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The boundedness of h-admissible Fourier integral operators on Bessel potential spaces

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Abstract: The aim of this work is to study the boundedness of h-admissible Fourier integral operators. These operators are bounded on the Bessel potential spaces if the weight of the amplitude is bounded.

Key words: h-admissible Fourier integral operators, symbol and phase, Sobolev and Bessel potential spaces

1. Introduction

A h-Fourier integral operator has the following form

$$[I_h(a,\varphi)u](x) = \iint e^{ih^{-1}\varphi(x,\xi,y)} a(x,\xi,y)\,u(y)\,dyd\xi,\ u \in S(\mathbb{R}^n), \tag{1.1}$$

where $h \in]0, h_0]$ is a semiclassical parameter. Two C^∞-functions appear in (1.1): the phase function φ and the amplitude a. In particular when $\varphi(x,\xi,y) = \langle x-y,\xi\rangle$, $I(a,\varphi) := Op(a)$ is called a h-pseudodifferential operator.

Historically, a systematic study of smooth Fourier integral operators with amplitudes in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

(i.e. $a(x,\xi) \in S^m_{\rho,\delta}$), and phase functions in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus 0)$ homogenous of degree 1 in the frequency variable ξ and with nonvanishing determinant of the mixed Hessian matrix (i.e. nondegenerate phase functions), was treated by Hörmander for the first time in [9], after being initially used by Lax, Maslov, Egorov, and others. The results in [9] were expanded in the paper [5] by Duistermaat and Hörmander, where they studied parametrices of pseudodifferential operators of principal type and propagation of singularities. In the meantime, Fourier integral operators had also been applied to the study of hyperbolic partial differential equation and spectral theory.

Later on, Hörmander's local L^2 result was extended by Beals [3] and Greenleaf and Uhlmann [6] to the case of amplitudes in $S^0_{\frac{1}{2},\frac{1}{2}}$.

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On the other hand, other classes of amplitudes and phase functions were studied. In [8] and [11], Robert and Helffer treated the symbol class Γ_ρ^μ (2.1)(see below) and they considered phase functions satisfying certain properties. In [2], Aitemrar and Senoussaoui treated the L^2 boundedness and L^2 compactness of h-admissible Fourier integral operator (2.3) (see below) with symbol class just defined and $\varphi(x,\xi,y) = S(x,\xi) - \langle y,\xi\rangle$.

The aim of this work is to extend results obtained in [2], the same hypothesis on the phase function is kept, and we mainly prove the boundedness of h-admissible Fourier integral operator on Bessel potential space when the weight of the amplitude a is bounded. Using the estimate given in [1] for h-pseudodifferential operators, we also establish an estimate of $\|F\|_{\mathcal{L}(H^s(\mathbb{R}^n))}$.

It should be noted that in Hörmander's class this result is not true in general. In fact, in [7] the author gave an example of h-Fourier integral operators with symbol belonging to $\bigcap_{0<\rho<1} S^0_{\rho,1}$ that cannot be extended as a bounded operator on L^2 (same for Bessel potential space).

2. Preliminaries

Definition 2.1 *Let Ω be an open set in $\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$. The space of amplitudes $\Gamma^\mu_\rho(\Omega)$ is the set of smooth functions $a : \Omega \to \mathbb{C}$ such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma a(x,\xi,y)\right| \le C_{\alpha\beta\gamma} \lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y) \tag{2.1}$$

uniformly in Ω for all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$. Moreover, let for all $k \in \mathbb{N}$

$$|a|_k^{\mu,\rho} := \max_{|\alpha|+|\beta|+|\gamma|\le k} \sup_{(x,\xi,y)\in\Omega} \lambda^{-\mu+\rho(|\alpha|+|\beta|+|\gamma|)}(x,\xi,y) \left|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma a(x,\xi,y)\right|$$

be the associated sequence of seminorms.

Here $\lambda(x,\xi,y) = \left(1+|x|^2+|\xi|^2+|y|^2\right)^{1/2}$ is called the weight.

Remark 2.2 *For short we write Γ^μ_ρ instead of $\Gamma^\mu_\rho\left(\mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y\right)$*

Proposition 2.3 *Let $\mu,\nu \in \mathbb{R}$ and $\rho,\delta \in [0,1]$*

(*i*) *if $a \in \Gamma^\mu_\rho(\Omega)$ then $\partial^\alpha a \in \Gamma^{\mu-\rho|\alpha|}_\rho(\Omega)$;*

(*ii*) *if $a \in \Gamma^\mu_\rho(\Omega)$ and $b \in \Gamma^\nu_\rho(\Omega)$ then $ab \in \Gamma^{\mu+\nu}_\rho(\Omega)$;*

(*iii*) *if $\delta \le \rho$ then $\Gamma^\mu_\rho(\Omega) \subset \Gamma^\mu_\delta(\Omega)$;*

(*iv*) *Let $a \in \Gamma^\mu_\rho(\Omega)$. If there exists $C > 0$ and $\delta \in \mathbb{R}$ such that $|a| \ge C\lambda^\delta$ uniformly on Ω then $a^{-1} \in \Gamma^{\mu-2\delta}_\rho(\Omega)$.*

Proof The proof is based on Leibniz's formula. □

Now, we take an interest in giving a sense of the integral of type

$$[I_h(a,\varphi) f](x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{ih^{-1}\varphi(x,\xi,y)} a(x,\xi,y) f(y)\, dy d\xi, \quad f \in S(\mathbb{R}^n), \tag{2.2}$$

where $h \in]0, h_0], a \in \Gamma_\rho^\mu$ and φ be a phase function which satisfies the following conditions:

$(H1)$ $\varphi \in C^\infty\left(\mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n; \mathbb{R}\right)$

$(H2)$ $\forall (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n, \exists C_{\alpha\beta\gamma} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma \varphi(x, \xi, y)\right| \leq C_{\alpha\beta\gamma} \lambda^{(2-|\alpha|-|\beta|-|\gamma|)}(x, \xi, y);$$

$(H3)$ $\exists C_1, C_2 > 0$ such that

$$C_1 \lambda(x, \xi, y) \leq \lambda(\partial_y \varphi, \partial_\xi \varphi, y) \leq C_2 \lambda(x, \xi, y), \quad \forall (x, \xi, y) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n$$

$(H'3)$ $\exists C_1', C_2' > 0$ such that

$$C_1' \lambda(x, \xi, y) \leq \lambda(x, \partial_\xi \varphi, \partial_x \varphi) \leq C_2' \lambda(x, \xi, y), \quad \forall (x, \xi, y) \in \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n.$$

It is obvious that the phase function $\varphi(x, y, \xi) = \langle x - y, \xi \rangle$ satisfies the hypothesis $(H1), (H2), (H3)$, and $(H'3)$

In order to generalize the notion of h-admissible operators [11], we give the following definitions.

Definition 2.4 *We call h-admissible symbol of weight (μ, ρ), every application $a(h)$ of $]0, h_0]$ in Γ_ρ^μ, such that for all $N \in \mathbb{N}$*

$$a(h) = \sum_{j=0}^{N} h^j a_j + h^{N+1} r_{N+1}(h),$$

where $a_j \in \Gamma_\rho^{\mu-2\rho j}$, and $\{r_{N+1}(h), h \in]0, h_0]\}$ is bounded in $\Gamma_\rho^{\mu-2\rho(N+1)}$.

Definition 2.5 *We call h-admissible Fourier integral operator, every C^∞ application A of $]0, h_0]$ in $\mathcal{L}(S(\mathbb{R}^n); L^2(\mathbb{R}^n))$, for which there exists a sequence $(a_j)_j \in \Gamma_0^\mu$ satisfying for all $N \in \mathbb{N}$ and N large enough*

$$A(h) = \sum_{j=0}^{N} h^j I_h(a_j, \varphi) + h^{N+1} R_{N+1}(h), \tag{2.3}$$

where

$$[I_h(a_j, \varphi) f](x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} e^{ih^{-1}\varphi(x,\xi,y)} a_j(x, \xi, y) f(y) dy d\xi,$$

$$\sup_{h \in [0, h_0]} \|R_{N+1}(h)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < \infty.$$

To give a meaning to the right-hand side of (2.2), we consider $g \in S\left(\mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n\right)$ and $g(0,0,0) = 1$. If $a \in \Gamma_0^\mu$, we define

$$a_r(x, \xi, y) = g\left(\frac{x}{r}, \frac{\xi}{r}, \frac{y}{r}\right) a(x, \xi, y), \quad r > 0.$$

Theorem 2.6 *Let φ be a phase function satisfying $(H1), (H2), (H3)$, and $(H'3)$. Then,*

1. *For all $f \in S(\mathbb{R}^n)$, $\lim_{r\to+\infty}[I_h(a_r,\varphi)f](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We then set*

$$[I_h(a,\varphi)f](x) := \lim_{r\to+\infty}[I_h(a_r,\varphi)f](x) \quad \forall x \in \mathbb{R}^n.$$

2. *$I_h(a,\varphi)$ defines a linear bounded operator on $S(\mathbb{R}^n)$ and on $S'(\mathbb{R}^n)$.*

Proof Let $\psi \in C_0^\infty(\mathbb{R})$ such that *supp* $\psi \subset [-2,2]$ and $\psi \equiv 1$ on $[-1,1]$.

For all $\epsilon > 0$, we set

$$\omega_\epsilon(x,\xi,y) = \psi\left(\frac{|\partial_y\varphi|^2 + |\partial_\xi\varphi|^2}{\epsilon\lambda^2(x,\xi,y)}\right).$$

The hypothesis $(H3)$ implies that there exists $\gamma > 0$ such that we have on the support of ω_ϵ

$$\lambda(x,\xi,y) \le \gamma\left(\lambda^{\frac{1}{2}}(y) + \epsilon^{\frac{1}{2}}\lambda(x,\xi,y)\right).$$

Therefore, there exists ϵ_0 and γ_0, such that for all $\epsilon \le \epsilon_0$ we have the inequality

$$\lambda(x,\xi,y) \le \gamma_0\lambda^{\frac{1}{2}}(y).$$

on the support of ω_ϵ.

Now, we fix $\epsilon = \epsilon_0$. Then it is immediate that $I_h(\omega_{\epsilon_0}a_r,\varphi)f$ is an absolutely convergent integral and we have

$$\lim_{r\to+\infty}[I_h(\omega_{\epsilon_0}a_r,\varphi)f](x) = [I_h(\omega_{\epsilon_0}a,\varphi)f](x), \quad \forall x \in \mathbb{R}^n \tag{2.4}$$

using $(H2)$, we also prove that $I_h(\omega_{\epsilon_0}a,\varphi)$ is a bounded operator from $S(\mathbb{R}^n)$ into itself.

To study $\lim_{r\to+\infty}[I_h((1-\omega_{\epsilon_0})a_r,\varphi)f](x)$ for all $x \in \mathbb{R}^n$, we create the operator

$$L_h = -ih\left(|\Delta_y\varphi|^2 + |\Delta_\xi\varphi|^2\right)^{-1}\left(\sum_{j=1}^{n}\frac{\partial\varphi}{\partial y_j}\frac{\partial}{\partial y_j} + \sum_{j=1}^{N}\frac{\partial\varphi}{\partial\xi_j}\frac{\partial}{\partial\xi_j}\right).$$

Clearly we have

$$L_h\left(e^{ih^{-1}\varphi}\right) = e^{ih^{-1}\varphi}. \tag{2.5}$$

Let Ω_0 be the open subset of $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$ defined by

$$\Omega_0 = \left\{(x,\xi,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n, |\Delta_y\varphi|^2 + |\Delta_\xi\varphi|^2 > \frac{\epsilon_0}{2}\lambda^2(x,\xi,y)\right\}.$$

We need the following lemma.

Lemma 2.7 *For all $q \in \mathbb{N}_0$, and $b \in C^\infty\left(\mathbb{R}^N_\xi \times \mathbb{R}^n_y\right)$, we have*

$$\left({}^tL_h\right)^q((1-\omega_{\epsilon_0})b) = \sum_{|\alpha|+|\beta|\le q} g^q_{\alpha,\beta}\partial^\alpha_\xi\partial^\beta_y((1-\omega_{\epsilon_0})b),$$

where $g^q_{\alpha,\beta} \in \Gamma_0^{-q}(\Omega_0)$ and depend only on φ.
($\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and ${}^tL_h$ is the transpose of L_h).

Proof We prove the lemma by recurrence,

It is evident for $q=0$. Now we see easily that

$$ {}^{t}L_h = \sum_{j=1}^{n} F_{h,j} \frac{\partial}{\partial y_j} + \sum_{k=1}^{N} G_{h,k} \frac{\partial}{\partial \xi_k} + H_h, \tag{2.6} $$

where

$$ F_{h,j} \in \Gamma_0^{-1}(\Omega_0), \quad \forall j \in \{1,\dots,n\} $$

$$ G_{h,k} \in \Gamma_0^{-1}(\Omega_0), \quad \forall k \in \{1,\dots,N\} $$

and

$$ H_h \in \Gamma_0^{-2}(\Omega_0) $$

(which results from the hypothesis $(H2)$). Therefore, the recurrence is immediately proved. □

We have from (2.5)

$$ [I_h((1-\omega_{\epsilon_0})a_r,\varphi)f](x) = \iint e^{ih^{-1}\varphi(x,\xi,y)} \left({}^{t}L_h\right)^q ((1-\omega_{\epsilon_0})a_r f(y))\, dyd\xi, \tag{2.7} $$

for all $q \in \mathbb{N}_0$.

Now $\left({}^{t}L_h\right)^q((1-\omega_{\epsilon_0}))a_r f$ described (when r varies) a bound of $\Gamma_0^{\mu-q}$, and

$$ \lim_{r\to+\infty} \left({}^{t}L_h\right)^q((1-\omega_{\epsilon_0})a_r f)(x,\xi,y) = \left({}^{t}L_h\right)^q((1-\omega_{\epsilon_0})af)(x,\xi,y), \tag{2.8} $$

for all $(x,\xi,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$.

Finally, we have

$$ \lambda^{s-n-N}(x) \iint \lambda^{-s}(x,\xi,y)\, d\xi dy \le \gamma_s, \tag{2.9} $$

for all $s > n+N$.

It results so from (2.7) – (2.9) and using Lebesgue's dominated convergence theorem we can see that

$$ \lim_{r\to+\infty} [I_h((1-\omega_{\epsilon_0})a_r,\varphi)f](x) = \iint e^{ih^{-1}\varphi(x,\xi,y)} \left({}^{t}L_h\right)^q ((1-\omega_{\epsilon_0})af(y))\, dyd\xi, \tag{2.10} $$

where q satisfies $q > n+N+\mu$.

From (2.4) and (2.10), we can prove the first part of the theorem.

Now let us show that $I_h((1-\omega_{\epsilon_0})a,\varphi)$ is bounded.

Taking account of (2.6) and (2.10), we find

$$ [I_h((1-\omega_{\epsilon_0})a,\varphi)f](x) = \sum_{|\gamma|\le q} \iint e^{ih^{-1}\varphi(x,\xi,y)} b_\gamma^{(q)}(x,\xi,y)\, \partial_y^\gamma f(y)\, dyd\xi, \tag{2.11} $$

with $b_\gamma^{(q)} \in \Gamma_0^{\mu-q}$.

On the other hand, we have

$$x^{\alpha}\partial_x^{\beta}\left(e^{ih^{-1}\varphi(x,\xi,y)}b_{\gamma}^{(q)}(x,\xi,y)\right)\in\Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.12}$$

We deduce from (2.11) and (2.12) that, for all $q>n+N+\mu+|\alpha|+|\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$\left|x^{\alpha}\partial_x^{\beta}I_h\left(\left(1-\omega_{\epsilon_0}\right)a,\varphi\right)f(x)\right|\leq C_{\alpha,\beta,q}\sup_{|\gamma|\leq q}\sup_{x\in\mathbb{R}^n}\left|\partial_x^{\gamma}f(x)\right|,$$

which proves the boundedness of $I_h\left(\left(1-\omega_{\epsilon_0}\right)a,\varphi\right)$ from $S\left(\mathbb{R}^n\right)$ into itself.

The continuity of $I_h(a,\varphi)$ on the space $S'\left(\mathbb{R}^n\right)$ is obtained by the same way via the assumption $(H'3)$.

□

In this sequel we give a theorem where it is shown that these types of operators are stable by composition:

Theorem 2.8 *Let φ_1,φ_2 be two phases functions satisfying $(H1),(H2),(H3)$, and $(H'3)$. Set*

$$\varphi(x,\xi,y)=\varphi_1\left(x,\xi_1,z\right)+\varphi_2\left(z,\xi_2,y\right),$$

with $x,y\in\mathbb{R}^n,\xi=\left(\xi_1,z,\xi_2\right)\in\mathbb{R}^{N_1}\times\mathbb{R}^n\times\mathbb{R}^{N_2}$. Then φ verifies $(H1),(H2),(H3)$, and $(H'3)$, for all $a_1\in\Gamma_0^{\mu_1},a_2\in\Gamma_0^{\mu_2}$, we have

$$I_h\left(a_1,\varphi_1\right)I_h\left(a_2,\varphi_2\right)=I_h\left(a_1\times a_2,\varphi\right)$$

with

$$\left(a_1\times a_2\right)(x,\xi,y)=a_1\left(x,\xi_1,z\right)a_2\left(z,\xi_2,y\right).$$

Proof See [2]. □

3. Bessel potential spaces

Definition 3.1 *Let $s\in\mathbb{R}$. Then the Bessel Potential space $H^s\left(\mathbb{R}^n\right)$ is defined as*

$$H^s\left(\mathbb{R}^n\right):=\left\{u\in S'\left(\mathbb{R}^n\right):\Lambda^s\left(D_x\right)u\in L^2\left(\mathbb{R}^n\right)\right\}.$$

$$\|u\|_{H^s}:=\left\|\Lambda^s\left(D_x\right)u\right\|_{L^2},$$

where $D_x=-i\frac{\partial}{\partial x}$ and for all $u\in S'\left(\mathbb{R}^n\right)$, we have

$$\Lambda^s\left(D_x\right)u=\mathcal{F}^{-1}\left[\lambda^s(\xi)\hat{u}\right],$$

with $\mathcal{F}^{-1}$ is the inverse Fourier transformation.

Hence, $H^s\left(\mathbb{R}^n\right)$ normed by $\|.\|_{H^s}$ is a Banach space. Moreover,

$$\langle u,v\rangle_{H^s}:=\left\langle\Lambda^s\left(D_x\right)u,\Lambda^s\left(D_x\right)v\right\rangle_{L^2}$$

is inner product on $H^s\left(\mathbb{R}^n\right)$ and $\|u\|_{H^s}^2=\langle u,u\rangle_{H^s}$. Thus, $H^s\left(\mathbb{R}^n\right)$ is a Hilbert space.

Remark 3.2 1. *By definition $\Lambda^s\left(D_x\right)$ is an isomorphism from $H^s\left(\mathbb{R}^n\right)$ to $L^2\left(\mathbb{R}^n\right)$.*

2. *Since* $\Lambda^s(D_x) : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ *for all* $s \in \mathbb{R}$ *and* $S(\mathbb{R}^n)$ *is dense in* $L^2(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ *is dense in* $H^s(\mathbb{R}^n)$ *for all* $s \in \mathbb{R}$.

Proposition 3.3 *If* $x \to \varphi(x)$ *belongs to* $\Gamma_k^m(\mathbb{R}_x^n)$, *then* $(x,y) \to \varphi(x)$ *belongs to* $\Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$, *with* $k \in \{0,1\}$.

Proof We set $\psi(x,y) = \varphi(x)$ and we have to prove that $\psi(x,y) \in \Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$ with $k \in \{0,1\}$. Let $(\alpha,\beta) \in \mathbb{N}^n \times \mathbb{N}^N$.

If $|\beta| \geq 1, \left|\partial_x^\alpha \partial_y^\beta \psi(x,y)\right| = 0$

Else, if $|\beta| = 0$,

$$\begin{aligned} |\partial_x^\alpha \psi(x,y)| &= |\partial_x^\alpha \varphi(x)|. \\ &\leq C_\alpha \lambda^{m-|\alpha|}(x) \\ &\leq C_\alpha \lambda^{m-|\alpha|}(x,y) \end{aligned}$$

Thus, $\psi(x,y) \in \Gamma_k^m\left(\mathbb{R}_x^n \times \mathbb{R}_y^N\right)$. □

Example 3.4 *We have* $\xi \to \lambda^s(\xi)$ *belongs to* $\Gamma_0^s\left(\mathbb{R}_\xi^n\right)$.

Using Proposition (3.3), we deduce that

$$\psi(\xi,x) = \lambda^s(\xi) \in \Gamma_0^s\left(\mathbb{R}_\xi^n \times \mathbb{R}_x^n\right).$$

Lemma 3.5 *Let* $a \in \Gamma_0^m\left(\mathbb{R}^{2n}\right)$. *Then the operator* $Op(a) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ *is bounded.*

Proof Let $s, m \in \mathbb{R}$. Since $\Lambda^{s+m}(D_x) : H^{s+m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\Lambda^{-s}(D_x) : L^2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ are linear isomorphisms, $Op(a) : H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is a bounded operator if and only if

$$Op(b) := \Lambda^s(D_x)\, Op(a)\, \Lambda^{-s-m}(D_x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded operator.

For all $m_1, m_2 \in \mathbb{R}$

$$\Gamma_0^{m_1} \times \Gamma_0^{m_2} \ni (a_1,a_2) \to a_1 \sharp a_2 \in \Gamma_0^{m_1+m_2}.$$

If $a_1(x,\xi) = \lambda^s(\xi)$ and $a_2(x,\xi) = a(x,\xi)\,\lambda^{-s-m}(\xi)$.

By using Caldron–Vaillancourt's theorem [4], $Op(b)$ is bounded on $L^2(\mathbb{R}^n)$ for more details see [1]. □

Corollary 3.6 *If* $a \in \Gamma_0^0\left(\mathbb{R}^{2n}\right)$, *then* $Op(a) : H^s \to H^s$ *is a bounded linear operator. Moreover, there exists some* $k \in \mathbb{N}$ *and* $C_s > 0$ *such that*

$$\|Op(a)\|_{\mathcal{L}(H^s(\mathbb{R}^n))} \leq C_s\, |a|_k^{0,0}. \tag{3.1}$$

4. The special phase function

In this section, we will study a particular case on the phase function φ, which is very useful for solving Cauchy problems [10]. Let

$$\varphi(x,y,\xi) = S(x,\xi) - \langle y,\xi\rangle ,$$

where $S \in C^{\infty}\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi ; \mathbb{R}\right)$ satisfying

1. (G_1) There exists $\delta_0 > 0$ such that

$$\inf_{(x,\xi)\in\mathbb{R}^{2n}} \left| det \frac{\partial^2 S}{\partial x \partial \xi}(x,\xi)\right| \geq \delta_0;$$

2. (G_2) For all $(\alpha,\beta) \in \mathbb{N}^{2n}$, there exists $C_{\alpha,\beta} > 0$, such that

$$\left|\partial_x^\alpha \partial_\xi^\beta S(x,\xi)\right| \leq C_{\alpha,\beta} \lambda^{(2-|\alpha|-|\beta|)}(x,\xi)$$

Proposition 4.1 *If S satisfies (G_1) and (G_2). Then the phase function $\varphi(x,y,\xi) = S(x,\xi) - \langle y,\xi\rangle$ satisfies $(H_1),(H_2),(H_3)$, and (H_3'). In addition, there exists $C_3 > 0$ such that*

$$|x - x'| \leq C_3 \left|(\partial_\xi S)(x,\xi) - (\partial_\xi S)(x',\xi)\right| , \tag{4.1}$$

for all $(x,x',\xi) \in \mathbb{R}^{3n}$.

Proof (H_1) and (H_2) are trivially satisfied.

To prove (H_3) and (H_3'), we use the following lemma.

Lemma 4.2 *Let us assume that S satisfies (G_1) and (G_2), then S satisfies the following inequalities:*

There exists $C_1, C_2 > 0$, such that

$$\begin{cases} |x| \leq C_1 \lambda(\xi, \partial_\xi S), & \forall (x,\xi) \in \mathbb{R}^{2n}, \\ |\xi| \leq C_2 \lambda(x, \partial_x S), & \forall (x,\xi) \in \mathbb{R}^{2n}. \end{cases} \tag{4.2}$$

Also there exists $C_3 > 0$ such that for all $(x,\xi),(x',\xi') \in \mathbb{R}^{2n}$,

$$|x - x'| + |\xi - \xi'| \leq C_3 \left[\left|(\partial_\xi S)(x,\xi) - (\partial_\xi S)(x',\xi')\right| + |\xi - \xi'|\right]. \tag{4.3}$$

Proof The mappings

$$\xi \to f_x(\xi) = \partial_x S(x,\xi), \quad x \to g_\xi(x) = \partial_\xi S(x,\xi),$$

are diffeomorphisms of $\mathbb{R}^n$. From (G_1) and (G_2), it follows that $\left\|\left(f_x^{-1}\right)'\right\|, \left\|\left(g_\xi^{-1}\right)'\right\|$ are uniformly bounded on $\mathbb{R}^n$ and $\left\|\left(\psi_2^{-1}\right)'\right\|$ is uniformly bounded on $\mathbb{R}^{2n}$,

where

$$\psi_2(x,\xi) = (\xi, \partial_\xi S(x,\xi)).$$

Thus, (G_1) and the Taylor's theorem lead to the following estimate:

There exist $M, N > 0$, such that for all $(x,\xi), (x',\xi') \in \mathbb{R}^{2n}$,

$$\begin{aligned} |\xi| &= \left|f_x^{-1}(f_x(\xi)) - f_x^{-1}(f_x(0))\right| \\ &\leq M\left|\partial_x S(x,\xi) - \partial_x S(x,0)\right| \\ &\leq C_4\lambda(x,\partial_x S), \end{aligned}$$

with $C_4 > 0$;

$$\begin{aligned} |x| &= \left|g_\xi^{-1}(g_\xi(\xi)) - g_\xi^{-1}(g_\xi(0))\right| \\ &\leq N\left|\partial_\xi S(x,\xi) - \partial_\xi S(0,\xi)\right| \\ &\leq C_5\lambda(\partial_\xi S,\xi), \end{aligned}$$

with $C_5 > 0$;

$$\begin{aligned} |(x,\xi) - (x',\xi')| &= \left|h_2^{-1}(h_2(x,\xi)) - h_2^{-1}(h_2(x',\xi'))\right| \\ &\leq C_5\left|(\xi,\partial_\xi S(x,\xi)) - (\xi',\partial_\xi S(x',\xi'))\right|. \end{aligned}$$

□

From (4.2), we have

$$\begin{aligned} \lambda(x,y,\xi) &\leq \lambda(x,\xi) + \lambda(y) \\ &\leq C_6(\lambda(\xi,\partial_\xi S) + \lambda(y)), \end{aligned}$$

with $C_6 > 0$.

Also, we have $\partial_{y_j}\varphi = -\xi_j$; and $\partial_{\xi_j}\varphi = \partial_{\xi_j}S - y_j$, so

$$\begin{aligned} \lambda(\xi,\partial_\xi S) &= \lambda(\partial_y\varphi,\partial_\xi\varphi + y) \\ &\leq 2\lambda(\partial_y\varphi,\partial_\xi\varphi,y), \end{aligned}$$

which finally gives for some $C_7 > 0$,

$$\begin{aligned} \lambda(x,\xi,y) &\leq 2C_6\lambda(\partial_y\varphi,\partial_\xi\varphi,y) \\ &\leq \frac{1}{C_7}\lambda(\partial_y\varphi,\partial_\xi\varphi,y). \end{aligned}$$

The second inequality in (H_3) is a consequence of (4.2). By a similar argument we can show (H_3').

When $\xi = \xi'$ in (4.3), we obtain (4.1). □

Example 4.3 *Consider the function given by*

$$S(x,\xi) = \sum_{|\alpha|+|\beta|=2} K_{\alpha,\beta}x^\alpha\xi^\beta, \quad \forall (x,\xi) \in \mathbb{R}^{2n},$$

where $K_{\alpha,\beta} \in \mathbb{R} \quad \forall \alpha,\beta \in \mathbb{N}^n$,

$S(x,\xi)$ *verifies* (G_1) *and* (G_2).

5. The boundedness of F_h on the Bessel potential space

We have the following result concerning the boundedness on the Bessel potential space of the h-admissible Fourier integral operator defined by

$$\left(F_h u\right)(x) = \iint e^{ih^{-1}[S(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)\, u(y)\, dy\widehat{d\xi},$$

where $\widehat{d\xi} = (2\pi h)^{-n} d\xi$.

Proposition 5.1 *Let F_h be the integral operator of distribution kernel*

$$K_h(x,y) = \int_{\mathbb{R}^n} e^{ih^{-1}[S(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)\, \widehat{d\xi}, \tag{5.1}$$

with $a \in \Gamma_k^m\left(\mathbb{R}^{2n}\right)$, $k \in \{0,1\}$, $h \in \left]0, h_0\right]$ and S satisfies (G_1) and (G_2). Then $F_h F_h^*$ and $F_h^* F_h$ are h-admissible operators with symbol in $\Gamma_k^m\left(\mathbb{R}^{2n}\right)$, $k \in \{0,1\}$, given by

$$\sigma\left(F_h F_h^*\right)\left(x, \partial_x S(x,\xi)\right) \equiv |a(x,\xi)|^2 \left| \left(\det \frac{\partial^2 S}{\partial \xi \partial x}\right)^{-1} (x,\xi) \right|,$$

$$\sigma\left(F_h^* F_h\right)\left(\partial_\xi S(x,\xi), \xi\right) \equiv |a(x,\xi)|^2 \left| \left(\det \frac{\partial^2 S}{\partial \xi \partial x}\right)^{-1} (x,\xi) \right|,$$

we denote here $a \equiv b$ for $a, b \in \Gamma_k^{2\gamma}\left(\mathbb{R}^{2n}\right)$ if $(a-b) \in \Gamma_k^{2\gamma-2}\left(\mathbb{R}^{2n}\right)$ and σ stands for the symbol.

Proof If $u \in S(\mathbb{R}^n)$, then $F_h u$ is given by

$$\begin{aligned} \left(F_h u\right)(x) &= \int_{\mathbb{R}^n} K_h(x,y)\, u(y)\, dy. \qquad (5.2)\\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{ih^{-1}[S(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)\, \widehat{d\xi} \right) u(y)\, dy. \\ &= \iint_{\mathbb{R}^{2n}} e^{ih^{-1}[S(x,\xi)-\langle y,\xi\rangle]} a(x,\xi)\, u(y)\, dy\widehat{d\xi}. \\ &= \int_{\mathbb{R}^n} e^{ih^{-1}S(x,\xi)} a(x,\xi) \left(\int_{\mathbb{R}^n} e^{-ih^{-1}\langle y,\xi\rangle} u(y)\, dy \right) \widehat{d\xi}. \\ &= \int_{\mathbb{R}^n} e^{ih^{-1}S(x,\xi)} a(x,\xi)\, \mathcal{F}_h u(\xi)\, \widehat{d\xi}, \end{aligned}$$

where $\mathcal{F}_h$ is the Fourier transformation.

Here F_h is a continuous linear mapping from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$. Let $v \in S(\mathbb{R}^n)$, then

$$\begin{aligned}
\langle F_h u, v\rangle_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} e^{ih^{-1}S(x,\xi)} a(x,\xi)\,\mathcal{F}_h u(\xi)\,\widehat{d\xi}\right)\overline{v(x)}dx. \\
&= \iint_{\mathbb{R}^{2n}} e^{ih^{-1}S(x,\xi)} a(x,\xi)\,\mathcal{F}_h u(\xi)\,\overline{v(x)}dx\widehat{d\xi}. \\
&= \int_{\mathbb{R}^n}\mathcal{F}_h u(\xi)\left(\int_{\mathbb{R}^n} e^{ih^{-1}S(x,\xi)} a(x,\xi)\,\overline{v(x)}dx\right)\widehat{d\xi}. \\
&= \int_{\mathbb{R}^n}\mathcal{F}_h u(\xi)\left(\int_{\mathbb{R}^n} \overline{e^{-ih^{-1}S(x,\xi)}\overline{a(x,\xi)}v(x)\,dx}\right)\widehat{d\xi}.
\end{aligned}$$

Thus,

$$\langle F_h u, v\rangle_{L^2(\mathbb{R}^n)} = (2\pi h)^{-n}\langle \mathcal{F}_h u(.), \mathcal{F}_h((F^* v))(.)\rangle_{L^2(\mathbb{R}^n)},$$

where

$$\mathcal{F}_h((F_h^* v))(\xi) = \int_{\mathbb{R}^n} e^{-ih^{-1}S(\widetilde{x},\xi)}\overline{a}(\widetilde{x},\xi)\,v(\widetilde{x})\,d\widetilde{x}. \tag{5.3}$$

Hence, for all $v \in S(\mathbb{R}^n)$,

$$(F_h F_h^* v)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}\overline{a}(x,\xi)\,\overline{a}(\widetilde{x},\xi)\,v(\widetilde{x})\,d\widetilde{x}\widehat{d\xi}. \tag{5.4}$$

The main idea of showing that $F_h F_h^*$ is an h-admissible operator is to use the fact that $S(x,\xi) - S(\widetilde{x},\xi)$ can be expressed by the inner product $\langle x-\widetilde{x}, \zeta(x,\widetilde{x},\xi)\rangle$, after considering the change of variables $(x,\widetilde{x},\xi) \to (x,\widetilde{x},\zeta = \zeta(x,\widetilde{x},\xi))$. The kernel of $F_h F_h^*$ is

$$K_h(x,\widetilde{x}) = \int_{\mathbb{R}^n} e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]} a(\widetilde{x},\xi)\,\overline{a}(x,\xi)\,\widehat{d\xi}.$$

We obtain from (4.1) that if

$$|x-\widetilde{x}| \geq \frac{\varepsilon}{2}\lambda(x,\widetilde{x},\xi),\ \text{where } \varepsilon > 0 \text{ is sufficiently small.}$$

Then

$$|(\partial_\xi S)(x,\xi) - (\partial_\xi S)(\widetilde{x},\xi)| \geq \frac{\varepsilon}{2C_5}\lambda(x,\widetilde{x},\xi). \tag{5.5}$$

Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\omega(x) \geq 0, \quad \forall x \in \mathbb{R},$$

$$\omega(x) = 1,\ \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$supp\ \omega \subset \left]-1, 1\right[,$$

and setting

$$\begin{aligned} b(x,\widetilde{x},\xi) &:= a(x,\xi)\,\overline{a}(\widetilde{x},\xi) \\ &= b_{1,\varepsilon}(x,\widetilde{x},\xi)+b_{2,\varepsilon}(x,\widetilde{x},\xi), \end{aligned}$$

where

$$b_{1,\varepsilon}(x,\widetilde{x},\xi)=\omega\left(\frac{|x-\widetilde{x}|}{\varepsilon\lambda(x,\widetilde{x},\xi)}\right)b(x,\widetilde{x},\xi)$$

and

$$b_{2,\varepsilon}(x,\widetilde{x},\xi)=\left[1-\omega\left(\frac{|x-\widetilde{x}|}{\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\right]b(x,\widetilde{x},\xi).$$

We have $K_h(x,\widetilde{x})=K_{1,h}^{\varepsilon}(x,\widetilde{x})+K_{2,h}^{\varepsilon}(x,\widetilde{x}),$ where

$$K_{1,h}^{\varepsilon}(x,\widetilde{x})=\int_{\mathbb{R}^n}e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}b_{1,\varepsilon}(x,\widetilde{x},\xi)\,\widehat{d\xi},$$

and

$$K_{2,h}^{\varepsilon}(x,\widetilde{x})=\int_{\mathbb{R}^n}e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}b_{2,\varepsilon}(x,\widetilde{x},\xi)\,\widehat{d\xi}.$$

We will separately study the kernels $K_{1,h}^{\varepsilon}$ and $K_{2,h}^{\varepsilon}$.

On the support of $b_{2,\varepsilon}$, inequality (5.5) is satisfied and we have

$$K_{2,h}^{\varepsilon}\in S\left(\mathbb{R}^{2n}\right).$$

Let

$$L_h=-ih\left|\partial_\xi S(x,\xi)-\partial_\xi S(\widetilde{x},\xi)\right|^{-2}\sum_{l=1}^{n}\left[\frac{\partial S}{\partial\xi_l}(x,\xi)-\frac{\partial S}{\partial\xi_l}(\widetilde{x},\xi)\right]\frac{\partial}{\partial\xi_l}$$

using the oscillatory integral method to prove that L_h is a differential operator of order 1 such that

$$L_h\left(e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}\right)=e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}.$$

The transpose operator of L_h is

$${}^tL_h=\sum_{l=1}^{n}F_{h,l}(x,\widetilde{x},\xi)\frac{\partial}{\partial\xi_l}+G_h(x,\widetilde{x},\xi),$$

where

$$F_{h,l}(x,\widetilde{x},\xi)\in\Gamma_0^{-1}(\Omega_\varepsilon),\quad\forall l\in\{1,\dots,n\}$$

and

$$G_h(x,\widetilde{x},\xi)\in\Gamma_0^{-2}(\Omega_\varepsilon),$$

$$F_{h,l}(x,\widetilde{x},\xi)=-ih\left|(\partial_\xi S)(x,\xi)-(\partial_\xi S)(\widetilde{x},\xi)\right|^{-2}\left[\frac{\partial S}{\partial\xi_l}(x,\xi)-\frac{\partial S}{\partial\xi_l}(\widetilde{x},\xi)\right],$$

$$G_h(x,\widetilde{x},\xi) = -ih\sum_{l=1}^{n}\frac{\partial}{\partial\xi_l}\left[\left|(\partial_\xi S)(x,\xi)-(\partial_\xi S)(\widetilde{x},\xi)\right|^{-2}\left(\frac{\partial S}{\partial\xi_l}(x,\xi)-\frac{\partial S}{\partial\xi_l}(\widetilde{x},\xi)\right)\right],$$

$$\Omega_\varepsilon = \left\{(x,\widetilde{x},\xi)\in\mathbb{R}^{3n} : (\partial_\xi S)(x,\xi)-(\partial_\xi S)(\widetilde{x},\xi) > \frac{\varepsilon}{2C_5}\lambda(x,\widetilde{x},\xi)\right\}.$$

On the other hand, we prove by inducting on q that

$$\left({}^tL_h\right)^q b_{2,\varepsilon}(x,\widetilde{x},\xi) = \sum_{|\gamma|\le q,\gamma\in\mathbb{N}} g_{\gamma,q}(x,\widetilde{x},\xi)\,\partial_\xi^\gamma b_{2,\varepsilon}(x,\widetilde{x},\xi),$$

with $g_\gamma^{(q)}\in\Gamma_0^{-q}(\Omega_\varepsilon)$,

$$K_{2,h}^\varepsilon(x,\widetilde{x}) = \int_{\mathbb{R}^n} e^{ih^{-1}[S(x,\xi)-S(\widetilde{x},\xi)]}\left({}^tL_h\right)^q b_{2,\varepsilon}(x,\widetilde{x},\xi)\,\widehat{d\xi}.$$

Using Leibenitz's formula, (G_2) and the form $\left({}^tL_h\right)^q$, we can choose q large enough such that for all $\alpha,\alpha',\beta,\beta'\in\mathbb{N}^n, \exists C_{\alpha,\alpha',\beta,\beta'}>0$,

$$\sup_{x,\widetilde{x}\in\mathbb{R}^n}\left|x^\alpha\widetilde{x}^{\alpha'}\partial_x^\beta\partial_{\widetilde{x}}^{\beta'}K_{2,h}^\varepsilon(x,\widetilde{x})\right|\le C_{\alpha,\alpha',\beta,\beta'}.$$

Next, we study $K_{1,h}^\varepsilon$: this is more difficult and depends on the choice of the parameter ε. It follows from Taylor's formula that

$$S(x,\xi)-S(\widetilde{x},\xi) = \langle x-\widetilde{x},\zeta(x,\widetilde{x},\xi)\rangle$$

$$\zeta(x,\widetilde{x},\xi) = \int_0^1(\partial_x S)(x+t(\widetilde{x}-x),\xi)\,dt$$

We define the vectorial function

$$\widetilde{\zeta}_\varepsilon(x,\widetilde{x},\xi) = \omega\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\zeta(x,\widetilde{x},\xi)+\left(1-\omega\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\right)(\partial_x S)(\widetilde{x},\xi).$$

We have

$$\widetilde{\zeta}_\varepsilon(x,\widetilde{x},\xi) = \zeta(x,\widetilde{x},\xi),\ \text{on the supp } b_{1,\varepsilon}.$$

Moreover, for ε sufficiently small,

$$\lambda(x,\xi)\simeq\lambda(x,\widetilde{x},\xi),\ \text{on the supp } b_{1,\varepsilon}. \tag{5.6}$$

Let us consider the mappings

$$\mathbb{R}^{3n}\ni(x,\widetilde{x},\xi)\to\left(x,\widetilde{x},\widetilde{\zeta}_\varepsilon(x,\widetilde{x},\xi)\right), \tag{5.7}$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0\\ 0 & I_n & 0\\ \partial_x\widetilde{\zeta}_\varepsilon & \partial_{\widetilde{x}}\widetilde{\zeta}_\varepsilon & \partial_\xi\widetilde{\zeta}_\varepsilon\end{pmatrix}.$$

We have

$$\begin{aligned}\frac{\partial\widetilde{\zeta}_{j,\varepsilon}}{\partial\xi_i}(x,\widetilde{x},\xi) &= \frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)+\\ &+\omega\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\left(\frac{\partial\zeta_j}{\partial\xi_i}(x,\widetilde{x},\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right)-\\ &-\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\frac{\partial\lambda}{\partial\xi_i}(x,\widetilde{x},\xi)\,\lambda^{-1}(x,\widetilde{x},\xi)\,\omega'\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\left(\zeta_j(x,\widetilde{x},\xi)-\frac{\partial S}{\partial x_j}(\widetilde{x},\xi)\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\left|\frac{\partial\widetilde{\zeta}_{j,\varepsilon}}{\partial\xi_i}(x,\widetilde{x},\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right| &\leq \left|\omega\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\right|\left|\frac{\partial\zeta_j}{\partial\xi_i}(x,\widetilde{x},\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right|\\ &\quad+\lambda^{-1}(x,\widetilde{x},\xi)\left|\omega'\left(\frac{|x-\widetilde{x}|}{2\varepsilon\lambda(x,\widetilde{x},\xi)}\right)\right|\left|\zeta_j(x,\widetilde{x},\xi)-\frac{\partial S}{\partial x_j}(\widetilde{x},\xi)\right|.\end{aligned}$$

Now it follows from (G_2), (5.6), and Taylor's formula that

$$\begin{aligned}\left|\frac{\partial\zeta_j}{\partial\xi_i}(x,\widetilde{x},\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right| &\leq \int_0^1\left|\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right|dt \\ &\leq C_{10}|x-\widetilde{x}|\,\lambda^{-1}(x,\widetilde{x},\xi),\end{aligned} \tag{5.8}$$

with $C_{10}>0$.

$$\begin{aligned}\left|\zeta_j(x,\widetilde{x},\xi)-\frac{\partial S}{\partial x_j}(\widetilde{x},\xi)\right| &\leq \int_0^1\left|\frac{\partial S}{\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\xi)-\frac{\partial S}{\partial x_j}(\widetilde{x},\xi)\right|dt \\ &\leq C_{11}|x-\widetilde{x}|,\end{aligned} \tag{5.9}$$

with $C_{11}>0$.

From (5.8) and (5.9), there exists a positive constant $C_{12}>0$, such that

$$\left|\frac{\partial\widetilde{\zeta}_j}{\partial\xi_i}(x,\widetilde{x},\xi)-\frac{\partial^2 S}{\partial\xi_i\partial x_j}(\widetilde{x},\xi)\right|\leq C_{12}\varepsilon,\ \forall i,j\in\{1,\ldots,n\}. \tag{5.10}$$

If $\varepsilon<\frac{\delta_0}{2C}$, then (5.10) and (G_1) yields the estimate

$$\frac{\delta_0}{2}\leq-\widetilde{C}\varepsilon+\delta_0\leq-\widetilde{C}\varepsilon+\det\frac{\partial^2 S}{\partial\xi\partial x}\leq\det\partial_\xi\widetilde{\zeta}_\varepsilon(x,\widetilde{x},\xi). \tag{5.11}$$

with $\widetilde{C}>0$. If ε is such that (5.6) and (5.11) hold, then the mapping given in (5.7) is a diffeomorphism of $\mathbb{R}^{3n}$. Hence, there exists a mapping

$$\xi:\mathbb{R}^{3n}\ni(x,\widetilde{x},\zeta)\to\xi(x,\widetilde{x},\zeta)\in\mathbb{R}^n,$$

such that

$$\begin{gathered}\widetilde{\zeta}_{\varepsilon}\left(x, \widetilde{x}, \xi\left(x, \widetilde{x}, \zeta\right)\right)=\zeta . \\ \xi\left(x, \widetilde{x}, \widetilde{\zeta}_{\varepsilon}\left(x, \widetilde{x}, \xi\right)\right)=x . \\ \partial^{\alpha} \xi\left(x, \widetilde{x}, \zeta\right)=o\left(1\right), \ \forall \alpha \in \mathbb{N}^{3n} \backslash\left\{0\right\} .\end{gathered} \tag{5.12}$$

If we change the variable ζ by $\xi\left(x, \widetilde{x}, \zeta\right)$ in $K_{1,h}^{\varepsilon}\left(x, \widetilde{x}\right)$, we obtain

$$K_{1,h}^{\varepsilon}\left(x, \widetilde{x}\right)=\int e^{ih^{-1}\left\langle x-\widetilde{x}, \zeta\right\rangle} b_{1,\varepsilon}\left(x, \widetilde{x}, \xi\left(x, \widetilde{x}, \zeta\right)\right)\left|\det \frac{\partial \xi}{\partial \zeta}\left(x, \widetilde{x}, \zeta\right)\right| \widehat{d\zeta} . \tag{5.13}$$

From (5.12) we have, for $k \in\{0,1\}$, that $b_{1,\varepsilon}\left(x, \widetilde{x}, \xi\left(x, \widetilde{x}, \zeta\right)\right)\left|\det \frac{\partial \xi}{\partial \zeta}\left(x, \widetilde{x}, \zeta\right)\right|$ belongs to $\Gamma_{k}^{2m}\left(\mathbb{R}^{2n}\right)$.

Applying the stationary phase theorem to (5.13), we obtain the expressing of the symbol of the h-admissible operator $F_{h} F_{h}^{*}$,

$$\sigma\left(F_{h} F_{h}^{*}\right)=b_{1,\varepsilon}\left(x, \widetilde{x}, \xi\left(x, \widetilde{x}, \zeta\right)\right)\left|\det \frac{\partial \xi}{\partial \zeta}\left(x, \widetilde{x}, \zeta\right)\right|_{x=\widetilde{x}}+R\left(x, \zeta\right),$$

where $R\left(x, \zeta\right)$ belongs to $\Gamma_{k}^{2m-2}\left(\mathbb{R}^{2n}\right)$ if $a \in \Gamma_{k}^{m}\left(\mathbb{R}^{2n}\right)$, $k \in\{0,1\}$. For $x=\widetilde{x}$, we have

$$b_{1,\varepsilon}\left(x, x, \xi\left(x, x, \zeta\right)\right)=\left|a\left(x, \xi\left(x, x, \zeta\right)\right)\right|^{2},$$

where $\xi\left(x, x, \zeta\right)$ is the inverse of the mapping

$$\xi \rightarrow \partial_{x} S\left(x, \xi\right)=\zeta .$$

Thus,

$$\sigma\left(F_{h} F_{h}^{*}\right)\left(x, \partial_{x} S\left(x, \xi\right)\right) \equiv\left|a\left(x, \xi\right)\right|^{2}\left|\det \frac{\partial^{2} S}{\partial \xi \partial x}\left(x, \xi\right)\right|^{-1} .$$

From (5.2) and (5.3), we obtain the expression of $F_{h}^{*} F_{h}$:
$\forall v \in S\left(\mathbb{R}^{n}\right)$

$$\begin{aligned}\left(\mathcal{F}_{h}\left(F_{h}^{*} F_{h}\right) \mathcal{F}_{h}^{-1}\right) v\left(\xi\right) & =\int_{\mathbb{R}^{n}} e^{-ih^{-1} S(x, \xi)} \overline{a}\left(x, \xi\right)\left(F_{h}\left(\mathcal{F}_{h}^{-1} v\right)\right)\left(x\right) dx . \\ & =\int_{\mathbb{R}^{n}} e^{-ih^{-1} S(x, \xi)} \overline{a}\left(x, \xi\right)\left(\int_{\mathbb{R}^{n}} e^{-ih^{-1} S\left(x, \widetilde{\xi}\right)} a\left(x, \widetilde{\xi}\right) v\left(\widetilde{\xi}\right) \widehat{d\widetilde{\xi}}\right) dx . \\ & =\iint_{\mathbb{R}^{2n}} e^{-ih^{-1} S(x, \xi)} \overline{a}\left(x, \xi\right) e^{-ih^{-1} S\left(x, \widetilde{\xi}\right)} a\left(x, \widetilde{\xi}\right)\left(F_{h}\left(\mathcal{F}_{h}^{-1} v\right)\right)\left(\widetilde{\xi}\right) \widehat{d\widetilde{\xi}} dx . \\ & =\iint_{\mathbb{R}^{2n}} e^{-ih^{-1}\left[S(x, \xi)-S\left(x, \widetilde{\xi}\right)\right]} \overline{a}\left(x, \xi\right) a\left(x, \widetilde{\xi}\right) v\left(\widetilde{\xi}\right) \widehat{d\widetilde{\xi}} dx . \\ & =\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n}} e^{-ih^{-1}\left[S(x, \xi)-S\left(x, \widetilde{\xi}\right)\right]} \overline{a}\left(x, \xi\right) a\left(x, \widetilde{\xi}\right) \widehat{dx}\right) v\left(\widetilde{\xi}\right) d\widetilde{\xi} .\end{aligned}$$

Hence, the distribution kernel of the integral operator $\mathcal{F}_{h}\left(F_{h}^{*} F_{h}\right) \mathcal{F}_{h}^{-1}$ is

$$\widetilde{K}_h\left(\xi,\widetilde{\xi}\right)=\int e^{-ih^{-1}\left[S(x,\xi)-S\left(x,\widetilde{\xi}\right)\right]}\overline{a}\left(x,\xi\right)a\left(x,\widetilde{\xi}\right)\widehat{dx}.$$

We remark that we can deduce $\widetilde{K}_h\left(\xi,\widetilde{\xi}\right)=\widetilde{K}_h\left(x,\widetilde{x}\right)$ by replacing x by ξ. On the other hand, all assumptions used here are symmetrical on x and ξ; therefore, $\mathcal{F}_h\left(F_h^*F_h\right)\mathcal{F}_h^{-1}$ is a h-admissible operator with symbol

$$\sigma\left(F_h^*F_h\right)\left(\partial_\xi S\left(x,\xi\right),\xi\right)=\left|a\left(x,\xi\right)\right|^2\left|\det\frac{\partial^2 S}{\partial\xi\partial x}\left(x,\xi\right)\right|^{-1}.$$

□

Theorem 5.2 *Let* F_h *be the integral operator with the distribution kernel*

$$K_h\left(x,y\right)=\int_{\mathbb{R}^n}e^{ih^{-1}\left[S(x,\xi)-\langle y,\xi\rangle\right]}a\left(x,y\right)\widehat{d\xi},$$

where $a\in\Gamma_0^m\left(\mathbb{R}^{2n}\right)$ *and* S *satisfies* (G_1) *and* (G_2). *Then,*

For any m *such that* $m\leq 0$, F_h *can be extended as a bounded linear mapping on Bessel potential space.*

Proof It follows from Proposition 5.1 that

$F_h^*F_h$ is a pseudodifferfrential operator with symbol in $\Gamma_0^{2m}\left(\mathbb{R}^{2n}\right)$.

When $m\leq 0$, we can apply inequality (3.1) in the corollary (3.6) for $F_h^*F_h$ and obtain the existence of a positive constant C_s and $k\in\mathbb{N}$ such that

$$\left\|F_h^*F_h\right\|_{\mathcal{L}(H^s(\mathbb{R}^n))}\leq C_s\left|\sigma\left(F_h^*F_h\right)\right|_k^{0,0}<+\infty$$

Hence, for all $u\in S\left(\mathbb{R}^n\right)$,

$$\begin{aligned}\left\|F_hu\right\|^2_{H^s(\mathbb{R}^n)} &= \left\langle F_hu,F_hu\right\rangle_{H^s(\mathbb{R}^n)},\\ &= \left\langle u,F_h^*F_hu\right\rangle_{H^s(\mathbb{R}^n)},\\ &\leq \left\|u\right\|_{H^s(\mathbb{R}^n)}\left\|F_h^*F_hu\right\|_{H^s(\mathbb{R}^n)},\\ &\leq \left\|u\right\|_{H^s(\mathbb{R}^n)}\left\|F_h^*F_h\right\|_{\mathcal{L}(H^s(\mathbb{R}^n))}\left\|u\right\|_{H^s(\mathbb{R}^n)},\\ &\leq \left\|F_h^*F_h\right\|_{\mathcal{L}(H^s(\mathbb{R}^n))}\left\|u\right\|^2_{H^s(\mathbb{R}^n)}\end{aligned}$$

so

$$\begin{aligned}\left\|F_hu\right\|_{H^s(\mathbb{R}^n)} &\leq \left\|F_h^*F_h\right\|^{\frac{1}{2}}_{\mathcal{L}(H^s(\mathbb{R}^n))}\left\|u\right\|_{H^s(\mathbb{R}^n)},\\ &\leq \left(C_s\left|\sigma\left(F_h^*F_h\right)\right|_k^{0,0}\right)^{\frac{1}{2}}\left\|u\right\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Finally, F_h is a bounded linear operator on Bessel potential space.

□

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