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On ps-Drazin inverses in a ring

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Abstract: An element a in a ring R has a ps-Drazin inverse if there exists $b \in comm^2(a)$ such that $b = bab, (a-ab)^k \in J(R)$ for some $k \in \mathbb{N}$. Elementary properties of ps-Drazin inverses in a ring are investigated here. We prove that $a \in R$ has a ps-Drazin inverse if and only if a has a generalized Drazin inverse and $(a-a^2)^k \in J(R)$ for some $k \in \mathbb{N}$. We show Cline's formula and Jacobson's lemma for ps-Drazin inverses. The additive properties of ps-Drazin inverses in a Banach algebra are obtained. Moreover, we completely determine when a 2×2 matrix $A \in M_2(R)$ over a local ring R has a ps-Drazin inverse.

Key words: Generalized Drazin inverse, Cline's formula, Jacobson's lemma, 2×2 matrix, local ring

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$.

Recall that a has an s-Drazin inverse if there exists $b \in R$ such that

$$b = bab,\ b \in comm^2(a),\ a - ab \in N(R)$$

and a has a gs-Drazin inverse if there exists $b \in R$ such that

$$b = bab,\ b \in comm^2(a),\ a - ab \in R^{qnil},$$

where $N(R)$ and R^{qnil} denote the set of all nilpotent and quasinilpotents in R, respectively (see [4, 8]). The motivation of this paper is to introduce and study a kind of generalized inverses in a ring with respect to its Jacobson radical. An element a in a ring R has a ps-Drazin inverse if there exists $b \in comm^2(a)$ such that $b = bab$, $(a-ab)^k \in J(R)$. We easily see that the ps-Drazin inverse is located between the s-Drazin and gs-Drazin inverses. It is easy to see that an element having a ps-Drazin inverse has a pseudo-Drazin inverse (see [9]).

An element a in a ring R has a generalized Drazin inverse if there exists $b \in comm^2(a)$ such that $b = bab$, $a - a^2b \in R^{qnil}$. The preceding b is unique, if such an element exists, and is called the generalized Drazin inverse of a, denoting b by a^d. We always use a^π to define $1 - aa^d$. In Section 2, we investigate the

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elementary properties of ps-Drazin inverses in a ring. It is shown that $a \in R$ has a ps-Drazin inverse if and only if a has a generalized Drazin inverse and $(a-a^2)^k \in J(R)$ for some $k \in \mathbb{N}$.

In Section 3, we are concerned with Cline's formula and Jacobson's lemma for ps-Drazin inverses. Let $a, b \in R$. We prove that ab has a ps-Drazin inverse if and only if ba has a ps-Drazin inverse, and $1-ab \in R$ has a ps-Drazin inverse if and only if $1-ba \in R$ has a ps-Drazin inverse.

In Section 4, we investigate additive properties of ps-Drazin inverses in a Banach algebra. Let A be a Banach algebra. If $a, b \in A$ have ps-Drazin inverses and satisfy

$$a = ab^{\pi},\ b^{\pi}ba^{\pi} = b^{\pi}b,\ b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab,$$

we prove that $a+b$ has a ps-Drazin inverse.

Finally, in the last section, we completely determine 2×2 matrices over a local ring having a ps-Drazin inverse. Let R be a local ring, and let $A \in M_2(R)$. Then A has a ps-Drazin inverse if and only if $A = N+W$ with $N \in N(M_2(R))$, $W \in M_2(J(R))$, or $A = I_2+N+W$ with $N \in N(M_2(R))$, $W \in M_2(J(R))$, or A is similar to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $l_a - r_b,\ l_b - r_a$ are injective and $a \in 1+J(R),\ b \in J(R)$. Explicit results are obtained for 2×2 matrices over cobleached local rings.

Throughout this paper, all rings are associative with an identity. We use $J(R)$ and $U(R)$ to denote the Jacobson radical of R and the set of all units in R, respectively. $GL_2(R)$ denotes the sets of all 2×2 invertible matrices over R. $\mathbb{N}$ stands for the set of all natural numbers.

2. ps-Drazin inverses

In this section, we investigate elementary properties of ps-Drazin inverses. We begin with the following characterization.

Theorem 2.1 *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *a has a ps-Drazin inverse.*

(2) *There exists an idempotent $e \in comm^2(a)$ such that $(a-e)^k \in J(R)$ for some $k \in \mathbb{N}$.*

(3) *There exists a unique idempotent $e \in comm(a)$ such that $(a-e)^k \in J(R)$ for some $k \in \mathbb{N}$.*

Proof (1) $\Rightarrow$ (2) Assume that a has a ps-Drazin inverse. Then there exists $b \in comm^2(a)$ such that $b = b^2a$ and $(a-ab)^k \in J(R)$ for some $k \in \mathbb{N}$. Consider $e = ba$. Thus, we get the result.

(2) $\Rightarrow$ (1) Let $e^2 = e \in comm^2(a)$ such that $(a-e)^k \in J(R)$ for some $k \in \mathbb{N}$. Then $a+1-e \in U(R)$. Take $b = (a+1-e)^{-1}e$. Clearly, $b \in comm^2(a)$ since $e \in comm^2(a)$. Also, $a-ab = a-a(a+1-e)^{-1}e = a-(a+1-e)^{-1}(a+1-e)e = a-e$. Then $(a-ab)^k \in J(R)$. Lastly, $b^2a = (a+1-e)^{-2}ea = (a+1-e)^{-2}e(a+1-e) = b$ as asserted.

(2) $\Rightarrow$ (3) Assume that e and f satisfy the condition. Then $(a-e)^k \in J(R)$ and $(a-f)^k \in J(R)$ for some $k \in \mathbb{N}$. Hence, $((e-f)^2)^{2k+1} = (((a-e)-(a-f))^2)^{2k+1} \in J(R)$. Thus, $(e-f)^2 \in J(R)$ and so $1-(e-f)^2 \in U(R)$. Since $(e-f)^3 = e-f$, $(e-f)(1-(e-f)^2) = 0$. Therefore, $e-f = 0$, which implies $e = f$. Furthermore, clearly $e \in comm^2(a)$ implies that $e \in comm(a)$.

(3) $\Rightarrow$ (2) To see $e \in comm^2(a)$, let $ax = xa$. Assume that $p = e+(1-e)xe$. Then $p^2 = p \in comm(a)$

since $x \in comm(a)$ and $e \in comm(a)$. Furthermore, $(a-p)^{2k+1} = ((a-e)-(1-e)xe)^{2k+1} \in J(R)$ since $(a-e)^k \in J(R)$. Since e is unique, $e = p$. Hence, $xe = exe$. Similarly, $ex = exe$ and so $ex = xe$ as asserted. □

Corollary 2.2 *Let A be a Banach algebra, and let $a \in A$. Then the following are equivalent:*

(1) *a has a ps-Drazin inverse.*

(2) *There exists an idempotent $e \in comm(a)$ such that $(a-e)^k \in J(R)$ for some $k \in \mathbb{N}$.*

Proof (1) $\Rightarrow$ (2) This is obvious, by Theorem 2.1.

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $e \in comm(a)$ such that $(a-e)^k \in J(R)$ for some $k \in \mathbb{N}$. Suppose that $(a-f)^m \in J(R)$ for $f^2 = f \in comm(a)$ and $m \in \mathbb{N}$. Set $u = 1 + (a-e)$. Then $a = -(1-e) + u$, and so $ue = ae$. Hence, $e = u^{-1}ae$. Thus,

$$\begin{aligned} (1-e)fe &= (1-e)f(u^{-1}ae)^n \\ &= (1-e)fu^{-n}ea^n \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $fe = efe$. Likewise, $ef = efe$, and so $ef = fe$. It follows that $(e-f)^{k+m+1} = ((a-f)-(a-e))^{k+m+1} \in J(R)$; hence, $(e-f)^{2n} \in J(R)$ for some odd $n \in \mathbb{N}$. As $(e-f)^3 = e-f$, we see that $(e-f)(1-(e-f)^2) = 0$, and so $(e-f)(1-(e-f)^{2n}) = 0$. As $1-(e-f)^{2n} \in U(R)$, we deduce that $e = f$. According to Theorem 2.1, we complete the proof. □

Theorem 2.3 *Let R be a ring, and let $a \in R$. Then a has a ps-Drazin inverse if and only if:*

(1) *a has a generalized Drazin inverse;*

(2) *$(a-a^2)^k \in J(R)$ for some $k \in \mathbb{N}$.*

Proof $\Longrightarrow$ Let a have a ps-Drazin inverse. In view of Theorem 2.1, it has a p-Drazin inverse, and so a has a generalized Drazin inverse. Moreover, there exists $e^2 = e \in comm^2(a)$ such that $(a-e)^k \in J(R)$ by Theorem 2.1. Let $w = a-e$. Then $a = e+w$, and so $a-a^2 = (1-2e-w)w$. Therefore, $(a-a^2)^k = (1-2e-w)^k w^k \in J(R)$.

$\Longleftarrow$ Since a has a generalized Drazin inverse, we can find some $e^2 = e \in comm^2(a)$ such that $a-e \in U(R)$ and $ae \in R^{qnil}$. Since $(a-a^2)^k \in J(R)$, $\overline{a-a^2} \in N(R/J(R))$. In light of [10, Lemma 3.5], we can find $f \in \mathbb{Z}[a]$ such that $\overline{a-f} \in N(R/J(R))$ and $f-f^2 \in J(R)$. As $fa = af$, we see that $ef = fe$, and so $\overline{e-f} = \overline{(a-f)} - \overline{(a-e)} \in U(R/J(R))$. Clearly, $(\overline{e-f})^3 = \overline{e-f}$, and so $(\overline{e-f})^2 = \overline{1}$. Therefore, $f-(1-e) \in J(R)$. As $(a-f)^k \in J(R)$ for some $k \in \mathbb{N}$, we have $(a-(1-e))^{2k+1} = ((a-f)-((1-e)-f))^{2k+1} \in J(R)$. This completes the proof. □

The following shows that condition (2) of Theorem 2.3 cannot be superfluous.

Example 2.4 *Let $R = \mathbb{Z}_3$. Then $\overline{2} \in R$ has a generalized Drazin inverse. Since R is finite, every element in R has a generalized Drazin inverse, but $\overline{2}$ does not have a ps-Drazin inverse. In fact, $\overline{2} - \overline{2}^2 = \overline{1} \notin J(R)$.*

3. Cline's formula and Jacobson's lemma

In [6, Theorem 2.1], Liao et al. proved Cline's formula for generalized Drazin inverses. It was proved that ab has a generalized Drazin inverse if and only if ba has a generalized Drazin inverse. The aim of this section is to generalize Cline's formula from generalized Drazin inverses to ps-Drazin inverses.

Theorem 3.1 *Let R be a ring, and let $a, b \in R$. Then ab has a ps-Drazin inverse if and only if ba has a ps-Drazin inverse.*

Proof Suppose that $ab \in R$ has a ps-Drazin inverse. In view of Theorem 2.3, $ab \in R$ has a generalized Drazin inverse and $(ab-(ab)^2)^k \in J(R)$. According to [6, Theorem 2.1], $ba \in R$ has a generalized Drazin inverse. It is easy to check that

$$(ba-(ba)^2)^{k+1} = b(ab-(ab)^2)^k(a-aba) \in J(R).$$

Therefore, $ba \in R$ has a ps-Drazin inverse by Theorem 2.3. The converse is symmetric. □

Corollary 3.2 *Let R be a ring, let $k \in \mathbb{N}$, and let $a, b \in R$. If $(ab)^k$ has a ps-Drazin inverse, then so does $(ba)^k$.*

Proof In case of $k = 1$, the proof is clear by Theorem 3.1. We can assume that $k \geq 2$ and $(ab)^k$ has a ps-Drazin inverse. Since $(ab)^k = ab(ab)^{k-1}$, $b(ab)^{k-1}a = (ba)^k$ has a ps-Drazin inverse by Theorem 3.1. □

Theorem 3.3 *Let R be a ring, $a, b \in R$, and $ab = ba$. If a, b have ps-Drazin inverses, then ab has a ps-Drazin inverse.*

Proof Since a, b have ps-Drazin inverses, it follows by Theorem 2.3 that a, b have generalized Drazin inverses, and

$$(a-a^2)^m,\ (b-b^2)^n \in J(R) \text{ for some } m, n \in \mathbb{N}.$$

We check that

$$(ab-(ab)^2)^{m+n+1} = ((a-a^2)b + a^2(b-b^2))^{m+n+1} \in J(R).$$

Therefore, ab has a ps-Drazin inverse by Theorem 2.3. □

As an immediate consequence of Theorem 3.3, we now derive the following:

Corollary 3.4 *Let R be a ring, and let $a \in R$. If $a \in A$ has a ps-Drazin inverse, then a^n has a ps-Drazin inverse.*

Jacobson's lemma was initially a statement for the classical inverse in a ring. It claims that $1-ab$ is invertible if and only if $1-ba$ is invertible. It was then extended to inner inverses, group inverses, and Drazin inverses. More recently, it was generalized to generalized Drazin inverses. That is, $1-ab$ has a generalized Drazin inverse if and only if $1-ba$ has a generalized Drazin inverse (see [11, Theorem 2.3]).

Theorem 3.5 *Let R be a ring, and let $a, b \in R$. Then $1-ab$ has a ps-Drazin inverse if and only if $1-ba$ has a ps-Drazin inverse.*

Proof Suppose that $1-ab$ has a ps-Drazin inverse. Then $1-ab$ has a generalized Drazin inverse and

$$((1-ab)-(1-ab)^2)^k \in J(R)$$

for some $k \in \mathbb{N}$. In light of [11, Theorem 2.1], $1-ba \in R$ has a generalized Drazin inverse. Moreover, we see that

$$((1-ba)-(1-ba)^2)^{k+1} \in J(R).$$

According to Theorem 2.3, $1-ba$ has a ps-Drazin inverse.

The converse is symmetric. □

Corollary 3.6 *Let R be a ring, let $n \in \mathbb{N}$, and let $a,b \in R$. Then $(1-ab)^n$ has a ps-Drazin inverse if and only if $(1-ba)^n$ has a ps-Drazin inverse.*

Proof Suppose $(1-ab)^n$ has a ps-Drazin inverse. We verify that

$$\begin{array}{rcl} (1-ab)^n & = & 1-a(1+(1-ba)+\cdots+(1-ba)^{n-1})b; \\ (1-ba)^n & = & 1-b(1+(1-ab)+\cdots+(1-ab)^{n-1})a \\ & = & 1-(1+(1-ba)+\cdots+(1-ba)^{n-1})ba. \end{array}$$

In view of Theorem 3.5, $(1-ba)^n$ has a ps-Drazin inverse.

The converse is symmetric. □

4. Additive properties in Banach algebra

Let A be a complex Banach algebra with unit 1. In this section, we are concerned with the additive properties of ps-Drazin inverses in a Banach algebra A. Since every element having a ps-Drazin inverse in a ring has a generalized Drazin inverse, we easily check the following useful lemma.

Lemma 4.1 *Let A be a Banach algebra, $e^2 = e$, $b \in A$ and $eb = 0$. If b has ps-Drazin inverses, then:*

(1) $eb^d = 0$;

(2) $(be)^d = 0$;

(3) $(b(1-e))^d = b^d(1-e)$;

(4) $b(1-e)(b(1-e))^d = bb^d$.

Proof Straightforward. □

Lemma 4.2 *Let A be a Banach algebra, $a,b \in A$ and $ab = 0$. If a,b have ps-Drazin inverses, then $a+b$ has a ps-Drazin inverse.*

Proof Let $A = \begin{pmatrix} 1 \\ a \end{pmatrix}$ and $B = (b, 1)$. Set

$$C = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in M_{3\times 3}(A).$$

Then we observe that

$$CD = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 1 \\ 0 & 0 & a \end{pmatrix}, DC = \begin{pmatrix} a+b & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In view of Corollary 2.2, $a, b \in A$ have p-Drazin inverses. Thus, $\begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ has a p-Drazin inverse by [9, Theorem 5.3], and so it has a generalized Drazin inverse. By using Theorem 2.3, $\begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}$ has a ps-Drazin inverse. We then easily see that CD has ps-Drazin inverse. By virtue of Theorem 3.1, DC has a ps-Drazin inverse. Therefore, $a+b$ has a ps-Drazin inverse, as asserted. □

Theorem 4.3 *Let A be a Banach algebra, and let $a, b \in A$ have ps-Drazin inverses. If*

$$a = ab^{\pi},\ b = ba^{\pi},\ b^{\pi}aba^{\pi} = 0,$$

then $a+b$ has a ps-Drazin inverse.

Proof Since $a = ab^{\pi},\ b = ba^{\pi}$, we see that $ab^d = 0 = ba^d$. Moreover, we have $(1-bb^d)ab(1-aa^d) = 0$. Let $e = aa^d$ and $f = bb^d$. Then $eb = 0 = af$. Thus, we have $a = fa + (1-f)a$ and $b = b_1 + b_2$ where $b_1 = be$ and $b_2 = b(1-e)$. In view of Lemma 4.1, we have

$$b_2^d = b^d(1-e),\ b_2b_2^d = bb^d,$$
$$b_2^2b_2^d = b^2b^d,\ b_2(1-b_2b_2^d) = b(1-e)(1-f) = b(1-e-f).$$

We see that $a+b = x+y$, where $x = aa^d + b_1$ and $y = a - aa^d + b_2$.

Step 1. $xy = 0$. Since $eb = 0 = af$, we are done by Lemma 4.1.

Step 2. x has a ps-Drazin inverse. Clearly, aa^d, b_1 have ps-Drazin inverses. Moreover, $aa^db_1 = 0$. In light of Lemma 4.2, $x = aa^d + b_1$ has a ps-Drazin inverse.

Step 3. y has a ps-Drazin inverse. Set $u = (1-f)(a-aa^d) + b_2(1-b_2b_2^d)$ and $v = f(a-aa^d) + b_2^2b_2^d$. Then $y = (a-aa^d) + b_2 = u+v$. Since $af = 0$, we see that $uv = 0$. One can easily check that $(1-f)(a-aa^d)$, $b_2(1-b_2b_2^d)$, $v = f(a-aa^d)$, and $b_2^2b_2^d$ have ps-Drazin inverses. Moreover, $(1-f)(a-aa^d)b_2(1-b_2b_2^d) = f(a-aa^d)b_2^2b_2^d = 0$. According to Lemma 4.2, u, v have ps-Drazin inverses. It follows from Lemma 4.2 that $y = u+v$ has a ps-Drazin inverse.

By using Lemma 4.2 again, we complete the proof. □

Let $p \in A$ be an idempotent, and let $x \in A$. Then we write

$$x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p),$$

and we induce a representation given by the matrix

$$x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}_p,$$

and so we may regard such a matrix as an element in A. The following lemma is crucial.

Lemma 4.4 *Let A be a Banach algebra, let $a \in A$, and let*

$$x = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}_p,$$

relative to $p^2 = p \in A$. If $a \in pAp$ and $b \in (1-p)A(1-p)$ have ps-Drazin inverses, then so does x in A.

Proof In light of Corollary 2.2, $a, b \in A$ have p-Drazin inverses. In view of [9, Theorem 5.3], $x \in A$ has a p-Drazin inverse, and so it has a generalized Drazin inverse. By virtue of Theorem 2.3, $(a-a^2)^m \in J(pAp)$ and $(b-b^2)^n \in J((1-p)A(1-p))$. Hence,

$$(x-x^2)^{2(m+n)} = \begin{pmatrix} (a-a^2)^{m+n} & * \\ 0 & (b-b^2)^{m+n} \end{pmatrix}^2 \in J(M_2(A)).$$

Therefore, x has a ps-Drazin inverse by Theorem 2.3. □

Lemma 4.5 *Let A be a Banach algebra. Suppose that $a^k \in J(A)$ and $b \in A$ has a ps-Drazin inverse. If*

$$a = ab^{\pi}, b^{\pi}ba = b^{\pi}ab,$$

then $a+b$ has a ps-Drazin inverse.

Proof Since $b \in A$ has a ps-Drazin inverse, we can find $x \in comm^2(b)$ such that $x = x^2b$ and $(b-bx)^k \in J(A)$. Hence,

$$(b-b^2x)^k = (b-bx)^k(1-bx)^k \in J(A).$$

$p = 1-b^{\pi}$ and $b^{\pi} = 1-bx$. Then we have

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, a = \begin{pmatrix} a_{11} & a_1 \\ a_{21} & a_2 \end{pmatrix}_p$$

where $b_2^k = ((1-bx)b(1-bx))^k = (b-b^2x)^k \in J(A)$. From $a = ab^{\pi}$, we see that $a_{11} = a_{21} = 0$, and so

$$a+b = \begin{pmatrix} b_1 & a_1 \\ 0 & a_2+b_2 \end{pmatrix}_p.$$

It follows from $b^{\pi}ba = b^{\pi}ab$ that $a_2b_2 = b_2a_2$. Since $a^k \in J(A)$, we see that $a_2^k = (b^{\pi}ab^{\pi})^k = b^{\pi}a^k \in J(A)$. Thus, $(a_2+b_2)^{2k} \in J(A)$; hence, a_2+b_2 has a ps-Drazin inverse. Since b has a ps-Drazin inverse in A, we see that $b_1 = pbp = bp$ with $p^2 = p \in comm^2(b)$, and so b_1 has a ps-Drazin inverse. According to Lemma 4.4, $a+b$ has a ps-Drazin inverse. □

It is convenient at this stage to include the following additive property.

Theorem 4.6 *Let A be a Banach algebra. If $a, b \in A$ have ps-Drazin inverses and satisfy*

$$a = ab^{\pi},\ b^{\pi}ba^{\pi} = b^{\pi}b,\ b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab,$$

then $a+b$ has a ps-Drazin inverse.

Proof Since $b \in A$ has a ps-Drazin inverse, we can find $x \in comm^2(b)$ such that $x = x^2b$ and $(b-bx)^k \in J(A)$. As in the proof of Lemma 4.5, we have

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p, \ a = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}_p$$

where $p = 1 - b^{\pi}$ and $b^{\pi} = 1 - bx$. Hence, $b_2 = (1-p)b(1-p) = b(1-bx) = b - b^2x$, and so $b_2^k \in J(A)$. We see that

$$a + b = \begin{pmatrix} b_1 & a_1 \\ 0 & a_2 + b_2 \end{pmatrix}_p.$$

Since $b^{\pi}ba^{\pi} = b^{\pi}b$ and $b^{\pi}a^{\pi}ba = b^{\pi}a^{\pi}ab$, we see that $a_2^{\pi}a_2b_2 = a_2^{\pi}b_2a_2$ and $b_2 = b_2a_2^{\pi}$. In light of Lemma 4.5, $a_2 + b_2 \in (1-p)A(1-p)$ has a ps-Drazin inverse. Since $b_1 = pbp = b(bx)$ has a ps-Drazin inverse, it follows by Lemma 4.4 that $a + b$ has a ps-Drazin inverse. This completes the proof. □

5. Matrices over local rings

The goal of this section is to characterize the ps-Drazin inverse of a 2×2 matrix over local rings. We record the following:

Lemma 5.1 (see [3, Lemma 3.2]) *Let R be a local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) $A^k \in M_2(J(R))$ *for some* $k \in \mathbb{N}$.

(2) $A^2 \in M_2(J(R))$.

(3) $A = N + W$ *with* $N \in N(M_2(R))$, $W \in M_2(J(R))$.

Theorem 5.2 *Let R be a local ring, and let $A \in M_2(R)$. Then A has a ps-Drazin inverse if and only if:*

(1) $A = N + W$ *with* $N \in N(M_2(R))$, $W \in M_2(J(R))$*; or*

(2) $A = I_2 + N + W$ *with* $N \in N(M_2(R))$, $W \in M_2(J(R))$*; or*

(3) A *is similar to* $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, *where* $l_a - r_b$, $l_b - r_a$ *are injective and* $a \in 1 + J(R)$, $b \in J(R)$.

Proof $\Longrightarrow$ Since A has a ps-Drazin inverse, we may write $(A-E)^k \in M_2(J(R))$ for some $k \in \mathbb{N}$, where $E^2 = E \in comm^2(A)$. In light of Theorem 2.1, A has a p-Drazin inverse. By virtue of [3], we have the following:

Case 1. $A \in GL_2(R)$. Then $E \in GL_2(R)$, and so $E = I_2$. Hence, $(A - I_2)^k \in M_2(J(R))$. By virtue of Lemma 5.1, $A = I_2 + N + W$ with $N \in N(M_2(R))$, $W \in M_2(J(R))$.

Case 2. $A^2 \in M_2(J(R))$. Then $A = N + W$ with $N \in N(M_2(R))$, $W \in M_2(J(R))$, by Lemma 5.1.

Case 3. A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $l_{\lambda} - r_{\mu}$, $l_{\mu} - r_{\lambda}$ are injective and $\lambda \in U(R), \mu \in J(R)$. By virtue of Theorem 2.3 and Lemma 5.1,

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^2 \right)^2 \in M_2(J(R)),$$

and so $(\lambda - \lambda^2)^2 \in J(R)$. Hence, $\lambda \in 1 + J(R)$, as desired.

$\Longleftarrow$ If $A=N+W$ with $N\in N(M_2(R)),\ W\in M_2(J(R))$, then $A^2\in M_2(J(R))$ by Lemma 5.1. Then A has a ps-Drazin inverse.

If $A=I_2+N+W$ with $N\in N(M_2(R)),\ W\in M_2(J(R))$, then $(I_2-A)^2\in M_2(J(R))$ by Lemma 5.1, and so A has a ps-Drazin inverse.

If A is similar to $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, where $l_\lambda-r_\mu,\ l_\mu-r_\lambda$ are injective, where $\lambda\in 1+J(R),\ \mu\in J(R)$, then one easily checks that

$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}\lambda-1 & 0\\ 0 & \mu\end{pmatrix},$$

where $\begin{pmatrix}\lambda-1 & 0\\ 0 & \mu\end{pmatrix}\in M_2(J(R))$. Let $\begin{pmatrix}x & s\\ t & y\end{pmatrix}\in comm\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$. Then

$$\lambda s=s\mu \text{ and } \mu t=t\lambda.$$

Since $l_\lambda-r_\mu, l_\mu-r_\lambda$ are injective, $s=t=0$, and so

$$\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix}.$$

That is, $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\in comm^2\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$. This completes the proof. □

Corollary 5.3 *Let R be a commutative local ring, and let $A\in M_2(R)$. Then A has a ps-Drazin inverse if and only if:*

(1) $A=N+W$ *with* $N\in N(M_2(R)),\ W\in M_2(J(R))$*; or*

(2) $A=I_2+N+W$ *with* $N\in N(M_2(R)),\ W\in M_2(J(R))$*; or*

(3) A *is similar to* $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$, *where* $a\in 1+J(R),\ b\in J(R)$.

Proof Since R is commutative, we see that $l_a-r_b,\ l_b-r_a$ are injective for all $a\in 1+J(R), b\in J(R)$. This completes the proof, by Theorem 5.2. □

We come now to completely characterize ps-Drazin inverses in terms of the quadratic equations.

Theorem 5.4 *Let R be a cobleached local ring, and let $A\in M_2(R)$. Then A has a ps-Drazin inverse if and only if:*

(1) $A=N+W$ *with* $N\in N(M_2(R)),\ W\in M_2(J(R))$*; or*

(2) $A=I_2+N+W$ *with* $N\in N(M_2(R)),\ W\in M_2(J(R))$*; or*

(3) A *is similar to* $\begin{pmatrix}0 & \lambda\\ 1 & \mu\end{pmatrix}$, *where* $\lambda\in J(R),\ \mu\in U(R)$, *and the equation* $x^2-\mu x-\lambda=0$ *has a root in* $1+J(R)$ *and a root in* $J(R)$.

Proof $\Longrightarrow$ As in the proof of Theorem 5.2, we may assume

$$U^{-1}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}U=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for some $U\in GL_2(R)$. Write $U^{-1}=\begin{pmatrix} x & y \\ s & t \end{pmatrix}$. Then we have

$$\begin{aligned} y&=\alpha x;\\ x\lambda+y\mu&=\alpha y;\\ t&=\beta s;\\ s\lambda+t\mu&=\beta t. \end{aligned}$$

Thus, we see that $t\in J(R), y,s,x\in U(R)$.

Let $\delta=x^{-1}\alpha x$ and $\gamma=s^{-1}\beta s$. Then $\delta\in 1+J(R)$, $\gamma\in J(R)$. Moreover, $\delta^2-\delta\mu=\lambda$, and so $\delta^2-\delta\mu-\lambda=0$. Similarly, $\gamma^2-\gamma\mu=\lambda$, as needed.

$\Longleftarrow$ Suppose that the equation $x^2-x\mu-\lambda=0$ has a root $\alpha\in 1+J(R)$ and a root $\beta\in J(R)$. Then $\alpha^2=\alpha\mu+\lambda$; $\beta^2=\beta\mu+\lambda$. We see that

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}=\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix},$$

where

$$\begin{pmatrix} 1 & \alpha \\ 1 & \beta \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 1 & \beta-\alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\in GL_2(R).$$

Therefore, $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha\in 1+J(R)$ and $\beta\in J(R)$. By virtue of Theorem 5.2, A has a ps-Drazin inverse. □

Corollary 5.5 *Let R be a commutative local ring, and let $A\in M_2(R)$. Then A has a ps-Drazin inverse if and only if:*

(1) $A=N+W$ *with* $N\in N(M_2(R)), W\in M_2(J(R))$*; or*

(2) $A=I_2+N+W$ *with* $N\in N(M_2(R)), W\in M_2(J(R))$*; or*

(3) $x^2-tr(A)x+det(A)$ *has a root* $\alpha\in 1+J(R)$ *and a root* $\beta\in J(R)$.

Proof $\Longrightarrow$ By virtue of Theorem 5.4, we assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda\in J(R)$, $\mu\in U(R)$, and the equation $x^2-\mu x-\lambda=0$ has a root in $1+J(R)$ and a root in $J(R)$. Then $\lambda=-det(A)$ and $\mu=tr(A)$, as required.

$\Longleftarrow$ If (1) and (2) hold, then A has a ps-Drazin inverse by Theorem 5.2. If (3) holds, then $det(A)=\alpha\beta\in J(R)$ and $tr(A)=\alpha+\beta\in 1+J(R)$. Hence,

$$det(I_2-A)=1-tr(A)+det(A)\in J(R).$$

In view of [5, Lemma 2.4], A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R)$, $\mu \in 1+J(R)$. It follows that $\lambda = -det(A)$ and $tr(A) = \mu$, and so the equation $x^2 - \mu x - \lambda = 0$ has a root in $1+J(R)$ and a root in $J(R)$. This completes the proof, by Theorem 5.4. $\square$

Example 5.6 *Let* $\mathbb{Z}_{(2)} = \{\frac{m}{n} \mid m,n \in \mathbb{Z}, (m,n) = 1, 2 \nmid n\}$, *and let* $A = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. *Then* $A \in M_2(\mathbb{Z}_{(2)})$ *has a ps-Drazin inverse, but* B *does not.*

Proof Clearly, $\mathbb{Z}_{(2)}$ is a commutative local ring with $J(\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{(2)}$. As $tr(A) = 7$ and $det(A) = -8$, we see that the equation $x^2 - tr(A)x + det(A) = 0$ has a root $-1 \in 1 + J(\mathbb{Z}_{(2)})$ and a root $8 \in J(\mathbb{Z}_{(2)})$. According to Corollary 5.5, $A \in M_2(\mathbb{Z}_{(2)})$ has a ps-Drazin inverse. Clearly, B^2, $(I_2 - B)^2 \notin M_2(J(R))$. In light of Lemma 5.1, conditions (1) and (2) in Corollary 5.5 are not satisfied. Clearly, $tr(B) = 5$ and $det(B) = -2$. We take $\sigma : \mathbb{Z}_{(2)} \to \mathbb{Q}$ to be the natural map, and $p(x) = x^2 - 5x - 2 \in \mathbb{Z}_{(2)}[x]$. Clearly, $p^*(x) = x^2 - 5x - 2 \in \mathbb{Q}[x]$ is irreducible, and so $x^2 - 5x - 2 = 0$ is not solvable in $\mathbb{Z}_{(2)}$. By virtue of Corollary 5.5, $B \in M_2(\mathbb{Z}_{(2)})$ does not have a ps-Drazin inverse. $\square$

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