$\frac{\alpha^k - \beta^k}{\sqrt{5}} = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9}$$

to obtain

$$\frac{d_1 d_2 10^{m+n}}{81} - \frac{\alpha^k}{\sqrt{5}} = -\frac{\beta^k}{\sqrt{5}} + \frac{d_1 d_2 10^m}{81} + \frac{d_1 d_2 10^n}{81} - \frac{d_1 d_2}{81}. \tag{9}$$

Taking absolute values of both sides of (9), we get

$$\left| \frac{d_1 d_2 10^{m+n}}{81} - \frac{\alpha^k}{\sqrt{5}} \right| \leq \frac{|\beta|^k}{\sqrt{5}} + \frac{d_1 d_2 10^m}{81} + \frac{d_1 d_2 10^n}{81} + \frac{d_1 d_2}{81}. \tag{10}$$

Dividing both sides of (10) by  $\frac{d_1 d_2 10^{m+n}}{81}$  gives us the following inequality:

$$\begin{aligned} \left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-(m+n)}}{d_1 d_2 \sqrt{5}} \right| &\leq \frac{81 |\beta|^k}{\sqrt{5} d_1 d_2 10^{m+n}} + \frac{1}{10^n} + \frac{1}{10^m} + \frac{1}{10^{m+n}} \\ &< 4 \cdot 10^{-m}. \end{aligned}$$

From this, it follows that

$$\left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-(m+n)}}{d_1 d_2 \sqrt{5}} \right| < 4 \cdot 10^{-m}. \tag{11}$$

Now, let us apply Lemma 1 with  $\gamma_1 := \alpha$ ,  $\gamma_2 := 10$ ,  $\gamma_3 := 81/d_1 d_2 \sqrt{5}$  and  $b_1 := k, b_2 := -(m+n), b_3 := 1$ . Note that the numbers  $\gamma_1, \gamma_2$ , and  $\gamma_3$  are positive real numbers and elements of the field  $\mathbb{K} = \mathbb{Q}(\sqrt{5})$ . The degree of the field  $\mathbb{K}$  is 2, so  $D = 2$ . Now we show that

$$\Lambda_1 := 1 - \frac{81 \cdot \alpha^k \cdot 10^{-(m+n)}}{d_1 d_2 \sqrt{5}}$$

is nonzero. On the contrary, assume that  $\Lambda_1 = 0$ . Then  $\alpha^k = 10^{(m+n)} d_1 d_2 \sqrt{5} / 81$ . Conjugating in  $\mathbb{Q}(\sqrt{5})$  gives us  $\beta^k = -10^{(m+n)} d_1 d_2 \sqrt{5} / 81$ . This implies that  $L_k = \alpha^k + \beta^k = 0$ , which is a contradiction. Since

$$h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) = h(81/d_1d_2\sqrt{5}) \leq h(81) + h(\sqrt{5}) + h(d_1) + h(d_2) < 9.6$$

by (7), we can take  $A_1 := 0.5$ ,  $A_2 := 4.7$ , and  $A_3 = 19.2$ . Also, since  $k < 10n + 2$ ,  $m \leq n$ , and  $B \geq \max\{|k|, |-(n+m)|, |1|\}$ , we can take  $B := 10n + 2$ . Thus, taking into account inequality (11) and using Lemma 1, we obtain

$$4 \cdot 10^{-m} > |\Lambda_1| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log(10n + 2)) \cdot C),$$

where  $C = (0.5)(4.7)(19.2)$ . By a simple computation, it follows that

$$m \log 10 < 4.38 \cdot 10^{13} \cdot (1 + \log(10n + 2)) + \log 4. \tag{12}$$

Rearranging equation (3) as

$$\frac{d_2 10^n}{9} - \frac{9 \cdot \alpha^k}{d_1(10^m - 1)\sqrt{5}} = -\frac{9\beta^k}{d_1(10^m - 1)\sqrt{5}} + \frac{d_2}{9} \tag{13}$$

and taking absolute values of both sides of (13), we get

$$\left| \frac{d_2 10^n}{9} - \frac{9 \cdot \alpha^k}{d_1(10^m - 1)\sqrt{5}} \right| \leq \frac{9 \cdot |\beta|^k}{d_1(10^m - 1)\sqrt{5}} + \frac{d_2}{9}. \tag{14}$$

Dividing both sides of (14) by  $d_2 10^n / 9$  gives us the following inequality:

$$\left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-n}}{d_1 d_2 (10^m - 1) \sqrt{5}} \right| \leq \frac{81 \cdot |\beta|^k}{d_1 d_2 10^n (10^m - 1) \sqrt{5}} + \frac{1}{10^n} < 2 \cdot 10^{-n}.$$

From this, it follows that

$$\left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-n}}{d_1 d_2 (10^m - 1) \sqrt{5}} \right| < 2 \cdot 10^{-n}. \tag{15}$$

Taking  $\gamma_1 := \alpha$ ,  $\gamma_2 := 10$ ,  $\gamma_3 := 81/d_1d_2(10^m - 1)\sqrt{5}$ , and  $b_1 := k, b_2 := -n, b_3 := 1$ , we can apply Lemma 1. The numbers  $\gamma_1, \gamma_2$ , and  $\gamma_3$  are positive real numbers and elements of the field  $\mathbb{K} = \mathbb{Q}(\sqrt{5})$  and so  $D = 2$ . As before, it can be shown that

$$\Lambda_2 := 1 - \frac{81 \cdot \alpha^k \cdot 10^{-n}}{d_1 d_2 (10^m - 1) \sqrt{5}}$$

is nonzero. By using (5) and the properties of the logarithmic height, we get  $h(\gamma_1) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}$ ,  $h(\gamma_2) = \log 10$ , and

$$\begin{aligned} h(\gamma_3) &= h(81/d_1d_2(10^m - 1)\sqrt{5}) \leq h(81) + h(\sqrt{5}) + h(d_1) + h(d_2) + h(10^m - 1) \\ &< 10.3 + m \log 10. \end{aligned}$$

We can thus take  $A_1 := 0.5$ ,  $A_2 = 4.7$ , and  $A_3 := 20.6 + 2m \log 10$ . Since  $k < 10n + 2$  and  $B \geq \max\{|k|, | -n|, |1|\}$ , we can take  $B := 10n + 2$ . Thus, taking into account inequality (15) and using Lemma 1, we obtain

$$2 \cdot 10^{-n} > |\Lambda_2| > \exp(C \cdot (1 + \log(10n + 2)) (0.5)(4.7)(20.6 + 2m \log 10)),$$

where  $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$ . Thus,

$$n \log 10 - \log(2) < 2.28 \cdot 10^{12} \cdot (1 + \log(10n + 2)) (20.6 + 2m \log 10). \tag{16}$$

Using inequalities (12) and (16), a computer search with Mathematica gives us that  $n < 4.45 \cdot 10^{29}$ . Now let us try to reduce the upper bound on  $n$  by applying Lemma 2. Let

$$z_1 := k \log \alpha - (n + m) \log 10 + \log(81/d_1 d_2 \sqrt{5}) \text{ and } x = e^{z_1} - 1.$$

From (11), we have

$$|x| = |e^{z_1} - 1| < \frac{4}{10^m} < \frac{1}{2}$$

for  $m \geq 1$ . Choosing  $a := \frac{1}{2}$ , we get the inequality

$$|z_1| = |\log(x + 1)| < \frac{\log 2}{(1/2)} \cdot \frac{4}{10^m} < 5.55 \cdot 10^{-m}$$

by Lemma 3. Thus, it follows that

$$0 < \left| k \log \alpha - (n + m) \log 10 + \log(81/d_1 d_2 \sqrt{5}) \right| < 5.55 \cdot 10^{-m}.$$

Dividing this inequality by  $\log 10$ , we get

$$0 < \left| k \left( \frac{\log \alpha}{\log 10} \right) - (n + m) + \left( \frac{\log(81/d_1 d_2 \sqrt{5})}{\log 10} \right) \right| < 2.5 \cdot 10^{-m}. \tag{17}$$

Take  $\gamma := \frac{\log \alpha}{\log 10} \notin \mathbb{Q}$  and  $M := 4.45 \cdot 10^{30}$ . Then we find that  $q_{63}$ , the denominator of the 63rd convergent of  $\gamma$ , exceeds  $6M$ . Now take

$$\mu := \frac{\log(81/d_1 d_2 \sqrt{5})}{\log 10}.$$

In this case, considering the fact that  $1 \leq d_1, d_2 \leq 9$ , a quick computation with Mathematica gives us the inequality  $0 < \epsilon = \epsilon(\mu) := \|\mu q_{63}\| - M \|\gamma q_{63}\| < 0.485826$ . Let  $A := 2.5$ ,  $B := 10$ , and  $w := m$  in Lemma 2. Thus, with the help of Mathematica, we can say that inequality (17) has no solution for

$$m = w \geq \frac{\log(Aq_{63}/\epsilon(\mu))}{\log B} \geq 33.0598.$$

Thus,

$$m \leq 33.$$

Substituting this upper bound for  $m$  into (16), we obtain  $n < 6.8 \cdot 10^{15}$ . Now, let

$$z_2 := k \log \alpha - n \log 10 + \log \left( 81/(10^m - 1)d_1 d_2 \sqrt{5} \right) \text{ and } x = e^{z_2} - 1.$$

From (15), we have

$$|x| = |e^{z_2} - 1| < \frac{2}{10^n} < \frac{1}{10}$$

for  $n \geq 23$ . Choosing  $a := \frac{1}{10}$ , we get the inequality

$$|z_2| = |\log(x + 1)| < \frac{\log(10/9)}{(1/10)} \cdot \frac{2}{10^n} < 2.2 \cdot 10^{-n}$$

by Lemma 3. Thus, it follows that

$$0 < \left| k \log \alpha - n \log 10 + \log \left( 81/(10^m - 1)d_1d_2\sqrt{5} \right) \right| < 2.2 \cdot 10^{-n}.$$

Dividing both sides of the above inequality by  $\log 10$ , we obtain

$$0 < \left| k \left( \frac{\log \alpha}{\log 10} \right) - n + \frac{\log \left( 81/(10^m - 1)d_1d_2\sqrt{5} \right)}{\log 10} \right| < 0.96 \cdot 10^{-n}. \tag{18}$$

Put  $\gamma := \frac{\log \alpha}{\log 10}$  and take  $M := 6.8 \cdot 10^{16}$ . Then we find that  $q_{42}$ , the denominator of the 42nd convergent of  $\gamma$ , exceeds  $6M$ . Taking

$$\mu := \frac{\log \left( 81/(10^m - 1)d_1d_2\sqrt{5} \right)}{\log 10}$$

and considering the fact that  $m \leq 33$  and  $1 \leq d_1, d_2 \leq 9$ , a quick computation with Mathematica gives us the inequality  $0 < \epsilon = \epsilon(\mu) = \left| \mu q_{42} \right| - M \left| \gamma q_{42} \right| < 0.499112$ . Let  $A := 0.96$ ,  $B := 10$ , and  $w := n$  in Lemma 2. Thus, with the help of Mathematica, we can say that inequality (18) has no solution for

$$n = w \geq \frac{\log(Aq_{42}/\epsilon(\mu))}{\log B} \geq 22.6951.$$

Therefore,  $n \leq 22$ , which contradicts our assumption that  $n \geq 23$ . This completes the proof. □

**Theorem 5** *Let  $m, n, d_1, d_2$  be positive integers with  $m \leq n$  and  $d_1, d_2 \leq 9$ . Let*

$$M = \frac{d_1(10^m - 1)}{9} \text{ and } N = \frac{d_2(10^n - 1)}{9}.$$

*If  $L_k = M \cdot N$ , then*

$$(k, L_k, M, N) = (0, 2, 1, 2), (1, 1, 1, 1), (2, 3, 1, 3), (3, 4, 1, 4), (3, 4, 2, 2)$$

*and*

$$(k, L_k, M, N) = (4, 7, 1, 7), (5, 11, 1, 11), (6, 18, 2, 9), (6, 18, 3, 6).$$

**Proof** Assume that  $L_k = M \cdot N$ . Now assume that  $1 \leq m \leq n \leq 40$ . Then by using the Mathematica program, we see that  $k \leq 385$ . In this case, we obtain only the solutions  $L_k \in \{1, 2, 3, 4, 7, 11, 18\}$ . From now on, assume that  $n > 40$ . Then we get the inequality

$$\alpha^{4n-4} < 10^{n-1} \leq \frac{d_2(10^n - 1)}{9} \leq \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9} = L_k \leq 2\alpha^k < \alpha^{k+2}$$



and therefore  $k > 4n - 6$ . That is,  $k > 158$  for  $n > 40$ . Combining the left side of (2) with (4), we obtain

$$\alpha^{k-1} \leq L_k = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9} \leq (10^n - 1)^2 < 10^{2n} \leq \alpha^{10n}.$$

This implies that  $k < 10n + 1$ . Now we rewrite equation (4) as

$$\alpha^k + \beta^k = \frac{d_1(10^m - 1)}{9} \cdot \frac{d_2(10^n - 1)}{9}$$

to obtain

$$\frac{d_1 d_2 10^{m+n}}{81} - \alpha^k = -\beta^k + \frac{d_1 d_2 10^m}{81} + \frac{d_1 d_2 10^n}{81} - \frac{d_1 d_2}{81}. \tag{19}$$

Taking absolute values of both sides of equation (19), we get

$$\left| \frac{d_1 d_2 10^{m+n}}{81} - \alpha^k \right| \leq |\beta|^k + \frac{d_1 d_2 10^m}{81} + \frac{d_1 d_2 10^n}{81} + \frac{d_1 d_2}{81}. \tag{20}$$

Dividing both sides of (20) by  $d_1 d_2 10^{m+n}/81$  gives us the following inequality:

$$\begin{aligned} \left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-(m+n)}}{d_1 d_2} \right| &\leq \frac{81 \cdot |\beta|^k}{d_1 d_2 10^{m+n}} + \frac{1}{10^n} + \frac{1}{10^m} + \frac{1}{10^{m+n}} \\ &< 4 \cdot 10^{-m}. \end{aligned}$$

From this, it follows that

$$\left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-(m+n)}}{d_1 d_2} \right| < 4 \cdot 10^{-m}. \tag{21}$$

Now let us apply Lemma 1 with  $\gamma_1 := \alpha, \gamma_2 := 10, \gamma_3 := 81/d_1 d_2$  and  $b_1 := k, b_2 := -(n + m), b_3 := 1$ . Note that the numbers  $\gamma_1, \gamma_2$ , and  $\gamma_3$  are positive real numbers and elements of the field  $K = \mathbb{Q}(\sqrt{5})$ . The degree of the field  $K$  is 2, so  $D = 2$ . It can be seen that  $\Lambda_3 := 1 - 81 \cdot \alpha^k \cdot 10^{-(m+n)}/d_1 d_2$  is nonzero. Moreover, since

$$h(\gamma_1) = h(\alpha) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}, h(\gamma_2) = h(10) = \log 10$$

and

$$h(\gamma_3) = h(81/d_1 d_2) \leq h(81) + h(d_1) + h(d_2) < 8.8$$

by (7), we can take  $A_1 := 0.5, A_2 := 4.7$ , and  $A_3 := 17.6$ . On the other hand, as  $k < 10n + 1, m \leq n$ , and  $B \geq \max\{|k|, |-(n + m)|, |1|\}$ , we can take  $B := 10n + 1$ . Thus, taking into account the inequality (21) and using Lemma 2, we obtain

$$4 \cdot 10^{-m} > |\Lambda_3| > \exp(C \cdot (1 + \log 2)(1 + \log(10n + 1)) (0.5) (4.7) (17.6)),$$

where  $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2$ . By a simple computation, it follows that

$$m \log 10 - \log 4 < 4.1 \cdot 10^{13} \cdot (1 + \log(10n + 1)). \tag{22}$$

Rearranging equation (4) as

$$\frac{d_2 10^n}{9} - \frac{9 \cdot \alpha^k}{d_1(10^m - 1)} = \frac{9 \cdot \beta^k}{d_1(10^m - 1)} + \frac{d_2}{9} \tag{23}$$

and taking absolute values of both sides of equation (23), we get

$$\left| \frac{d_2 10^n}{9} - \frac{9 \cdot \alpha^k}{d_1(10^m - 1)} \right| \leq \frac{9 \cdot |\beta|^k}{d_1(10^m - 1)} + \frac{d_2}{9}. \tag{24}$$

Dividing both sides of (24) by  $d_2 10^n / 9$  gives us the following inequality:

$$\begin{aligned} \left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-n}}{d_1 d_2 (10^m - 1)} \right| &\leq \frac{81 \cdot |\beta|^k}{d_1 d_2 10^n (10^m - 1)} + \frac{1}{10^n} \\ &< \frac{81 \cdot |\beta|^k}{10^n} + \frac{1}{10^n} < 2 \cdot 10^{-n}. \end{aligned}$$

From this, it follows that

$$\left| 1 - \frac{81 \cdot \alpha^k \cdot 10^{-n}}{d_1 d_2 (10^m - 1)} \right| < 2 \cdot 10^{-n}. \tag{25}$$

Taking  $\gamma_1 := \alpha, \gamma_2 := 10, \gamma_3 := 81/d_1 d_2 (10^m - 1)$ , and  $b_1 := k, b_2 := -n, b_3 := 1$ , we can apply Lemma 1. The numbers  $\gamma_1, \gamma_2$ , and  $\gamma_3$  are positive real numbers and elements of the field  $K = \mathbb{Q}(\sqrt{5})$  and so  $D = 2$ . One can verify that  $\Lambda_4 := 1 - 81 \cdot \alpha^k \cdot 10^{-n} / d_1 d_2 (10^m - 1) \neq 0$ . By using (5) and the properties of the logarithmic height, we get  $h(\gamma_1) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}$ ,  $h(\gamma_2) = \log 10$ , and

$$h(\gamma_3) = h(81/d_1 d_2 (10^m - 1)) \leq h(81) + h(d_1) + h(d_2) + h(10^m - 1) < 9.5 + m \log 10.$$

Thus, we can take  $A_1 := 0.5, A_2 = 4.7$ , and  $A_3 := 19 + 2m \log 10$ . As  $k < 10n + 1$  and  $B \geq \max\{|k|, |-n|, 1\}$ , we can take  $B := 10n + 1$ . Thus, taking into account inequality (25) and using Lemma 1, we obtain

$$2 \cdot 10^{-n} > |\Lambda_4| > \exp(C \cdot (1 + \log(10n + 1)) (0.5) (4.7) (19 + 2m \log 10)),$$

where  $C = -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)$ . Thus, it follows that

$$n \log 10 - \log 2 < 2.27 \cdot 10^{12} \cdot (1 + \log(10n + 1)) (19 + 2m \log 10). \tag{26}$$

Using inequalities (22) and (26), a computer search with Mathematica gives us that  $n < 4.15 \cdot 10^{29}$ . Now let us try to reduce the upper bound on  $n$  by applying Lemma 2. Let

$$z_3 := k \log \alpha - (n + m) \log 10 + \log(81/d_1 d_2) \text{ and } x = e^{z_3} - 1.$$

From (21), we have

$$|x| = |e^{z_3} - 1| < \frac{4}{10^m} < \frac{1}{2}$$

for  $m \geq 1$ . Choosing  $a := \frac{1}{2}$ , we get the inequality

$$|z_3| = |\log(x + 1)| < \frac{\log 2}{(1/2)} \cdot \frac{4}{10^m} < 5.55 \cdot 10^{-m}$$

by Lemma 3. Thus, it follows that

$$0 < |k \log \alpha - (n + m) \log 10 + \log(81/d_1 d_2)| < 5.55 \cdot 10^{-m}.$$

Dividing this inequality by  $\log 10$ , we get

$$0 < \left| k \left( \frac{\log \alpha}{\log 10} \right) - (n + m) + \left( \frac{\log(81/d_1 d_2)}{\log 10} \right) \right| < 2.5 \cdot 10^{-m}. \tag{27}$$

Let  $\gamma := \frac{\log \alpha}{\log 10} \notin \mathbb{Q}$  and  $M := 4.15 \cdot 10^{30}$ . Then we find that  $q_{67}$ , the denominator of the 67th convergent of  $\gamma$ , exceeds  $6M$ . Take

$$\mu := \frac{\log(81/d_1 d_2)}{\log 10}.$$

In this case, a quick computation with Mathematica gives us the inequality  $0 < \epsilon = \epsilon(\mu) := \|\mu q_{67}\| - M \|\gamma q_{67}\| < 0.4766$  for  $1 \leq d_1, d_2 \leq 9$ , except for  $d_1 = d_2 = 9$ . Let  $A := 2.5$ ,  $B := 10$ , and  $w := m$  in Lemma 2. Then with the help of Mathematica, we can say that inequality (27) has no solution for

$$m = w \geq \frac{\log(Aq_{67}/\epsilon(\mu))}{\log B} \geq 36.1498.$$

Thus,

$$m \leq 36.$$

In the case of  $d_1 = d_2 = 9$ , from (27) we have

$$0 < \left| k \left( \frac{\log \alpha}{\log 10} \right) - (n + m) \right| < 2.5 \cdot 10^{-m}.$$

If we divide this inequality by  $k$ , we get

$$0 < \left| \frac{\log \alpha}{\log 10} - \frac{n + m}{k} \right| < \frac{2.5}{10^m \cdot k}. \tag{28}$$

Assume that  $m \geq 36$ . Then it can be seen that  $\frac{10^m}{5} \geq 20 \cdot 10^{34} > 10n + 1 \geq k$ , so we have  $\left| \frac{\log \alpha}{\log 10} - \frac{n+m}{k} \right| < \frac{2.5}{10^m \cdot k} < \frac{1}{2k^2}$ . From the known properties of the continued fraction, it is seen that the rational number  $\frac{n+m}{k}$  is a convergent to  $\gamma = \frac{\log \alpha}{\log 10}$ . Now let  $[a_0; a_1, a_2, \dots] = [0; 4, 1, 3, 1, 1, 1, 6, \dots]$  be the continued fraction expansion of  $\gamma$  and let  $\frac{p_r}{q_r}$  be its  $r$ th convergent. Assume that  $\frac{n+m}{k} = \frac{p_t}{q_t}$  for some  $t$ . Then we have  $13 \cdot 10^{31} > q_{62} > 12 \cdot 10^{31} > 10n + 1 \geq k$ . Thus,  $t \in \{0, 1, 2, \dots, 61\}$ . Furthermore,  $a_M = \max\{a_i | i = 0, 1, 2, \dots, 61\} = 106$ . Again, from the known properties of the continued fraction, we get

$$\left| \gamma - \frac{p_t}{q_t} \right| > \frac{1}{(a_M + 2)k^2} = \frac{1}{108 \cdot k^2}.$$

Thus, from (28), we obtain

$$\frac{2.5}{10^m \cdot k} > \frac{1}{108 \cdot k^2}.$$

This implies that

$$\frac{1}{40 \cdot 10^{34}} \geq \frac{2.5}{10^m} > \frac{1}{108 \cdot k} > \frac{1}{108 \cdot q_{62}} > \frac{1}{1.404 \cdot 10^{34}},$$

a contradiction. Therefore,  $m < 36$ . Consequently, taking  $m \leq 36$  and substituting this upper bound for  $m$  into (26), we obtain  $n < 7.26 \cdot 10^{15}$ . Now let

$$z_4 := k \log \alpha - n \log 10 + \log(81/d_1 d_2 (10^m - 1)) \text{ and } x = e^{z_4} - 1.$$

From (25), we have

$$|x| = |e^{z_4} - 1| < \frac{2}{10^n} < \frac{1}{10}$$

for  $n > 40$ . Choosing  $a := \frac{1}{10}$ , we get the inequality

$$|z_4| = |\log(x + 1)| < \frac{\log(10/9)}{(1/10)} \cdot \frac{2}{10^n} < 2.2 \cdot 10^{-n}$$

by Lemma 3. Thus, it follows that

$$0 < |k \log \alpha - n \log 10 + \log(81/d_1 d_2 (10^m - 1))| < 2.2 \cdot 10^{-n}.$$

Dividing both sides of the above inequality by  $\log 10$ , we get

$$0 < \left| k \left( \frac{\log \alpha}{\log 10} \right) - n + \frac{\log(81/d_1 d_2 (10^m - 1))}{\log 10} \right| < 0.96 \cdot 10^{-n}. \tag{29}$$

Let  $\gamma := \frac{\log \alpha}{\log 10}$  and  $M := 7.26 \cdot 10^{16}$ . Then we find that  $q_{53}$ , the denominator of the 53rd convergent of  $\gamma$ , exceeds  $6M$ . Taking

$$\mu := \frac{\log(81/d_1 d_2 (10^m - 1))}{\log 10}$$

and considering the fact that  $m \leq 34$  and  $1 \leq d_1, d_2 \leq 9$ , a quick computation with Mathematica gives us the equality  $0 < \epsilon = \epsilon(\mu) := ||\mu q_{53}|| - M ||\gamma q_{53}|| < 0.4999$  except for the cases  $d_1 \cdot d_2 = 9$  and  $m = 1$ . Let  $A := 0.96$ ,  $B := 10$ , and  $w := n$  in Lemma 2. Then with the help of Mathematica, we can say that inequality (29) has no solution for

$$n = w \geq \frac{\log(Aq_{53}/\epsilon(\mu))}{\log B} \geq 36.4655.$$

Therefore,  $n \leq 36$ . If  $d_1 \cdot d_2 = 9$  and  $m = 1$ , then from (29), we get

$$0 < \left| \left( \frac{\log \alpha}{\log 10} \right) - \frac{n}{k} \right| < \frac{0.96}{10^n \cdot k}. \tag{30}$$

If we follow the way after (28), we find that  $n \leq 40$ . These cases both contradict the assumption that  $n > 40$ . This completes the proof of the theorem. □

**Corollary 6** *The equations  $F_k = (10^n - 1)(10^m - 1)$  and  $L_k = (10^n - 1)(10^m - 1)$  have no solution  $(k, m, n)$  in positive integers.*

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