Turkish Journal of Mathematics

Volume 43 | Number 5 Article 2

1-1-2019

Further results on the join graph of a finite group

ZAHRA BAHRAMI

BIJAN TAERI

Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2097 – 2113

doi:10.3906/mat-1902-61

Further results on the join graph of a finite group

Zahra BAHRAMI, **Bijan TAERI***
Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

Received: 14.02.2019 • **Accepted/Published Online:** 20.06.2019 • **Final Version:** 28.09.2019

Abstract: Let G be a finite group which is not cyclic of prime power order. The join graph $\Delta(G)$ is an undirected simple whose vertices are the proper subgroups of G, which are not contained in the Frattini subgroup $\Phi(G)$ of G and two vertices H and K are joined by an edge if and only if $G = \langle H, K\rangle$. We classify finite groups whose join graphs have domination number ≤ 2 and independence number ≤ 3. We show that $\Delta(G) \cong \Delta(A_4)$ if and only if $G \cong A_4$. We also show that if the independence number of $\Delta(G)$ is less than 15, then G is solvable; moreover, if the equality holds and G is nonsolvable, then $G/\Phi(G) \cong A_5$.

Key words: Finite group, join graph, domination number, independence number, alternating group

1. Introduction and results

There are many ways to define a graph on a group. In this context, it is interesting to study the relation between the structure of the group and the structure of the graph. Bosák [6], in 1964, introduced the intersection graph of subsemigroups of a semigroup and Csákány and Pollák [8], in 1969, studied the intersection graph of subgroups of a finite group. Zelinka [21] studied the intersection graph of subgroups of a finite abelian group. Also, Shen [19] in 2009, classified finite groups with disconnected intersection graph of subgroups. Recall that the vertices of the intersection graph of a group G is the set of all proper nontrivial subgroups of G and two vertices H and K are joined by an edge, if and only if they have a nontrivial intersection, i.e. $H \cap K \neq 1$.

Note that in the lattice of subgroups $\mathscr{L}(G)$ of a group G, we have $H \cap K = \inf(H, K)$ and the identity subgroup $\{1\}$ is the smallest element of $\mathscr{L}(G)$. We can define the dual of intersection graph by considering $\sup(H, K) = \langle H, K\rangle$ and the largest element G of $\mathscr{L}(G)$. For a finite group G different from a cyclic group of prime power order, Ahmadi and the second author [1] defined an undirected simple graph $\Delta(G)$ whose vertices are the proper subgroups of G, which are not contained in the Frattini subgroup $\Phi(G)$ of G and two vertices H and K are joined by an edge if and only if $G = \langle H, K\rangle$. Note that when G is finite, the condition on subgroups avoids isolated vertices in $\Delta(G)$.

Let Δ be a graph with vertex set V. For a subset S of V, denote by $N(S)$ the set of vertices of Δ which are in S or adjacent to a member of S. If $N(S) = V$, then S is said to be a dominating set of vertices in Δ. The domination number of Δ, denoted by $\gamma(\Delta)$ is the minimum size of a dominating set of the vertices in Δ. A subset $\mathcal{A}$ of the vertices of Δ is called an independent set if the induced subgraph on $\mathcal{A}$ has no edges.

*Correspondence: b.taeri@cc.iut.ac.ir
2010 *AMS Mathematics Subject Classification:* 20D99, 05C25, 05C83

The maximum size of an independent set in a graph Δ is called the independence number of Δ and is denoted by $\alpha(\Delta)$.

Ahmadi and the second author [1] studied elementary properties of $\Delta(G)$ and showed that $\Delta(G)$ is connected and its clique number is equal to the number of maximal subgroups of G. They classified finite groups with planner join graph [2]. Moreover, they classified finite groups with regular join graph [3]. In this paper we continue the study of $\Delta(G)$. Ahmadi and the second author [1, Theorem 2.9] showed that finite groups whose join graphs have domination number 1 are solvable and classified finite p-groups, with $\gamma(\Delta(G)) = 1$. In this work we classify finite groups with $\gamma(\Delta(G)) \in \{1, 2\}$. As a corollary of this classification, we give a characterization of A_4 in terms of its join graph.

Theorem A *Let G be a finite group. Then $\Delta(G) \cong \Delta(A_4)$ if and only if $G \cong A_4$.*

It is shown that a finite group with $\alpha(\Delta(G)) \leq 7$ is solvable, see [1, Proposition 2.7]. We find a better and sharp bound and prove that every finite group with $\alpha(\Delta(G)) \leq 14$ is solvable.

Theorem B *If G is a finite group such that $\alpha(\Delta(G)) \leq 14$, then G is solvable.*

By using Theorem B, we obtain a characterization of A_5.

Corollary C *Let G be a finite nonsolvable group with $\Phi(G) = 1$. Then $\alpha(\Delta(G)) = 15$ if and only if $G \cong A_5$.*

Corollary D *If G is a finite nonsolvable group with $\Phi(G) \neq 1$ and $\alpha(\Delta(G)) = 15$, then $G/\Phi(G) \cong A_5$.*

In the last section we classify finite groups whose join graphs have independence number less than or equal to 3.

Throughout the paper we assume that all groups are finite different from a cyclic group of prime power order. For a group G, we denote by $\mathscr{M}(G)$ and $\pi(G)$, the set of all maximal subgroups of G and the set of all prime divisors of $|G|$, respectively. By $n(G)$ and $\mathrm{Syl}_p(G)$, where p is a prime divisor of $|G|$, we mean the number of all proper nontrivial subgroups of G and the set of all Sylow p-subgroups of G, respectively. The symbol $G = Y \rtimes X$ indicates that G is a split extension of a normal subgroup Y of G by a complement X. For elements $x, y \in G$, the conjugate of x by y is denoted by $x^y := y^{-1}xy$.

2. Finite groups with domination number less than three

Finite groups of prime power order with $\gamma(\Delta(G)) = 1$ are classified in [1, Theorem 2.9]. In this section we prove the Theorem A and classify finite groups with $\gamma(\Delta(G)) \in \{1, 2\}$.

We recall some well-known facts which play a fundamental role in this paper. Let G be a finite group. If $|\mathscr{M}(G)| \leq 2$, then G is cyclic. Moreover, if $|\mathscr{M}(G)| = 3$, then by [4, Lemma 1], G is a 2-group or a cyclic group. Every maximal subgroup of a finite nilpotent group is normal of prime index (see [17, Theorem 5.2.4]). Also, every maximal subgroup of a finite solvable group has prime power index (see [17, Theorem 5.4.3]).

Theorem 2.1 *Let G be a finite group, which is not of prime power order. Then $\gamma(\Delta(G)) = 1$ if and only if one of the following hold*

(1) $G \cong \mathbb{Z}_{p^n q}$,

(2) $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_{p^n}$, *where* $\mathbb{Z}_{p^n} \cong P \in \mathrm{Syl}_p(G)$ *is a maximal subgroup of* G *and* $|\Phi(G)| = p^{n-1}$,

(3) $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$,

where p *and* q *are distinct primes (note that (2) and (3) reduce to the same type if* $m = n = 1$*).*

Proof First let G be one of the groups of types (1)-(3). Thus, $S = \{P\}$ is a dominating set, where $P \in \mathrm{Syl}_p(G)$. Hence, $\gamma(\Delta(G)) = 1$.

Now to prove the converse suppose that $\gamma(\Delta(G)) = 1$. By [1, Theorem 2.9], G is solvable and there exists $P \in \mathrm{Syl}_p(G)$ such that P is a cyclic maximal subgroup of G and $\Phi(G)$ is a maximal subgroup of P. Let $|P| = p^n$. Then $|\Phi(G)| = p^{n-1}$.

If G is nilpotent, then $|G : P| = q$, where q is a prime distinct from p. Therefore, $|G| = p^n q$ and $G \cong P \times Q \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, where $Q \in \mathrm{Syl}_q(G)$, and thus G is of type (1).

Now suppose that G is nonnilpotent. The solvability of G implies that $|G : P| = q^m$, where q is a prime distinct from p, and so $|G| = p^n q^m$. We claim that order of each maximal subgroup of G is p^n or $p^{n-1}q^m$. Since P is a maximal subgroup of G, every Sylow p-subgroup is maximal. Thus, it is clear that G has no subgroup of order $p^n q^r$, for all $r \geq 1$. Let H be a maximal subgroup of G which is not a Sylow p-subgroup. Since G is solvable and $|\Phi(G)| = p^{n-1}$, $|G : H| = p$, so $|H| = p^{n-1}q^m$. Hence, all maximal subgroups of $\overline{G} = G/\Phi(G)$ are nilpotent. Since G is not nilpotent, $\overline{G}$ is nonnilpotent, so $\overline{G}$ is a minimal nonnilpotent group. Let $Q \in \mathrm{Syl}_q(G)$. Therefore, by Schmidt's Theorem [17, Theorem 9.1.9], we have the following cases:

Case (i). $\overline{G} \cong \overline{Q} \rtimes \overline{P}$. We have $Q\Phi(G) \trianglelefteq G$ and $Q \in \mathrm{Syl}_q(Q\Phi(G))$. By Frattini argument $G = N_G(Q)Q\Phi(G) = N_G(Q)$. Therefore, $Q \trianglelefteq G$, so $\Phi(Q) \leq \Phi(G)$. Since $|\Phi(G)| = p^{n-1}$ and $\Phi(Q)$ is a q-group, $\Phi(Q) = 1$; hence, Q is an elementary abelian q-group. Thus, G is of type (2).

Case (ii). $\overline{G} \cong \overline{P} \rtimes \overline{Q}$. Since $P\Phi(G) \trianglelefteq G$ and $\Phi(G) \leq P$, we have $P \trianglelefteq G$. Therefore, $|G : P| = q$, i.e., $m = 1$. Thus, G is of type (3). ■

Below are examples of each type of groups of Theorem 2.1 (SmallGroup(n, d) is the d'th group of order n in the GAP library [10]):

- The SmallGroup$(20, 1) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ and the alternating group A_4 are groups of type (2).
- The SmallGroup$(147, 1) = \mathbb{Z}_{49} \rtimes \mathbb{Z}_3$ and the Dihedral group $D_{2m} = \langle a,\ b \mid a^m = b^2 = 1,\ a^b = a^{-1}\rangle$ of order $2m$, where $m = p^n$ and p is a prime distinct from 2 are groups of type (3).

We denote the generalized quaternion group and the semidihedral group by Q_{2^n} and SD_{2^n}, respectively. Recall that $Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1,\ y^2 = x^{2^{n-2}},\ x^y = x^{-1}\rangle$, $n \geq 3$ and $SD_{2^n} = \langle x, a \mid x^2 = 1 = a^{2^{n-1}},\ a^x = a^{2^{n-2}-1}\rangle$, $n \geq 3$. For a subgroup H of G, we denote by $\overline{n}(H)$ the number of subgroups of H which are not contained in $\Phi(G)$. Now we obtain a characterization of A_4 by its join graph (see Figure 1).

Proof of Theorem A. If $G \cong A_4$, then it is clear that $\Delta(G) \cong \Delta(A_4)$. Now to prove the converse, let $\Delta(G) \cong \Delta(A_4)$. Then $\gamma(\Delta(G)) = \gamma(\Delta(A_4)) = 1$. We know that $|\mathscr{M}(A_4)| = 5$ and $A_4 \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$ is of type (2) of Theorem 2.1. Hence, by [1, Proposition 2.4], $|\mathscr{M}(G)| = 5$. If G is a p-group, then by [1, Theorem 2.9], G is isomorphic to one of the groups Q_{2^n}, SD_{2^n}, D_{2^n}, $\mathbb{Z}_{p^{n-1}} \times \mathbb{Z}_p$ or $\langle x, a \mid x^p = 1 = a^{p^{n-1}}, a^x = a^{1+p^{n-2}}\rangle$.

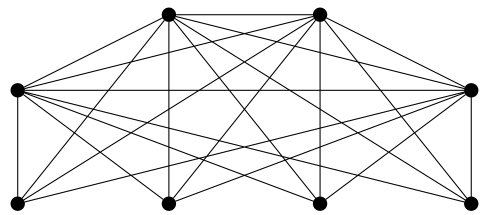

Figure 1. $\Delta(A_4)$.

The number of maximal subgroups of these groups are 3 or $p+1$, contradicting to $|\mathscr{M}(G)| = 5$. Thus, G is not p-group.
It is clear that G is not of type (1) of Theorem 2.1. Suppose, if possible, that G is of type (3), that is $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$. If $n = 1$, then by [1, Theorem 2.5], $\Delta(G)$ is complete, which is a contradiction. If $n \geq 2$, then $\Delta(G)$ has a unique vertex $P \cong \mathbb{Z}_{p^n}$ such that $N(P) = V(\Delta(G))$, but $\Delta(A_4)$ has 4 vertices with this property, which is a contradiction. Hence, G is of type (2), i.e. $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_{p^n}$, where $\mathbb{Z}_{p^n} \cong P \in \mathrm{Syl}_p(G)$ is a maximal subgroup of G and $|\Phi(G)| = p^{n-1}$. Since $4 = |\mathrm{Syl}_p(G)| = |G : N_G(P)| = |G : P| = q^m$, we conclude that $p = 3$, $q = 2$ and $m = 2$. Thus, $G \cong \mathbb{Z}_2^2 \rtimes \mathbb{Z}_3$, which completes the proof. ■

In the rest of this section we obtain a classification of finite groups G with $\gamma(\Delta(G)) = 2$.

Lemma 2.2 *Let G be a finite group. If $\gamma(\Delta(G)) = 2$, then there exists $M_1, M_2 \in \mathscr{M}(G)$ such that $\{M_1, M_2\}$ is a dominating set of $\Delta(G)$.*

Proof Let $S = \{A, B\}$ be a dominating set for $\Delta(G)$. We consider two following cases:
Case (i). There exists a maximal subgroup M_1 such that $A, B \leq M_1$. If $A < M_1$ and $B < M_1$, then since S is a dominating set, M_1 is adjacent either to A or B, which is a contradiction. Thus, $B = M_1$ and $A < M_1$, say. Hence, $\{A, M_1\}$ is a dominating set. Note that in this case A is a unique proper subgroup of M_1, which is not contained in $\Phi(G)$. Thus, there exists a maximal subgroup M_2 such that $A \nleq M_2$. Clearly the set $\{M_1, M_2\}$ is a dominating set.
Case (ii). There is no maximal subgroup of G containing A and B. In this case there exists distinct maximal subgroups M_1 and M_2 such that $A \leq M_1$ and $B \leq M_2$. It is clear that $M_1 \cap M_2$ is not adjacent to A and B and it is distinct from A and B. Since S is a dominating set, it follows that $M_1 \cap M_2 \leq \Phi(G)$. Thus, $M_1 \cap M_2 = \Phi(G)$. Put $S' = \{M_1, M_2\}$. Let H be a vertex of $\Delta(G)$. Since $M_1 \cap M_2 = \Phi(G)$ and $H \nleq \Phi(G)$, we conclude that $H \in S'$ or H is adjacent either to M_1 or M_2. Thus, we have dominating set $\{M_1, M_2\}$. This completes the proof. ■

Corollary 2.3 *If $\gamma(\Delta(G)) = 2$, then there exist maximal subgroups $M_1, M_2 \in \mathscr{M}(G)$ such that $\Phi(G) = M_1 \cap M_2$.*

Proof By Lemma 2.2, there exists a dominating set $\{M_1, M_2\}$, where $M_1, M_2 \in \mathscr{M}(G)$. Since $\{M_1, M_2\}$ is a dominating set and $M_1 \cap M_2$ is not adjacent to M_1 and M_2, we have $M_1 \cap M_2 \leq \Phi(G)$. Therefore, $\Phi(G) = M_1 \cap M_2$. ■

Suppose that $\Delta(G)$ is a regular graph. Then by [1, Proposition 3.2], we have $\Phi(G) = M_1 \cap M_2$, for every distinct $M_1, M_2 \in \mathscr{M}(G)$, and by Lemma 2.2, $\gamma(\Delta(G)) \leq 2$.

We need the following easy Lemma.

Lemma 2.4 *Let G be a finite noncyclic group. Then all proper subgroups of G are cyclic if and only if G is one of the following groups:*

(1) $\mathbb{Z}_p \times \mathbb{Z}_p$, *where p is a prime,*

(2) Q_8,

(3) $\mathbb{Z}_p \rtimes \mathbb{Z}_{q^m}$, *where p, q are distinct primes and $|\Phi(G)| = q^{m-1}$.*

Proof Suppose that all proper subgroups of G are cyclic. First suppose that $|G| = p^n$, where p is prime. Put $\overline{G} = G/\Phi(G)$ and let $M \in \mathscr{M}(G)$. Since M is cyclic, $\overline{M} = M/\Phi(G)$ is a cyclic maximal subgroup of order p^{m-1} of $\overline{G} \cong \mathbb{Z}_p^m$. Thus, $m = 2$, and so $|\Phi(G)| = p^{n-2}$. Since all maximal subgroups of G are cyclic and $|\Phi(G)| = p^{n-2}$, $\overline{n}(M) = 1$, which implies that $\Delta(G)$ is complete. By [1, Theorem 2.5], G is of type (1) or (2).

Now suppose that $|\pi(G)| \geq 2$. Since G is noncyclic and its Sylow subgroups are cyclic, it follows that G is not abelian. Therefore, G is minimal nonabelian, and so by [15], $G \cong \mathbb{Z}_p^n \rtimes \mathbb{Z}_{q^m}$, where p, q are distinct primes. By hypothesis, $n = 1$, and so $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^m}$. By [16], $\Phi(G) = \Phi(\mathbb{Z}_p) \times \langle x^q \rangle$, where $\mathbb{Z}_{q^m} = \langle x \rangle$. Therefore, $\Phi(G) \cong \mathbb{Z}_{q^{m-1}}$, and so $|\Phi(G)| = q^{m-1}$.

Conversely, if G is of type (1) or (2), then it is clear that every subgroup of G is cyclic. Now suppose that $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^m}$ and $|\Phi(G)| = q^{m-1}$. The order of maximal subgroups of G is q^m or pq^{m-1}. It is clear that maximal subgroups of order q^m are cyclic. Let M be a maximal subgroup of order pq^{m-1}. Thus, $M \cong \mathbb{Z}_p \times \Phi(G) \cong \mathbb{Z}_p \times \mathbb{Z}_{q^{m-1}}$, and so M is cyclic. Hence, all maximal subgroups of G are cyclic which implies that all proper subgroups of G are cyclic. ■

Clearly the domination number of the join graph of groups of types (1)-(3) in above Lemma is 1. Thus, the domination number of the join graph of a noncyclic group whose all subgroups are cyclic is 1.

Let M be a maximal subgroup of G. Then $\overline{n}(M) = 2$ if and only if $\{A, M\}$ is a dominating set, where A is a unique subgroup of M such that $A \not\leq \Phi(G)$. In the following Lemma we characterize finite groups with such dominating set.

Lemma 2.5 *Let $S = \{A, M\}$, where $M \in \mathscr{M}(G)$ and $A < M$, be a dominating set for $\Delta(G)$. Then $\gamma(\Delta(G)) = 2$ if and only if G is one of the following groups:*

(1) $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_{p^n}$, *where $|\Phi(G)| = p^{n-2}$ and $\mathbb{Z}_{p^n} \cong P \in \mathrm{Syl}_p(G)$ is a maximal subgroup,*

(2) $G \cong \mathbb{Z}_{p^n q^2}$, $n \geq 2$,

(3) $G \cong Q_8 \rtimes \mathbb{Z}_{p^n}$, *where $|\Phi(G)| = 2p^{n-1}$ and G has a cyclic maximal subgroup of order $2p^n$,*

(4) $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_{q^2}$, $n \geq 2$ *and $q \mid |\Phi(G)|$,*

where p, q are distinct primes.

Proof Since S is a dominating set, A is a unique proper subgroup of M, which is not contained in $\Phi(G)$. Thus, $|\mathscr{M}(M)| \leq 2$. We consider following cases:

Case (i). $|\mathscr{M}(M)| = 1$. In this case M is a cyclic Sylow p-subgroup of G. We have $\Phi(G) < A < M$ and $\mathscr{M}(M) = \{A\}$. Let $|M| = p^n$. Then $|\Phi(G)| = p^{n-2}$. If $M \trianglelefteq G$, then $\Phi(M) \leq \Phi(G)$, contradicting to $|\Phi(M)| = p^{n-1}$. Thus, M is not normal in G. Since M is a maximal subgroup of G, $M = N_G(M)$. We claim that G is p-nilpotent. Since M is an abelian Hall subgroup of G, by [18, Corollary 10.18], it is enough to show that every two distinct elements of M are not conjugate in G. Suppose, by contrary, that there exists distinct elements x and y of M which are conjugate in G. Thus, there exists $g \in G$ such that $y = x^g$. Put $A_1 = \{x\}$ and $A_2 = \{y\}$ so $A_2 = A_1^g$. Since M is abelian, $A_1^a = A_1$ and $A_2^a = A_2$, for every $a \in M$. By [11, Theorem 1.1, Chap. 7], A_1 and A_2 are conjugate in M, and so there exists $b \in M$ such that $y^b = x$. Since M is abelian, $x = y$. Thus, G is p-nilpotent. Since M is an abelian maximal subgroup, we see that G is solvable (see [17, Exercise 7, Page 309]). The solvability of G implies that $|G : M| = q^m$, where q is a prime distinct from p, so $|G| = p^n q^m$. Let $Q \in \mathrm{Syl}_q(G)$. Then $G \cong Q \rtimes M$. Since Q is normal in G, we have $\Phi(Q) \leq \Phi(G)$. Hence, $\Phi(Q) = 1$, and so Q is an elementary abelian q-group. Therefore, $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_{p^n}$ and clearly by Theorem 2.1, $\gamma(\Delta(G)) \neq 1$. Thus, G is of type (1).

Case (ii). $|\mathscr{M}(M)| = 2$. In this case $M \cong \mathbb{Z}_{p^n q^r}$ and $\mathscr{M}(M) = \{A, \Phi(G)\}$, where p and q are distinct primes. Without loss of generality we may assume that $|\Phi(G)| = p^{n-1}q^r$ and $|A| = p^n q^{r-1}$. If $H < A$, then $H < \Phi(G)$, and so $r = 1$. Hence, $|M| = p^n q$ and $|\Phi(G)| = p^{n-1}q$.

First suppose that G is nilpotent. Since $\pi(G) = \pi(G/\Phi(G))$, $|G : M| = q$ (see [17, Theorem 9.3.5]). Hence, $|G| = p^n q^2$. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Then $G \cong P \times Q$. Since Q is normal, we conclude that $\Phi(Q) = 1$ or q. If $\Phi(Q) = 1$, then Q is an elementary abelian q-group. Therefore, $G \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q^2$, and so $|\Phi(G)| = p^{n-1}$, a contradiction. Thus, $Q \cong \mathbb{Z}_{q^2}$ and we have $G \cong \mathbb{Z}_{p^n q^2}$. Thus, G is of type (2).

Now suppose that G is nonnilpotent. Note that G is solvable. Since $|G : M|$ is a power of prime r and $\pi(G) = \pi(G/\Phi(G))$, it follows that $r = q$. Thus, $|G| = p^n q^m$. We claim that all maximal subgroups of G have order $p^n q$ or $p^{n-1}q^m$. Let H be a maximal subgroup of G. Since $\Phi(G)$ is contained in every maximal subgroup, $p^{n-1}q$ divides $|H|$. Also, solvability of G implies that $|G : H| = p$ or power of q. Hence, $|H| = p^{n-1}q^m$ or $|H| = p^n q^s$. Suppose that $|H| = p^n q^s$. Since $\Phi(G)$ is cyclic of order $p^{n-1}q$, we can write $\Phi(G) = \langle a \rangle \times \langle b \rangle$, where the orders of a and b are p^{n-1} and q, respectively. Since $\Phi(G)$ is cyclic, $\langle b \rangle \trianglelefteq G$. On the other hand, we have $M = P \times \langle b \rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, where $P \in \mathrm{Syl}_p(G)$. Also $M^g = P^g \times \langle b \rangle$, for every $g \in G$. Since $|H| = p^n q^s$ and $\langle b \rangle \leq H$, it follows that there exists $g \in G$ such that $M^g \leq H$. Thus, $H = M^g$, and so $s = 1$. Let $\overline{G} = G/\Phi(G)$. Since G is nonnilpotent, $\overline{G}$ is nonnilpotent. All maximal subgroups of $\overline{G}$ are of orders p or q^{m-1}, and so are nilpotent. Therefore, $\overline{G}$ is minimal nonnilpotent. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. By Schmidt's Theorem [17, Theorem 9.1.9], we have the following cases:

Subcase (ii-a). $\overline{G} = \overline{Q} \rtimes \overline{P}$. Since $\overline{Q} \trianglelefteq \overline{G}$, it follows that $Q\Phi(G) \trianglelefteq G$. Since $Q \in \mathrm{Syl}_q(Q\Phi(G))$, by Frattini argument $G = N_G(Q)Q\Phi(G) = N_G(Q)$. Therefore, $G \cong Q \rtimes \mathbb{Z}_{p^n}$, where Q is an elementary abelian q-group or $|\Phi(Q)| = q$. First suppose that Q is an elementary abelian q-group. Hence, $G \cong \mathbb{Z}_q^m \rtimes \mathbb{Z}_{p^n}$. We know $\mathbb{Z}_{p^n}$ acts on $\mathbb{Z}_q^m$. Since $\Phi(G) = \langle a \rangle \times \langle b \rangle$ is cyclic, we have $\langle b \rangle$ is $\mathbb{Z}_{p^n}$-invariant. By [13, Theorem 8.4.6], there exists a $\mathbb{Z}_{p^n}$-invariant subgroup H of $\mathbb{Z}_q^m$ such that $\mathbb{Z}_q^m = H \times \langle b \rangle$. Hence, G has a maximal subgroup N of order $p^n q^{m-1}$ such that $H \leq N$. Since $\Phi(G) \leq N$, it follows that $\langle b \rangle \leq N$, and so $N = G$, a contradiction. Therefore, this case cannot happen.

Now suppose that $|\Phi(Q)| = q$. Suppose, if possible, that Q is cyclic. Since $|\Phi(Q)| = q$, $Q \cong \mathbb{Z}_{q^2}$. Hence, $G \cong \mathbb{Z}_{q^2} \rtimes \mathbb{Z}_{p^n}$. All maximal subgroups of G have orders $p^n q$ or $p^{n-1}q^2$. By hypothesis all maximal subgroups of order $p^n q$ are cyclic. Also since $\Phi(G)$ is cyclic of order $p^{n-1}q$, maximal subgroup of order $p^{n-1}q^2$ is cyclic. Thus, all maximal subgroups of G are cyclic, contradicting to Lemma 2.4.

Hence, Q is noncyclic. Note that $|Q| = q^m$ and $|\Phi(Q)| = q$. It is easy to see that $\Phi(Q)$ is the unique characteristic subgroup of Q. Now let $H = \langle a \in Q \mid a^q = 1 \rangle$, which is a characteristic subgroup of Q. Therefore, either $H = Q$ or $\Phi(Q) = H$. If $H = Q$, then Q is elementary abelian, a contradicting to $|\Phi(Q)| = q$. Hence, $H = \Phi(Q)$, which implies that $\Phi(Q)$ is the unique subgroup of Q of order q. By [17, Theorem 5.3.6], Q is cyclic or generalized quaternion group Q_{2^m}. Since Q is noncyclic, $Q \cong Q_{2^m}$. By [17, Theorem 5.3.4], Q has a cyclic maximal subgroup M. Let $\overline{Q} = Q/\Phi(Q)$, we know $\overline{Q} \cong \mathbb{Z}_q^r$, where $r \geq 2$. Thus, $M/\Phi(Q)$ is a cyclic maximal subgroup of order q^{r-1} of $\overline{Q}$. Since $\overline{Q}$ is an elementary abelian q-group, $r = 2$, and so $|\Phi(Q)| = q^{m-2}$. Hence, from $|\Phi(Q)| = q$, we have $m = 3$. Thus, $Q \cong Q_8$ and $G \cong Q_8 \rtimes \mathbb{Z}_{p^n}$. By Theorem 2.1, $\gamma(\Delta(G)) \neq 1$, and so G is of type (3).

Subcase (ii-b). $\overline{G} = \overline{P} \rtimes \overline{Q}$. As in the Subcase (ii-a), we have $P \trianglelefteq G$. Thus, $G \cong \mathbb{Z}_{p^n} \rtimes Q$. Note that $|Q| = q^m$ and G has a maximal subgroup of order $p^n q^{m-1}$. Hence, $m = 2$, and so $|Q| = q^2$. Since $\mathbb{Z}_{p^n} \trianglelefteq G$ and $|\Phi(G)| = p^{n-1}q$, we conclude that G has a unique subgroup of order q. Thus, Q is cyclic, and so $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_{q^2}$. If $n = 1$, then it is easy to see that G is a group of type (2) of Theorem 2.1. Hence, $n \geq 2$, and so by Theorem 2.1, $\gamma(\Delta(G)) \neq 1$. Thus, G is of type (4).

Conversely, if G is a group of types (1)–(3), then it is easy to see that $\gamma(\Delta(G)) = 2$. Now suppose that G is a group of type (4). Thus, $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_{q^2}$, $n \neq 1$ and $q \mid |\Phi(G)|$. Since $\mathbb{Z}_{p^n} \trianglelefteq G$, $p^{n-1} \mid |\Phi(G)|$. Therefore, $|\Phi(G)| = p^{n-1}q$ and clearly $\Phi(G)$ is cyclic. Let M be a maximal subgroup of order $p^n q$. Hence, $M \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$, and so M is cyclic. Therefore, $\{M, A\}$ is a dominating set, where A is a subgroup of order p^n of M. Since $n \neq 1$, by Theorem 2.1, $\gamma(\Delta(G)) \neq 1$. Thus, $\gamma(\Delta(G)) = 2$. ∎

Note that in type (3) of above Theorem, since $|\mathrm{Syl}_p(G)| = 4$ or 8, we have $p = 3$ or 7.

Using Lemma 2.5, we obtain a characterization of finite groups with $\gamma(\Delta(G)) = 2$:

Theorem 2.6 *$\gamma(\Delta(G)) = 2$ if and only if G is one of the following types:*

(1) *$\mathbb{Z}_{p^n q^m}$, where p, q are distinct primes and $n, m \geq 2$,*

(2) *G is a p-group of order p^n, where p is a prime, $|\Phi(G)| = p^{n-2}$ and $\overline{n}(M) \neq 1$, for every $M \in \mathscr{M}(G)$,*

(3) *G is nonnilpotent with $\Phi(G) = M_1 \cap M_2$ and $\overline{n}(M) \geq 3$, for every $M \in \mathscr{M}(G)$.*

(4) *G is a group of type (1), (3) or (4) of Lemma 2.5.*

Proof By Lemma 2.2, there exists dominating set $\{M_1, M_2\}$, where $M_1, M_2 \in \mathscr{M}(G)$. First suppose that G is nilpotent, by Corollary 2.3, $\Phi(G) = M_1 \cap M_2$. Let $\overline{G} = G/\Phi(G)$. The maximal subgroups $\overline{M_1} = M_1/\Phi(G)$ and

$\overline{M_2} = M_2/\Phi(G)$ of $\overline{G}$ have trivial intersection. Since $\overline{G}$ is nilpotent, $\overline{G} = \overline{M_1} \times \overline{M_2}$. Therefore, $|\overline{G} : \overline{M_1}| = p$ and $|\overline{G} : \overline{M_2}| = q$, where p and q are primes. Hence, $|\overline{M_1}| = q$ and $|\overline{M_2}| = p$. We consider two cases.

Case (i). If $p \neq q$, then $\overline{G} \cong \mathbb{Z}_{pq}$, and so G is cyclic. By [17, Theorem 9.3.5], $G \cong \mathbb{Z}_{p^n q^m}$. Since $\gamma(\Delta(G)) \neq 1$, by part (1) of Theorem 2.1, $n, m \geq 2$. Thus, G is of type (1).

Case (ii). If $p = q$, then $\overline{G} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now G is a p-group with $p+1$ maximal subgroups. Put $n_i = \overline{n}(M_i)$, where $M_i \in \mathscr{M}(G)$. Then the join graph of G is the complete $(p+1)$-partite graph $K_{n_1,\ldots,n_{p+1}}$. If there exists i such that $n_i = 1$, then $\gamma(\Delta(G)) = 1$. Hence, $n_i \neq 1$, for all $1 \leq i \leq p+1$. Thus, G is of type (2).

Now suppose that G is nonnilpotent. Since $\gamma(\Delta(G)) \neq 1$, it follows that $\overline{n}(M) \neq 1$, for every $M \in \mathscr{M}(G)$. If $\overline{n}(M) \geq 3$, for every $M \in \mathscr{M}(G)$, then G is of type (3). Now if there exists a maximal subgroup M of G such that $\overline{n}(M) = 2$, then we have the dominating set $\{A, M\}$, where A is a unique proper subgroup of M which is not contained in $\Phi(G)$. Hence, G is a group of type (1), (3), or (4) of Lemma 2.5.

Now we prove the converse. If G is a group of type (1), (3), or (4), then clearly $\gamma(\Delta(G)) = 2$. If G is of type (2), then it is easy to see that $\Phi(G) = M_i \cap M_j$, for every $M_i, M_j \in \mathscr{M}(G)$. Therefore, $\gamma(\Delta(G)) \leq 2$. Since $\overline{n}(M_i) \neq 1$, $\gamma(\Delta(G)) = 2$. ■

Below are examples of each type of groups of the above theorem:

- The SmallGroup$(20,3) = \mathbb{Z}_5 \rtimes \mathbb{Z}_4$ and SmallGroup$(72,19) = \mathbb{Z}_3^2 \rtimes \mathbb{Z}_8$ are groups of type (4) (of type (1) of Lemma 2.5).
- The SmallGroup$(72,3) = Q_8 \rtimes \mathbb{Z}_9$ is a group of type (4) (of type (3) of Lemma 2.5).
- The SmallGroup$(100,1) = \mathbb{Z}_{25} \rtimes \mathbb{Z}_4$ is a group of type (4) (of type (4) of Lemma 2.5).
- The SmallGroup$(32,2)$ is a group of type (2).
- The alternating group A_5 is a group of type (3).

3. Independence number and solvability

In this section we prove Theorem B and Corollaries C and D. Recall that a minimal simple group is a nonabelian simple group whose proper subgroups are solvable. Recall that $\mathrm{SL}(m,q)$ is the group of $m \times m$ matrices having determinant 1 whose entries lie in a field with q elements and that $\mathrm{PSL}(m,q) = \mathrm{SL}(m,q)/H$, where $H = \{kI \mid k^m = 1\}$. For any prime $q \geq 3$, the Suzuki group is denoted by $\mathrm{Sz}(2^q)$. The classification of minimal simple groups is given by Thompson:

Theorem 3.1 *([20, Corollary 1]) A finite group is a minimal simple group if and only if it is isomorphic to one of the following groups:*

(1) $\mathrm{PSL}(2, 2^p)$, *where p is any prime,*

(2) $\mathrm{PSL}(2, 3^p)$, *where p is an odd prime,*

(3) $\mathrm{PSL}(3, 3)$,

(4) $\mathrm{PSL}(2, p)$, *where $p^2 + 1 \equiv 0 \pmod 5$ and $p > 3$,*

(5) $\mathrm{Sz}(2^p)$, *where p is an odd prime.*

In order to obtain a bound for the independence number of the join graph of a nonabelian simple group, we use the following Theorem by Barry and Ward:

Theorem 3.2 *([5]) If G is a finite nonabelian simple group, then G contains a subgroup which is a minimal simple group.*

Lemma 3.3 *Let G be a finite group with $\Phi(G) = 1$. If H is a proper subgroup of G, then $n(H)+1 \leq \alpha(\Delta(G))$.*

Proof Since $\Phi(G) = 1$, it follows that each nontrivial subgroup of H is a vertex of $\Delta(G)$. Therefore, the set of all nontrivial subgroups of H is an independent set for $\Delta(G)$. Thus, $n(H) + 1 \leq \alpha(\Delta(G))$. ∎

Let G be a finite simple group. We want to find a bound for $\alpha(\Delta(G))$. By Theorem 3.2 and Lemma 3.3, it is enough to find a bound for the independence number of the join graphs of minimal simple groups. For this purpose, by using the following GAP program [10], we show that $\alpha(\Delta(A_5)) = 15$. Note that the package grape in GAP gives an independent set of size 15. We show that there exists no independent set of size > 15.

```
LoadPackage("Sonata");
LoadPackage("grape");
vert:=G->Filtered(Subgroups(G),x-> not IsSubgroup(FrattiniSubgroup(G),x));;
joingraph:=G->Graph(G,vert(G),OnPoints,
                function(x,y) return G=Subgroup(G,Union(x,y)); end,true);
```

Theorem 3.4 $\alpha(\Delta(A_5)) = 15$.

Proof We know that $n(A_5) = 57$, $|\mathscr{M}(A_5)| = 21$ and $n(M) \in \{4, 6, 8\}$, for every $M \in \mathscr{M}(A_5)$. Let $\mathcal{A}$ be an independent set of $\Delta(A_5)$.

First note that if there exists a maximal subgroup of A_5 in $\mathcal{A}$, then $|\mathcal{A}| \leq 9$, and the result follows. Thus, we may consider independent sets whose elements are not maximal subgroups of A_5. Therefore, we may delete maximal subgroups of A_5 from $\Delta(A_5)$ and consider the induced subgraph Γ_1 of $\Delta(A_5)$ whose vertex set is $V(\Delta(A_5)) \setminus \mathscr{M}(A_5)$. We must show the size of any independent set in Γ_1 is not bigger than 15. The subgraph Γ_1 has 36 vertices of the following types:
Type (i): 15 vertices of type $\langle(a_1, a_2)(b_1, b_2)\rangle$,
Type (ii): 10 vertices of type $\langle(a_1, a_2, a_3)\rangle$,
Type (iii): 5 vertices of type $\langle(a_1, a_2)(b_1, b_2), (c_1, c_2)(d_1, d_2)\rangle$,
Type (iv): 6 vertices of type $\langle(a_1, a_2, a_3, a_4, a_5)\rangle$.
By inspection we can see that for every subgroups H and K of type (i), $\langle H, K\rangle \neq A_5$. Hence, the set of all vertices of type (i) form an independent set of size 15 (note that GAP also gives this set as an independent set). Using GAP we see that the degrees of vertices of types (i), (ii), (iii), and (iv) in Γ_1 are 12, 18, 28, and 30, respectively. Let $\mathcal{A}_1$ be an independent set for Γ_1. If $\mathcal{A}_1$ consists of a vertex of type (iii) or (iv), then it is clear that $|\mathcal{A}_1| \leq 8$. Now suppose that $\mathcal{A}_1$ is an independent set which does not consist of vertices of types (iii) and (iv). As above, we can delete the subgroups of types (iii) and (iv) from Γ_1 and consider the subgraph Γ_2, which is the induced subgraph of Γ_1 with vertex set $V(\Gamma_1) \setminus \{$vertices of types (iii) and (iv)$\}$. This graph has 25 vertices which are of types (i) and (ii). The degrees of vertices of types (i) and (ii) in Γ_2 are 4 and 9, respectively. Suppose that $\mathcal{A}_2$ is an independent set consisting of a vertex of type (ii). Since the degree of this vertex is 9 and $V(\Gamma_2) = 25$, $|\mathcal{A}_2| \leq 16$. We show that there exist adjacent vertices in $\mathcal{A}_2$. By inspection, each vertex of type (ii) is nonadjacent to exactly 9 vertices of type (i). Suppose that $|\mathcal{A}_2| = 16$, then $\mathcal{A}_2$ consists

of at most 9 vertices of type (i) and at least 7 vertices of type (ii). Let H be a vertex of type (i) such that $H \in \mathcal{A}_2$. Then H is nonadjacent to all elements of $\mathcal{A}_2$. Hence, H is nonadjacent to at least 7 vertices of type (ii), so adjacent to less than 3 vertices of type (ii). Since all adjacent vertices to H are of type (ii), it follows that the degree of H is less than 3, which is a contradiction. Hence, H is adjacent to at least one vertex in $\mathcal{A}_2$. Thus, $|\mathcal{A}_2| \leq 15$. This completes the proof, i.e. $\alpha(\Delta(A_5)) = 15$. ■

We need the maximal subgroups of $\mathrm{PSL}(2, q)$, where $q = p^f$, which are obtained by Dickson [9]:

Theorem 3.5 *([12, Hauptsatz II.8.27]) Let $q = 2^f \geq 4$. Then the maximal subgroups of $\mathrm{PSL}(2, q)$ are:*

(1) $\mathbb{Z}_2^f \rtimes \mathbb{Z}_{q-1}$,

(2) $D_{2(q-1)}$,

(3) $D_{2(q+1)}$,

(4) $\mathrm{PGL}(2, q_0)$, *where $q = q_0^r$ for some prime r and $q_0 \neq 2$.*

Theorem 3.6 *([12, Hauptsatz II.8.27]) Let $q = p^f \geq 5$, where p is an odd prime. Then the maximal subgroups of $\mathrm{PSL}(2, q)$ are:*

(1) $\mathbb{Z}_p^f \rtimes \mathbb{Z}_{(q-1)/2}$,

(2) D_{q-1}, *for $q \geq 13$,*

(3) D_{q+1}, *for $q \neq 7, 9$,*

(4) $\mathrm{PGL}(2, q_0)$, *for $q = q_0^2$,*

(5) $\mathrm{PSL}(2, q_0)$, *for $q = q_0^r$, where r is an odd prime,*

(6) A_5, *for $q \equiv \pm 1 \pmod{10}$, where either $q = p$ or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$,*

(7) A_4, *for $q = p \equiv \pm 3 \pmod 8$ and $q \not\equiv \pm 1 \pmod{10}$,*

(8) S_4, *for $q = p \equiv \pm 1 \pmod 8$.*

Theorem 3.7 *If G is a finite nonabelian simple group, then $\alpha(\Delta(G)) \geq 15$.*

Proof It is enough to consider minimal simple groups.

(1) $\mathrm{PSL}(2, 2^p)$, p a prime. If $p = 2$, then $\mathrm{PSL}(2, 4) \cong A_5$. By Theorem 3.4, $\alpha(\Delta(A_5)) = 15$. If $p = 3$, then $\mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$ is a maximal subgroup of $\mathrm{PSL}(2, 8)$. This maximal subgroup has 24 nontrivial subgroups. Thus, $\alpha(\Delta(\mathrm{PSL}(2, 8))) \geq 24$. If $p \geq 5$, then $D_{2(2^p+1)}$ is a maximal subgroup of $\mathrm{PSL}(2, 2^p)$. Since $2^p + 1 \geq 33$, it is clear that the number of nontrivial subgroups of $D_{2(2^p+1)}$ is at least 35. Thus, $\alpha(\Delta(\mathrm{PSL}(2, 2^p))) > 15$, where $p \geq 3$.

(2) $\mathrm{PSL}(2, 3^p)$, p an odd prime. If $p = 3$, then D_{28} is a maximal subgroup of $\mathrm{PSL}(2, 27)$. Since D_{28} has 27 nontrivial subgroups, $\alpha(\Delta(\mathrm{PSL}(2, 27))) \geq 27$. If $p \geq 5$, then $D_{3^p+1} = D_{2n}$ is a maximal subgroup which $n \geq 122$. So $\alpha(\Delta(\mathrm{PSL}(2, 3^p))) > 15$ for all odd prime p.

(3) $\mathrm{PSL}(3,3)$. This group has a maximal subgroup of order 432 which has 645 nontrivial subgroups. Thus, $\alpha(\Delta(\mathrm{PSL}(3,3))) \geq 645 > 15$.

(4) $\mathrm{PSL}(2,p)$, p a prime with $p > 3$ and $p^2 + 1 \equiv 0 \pmod 5$. If $p = 7, 17, 23$, then S_4 is a maximal subgroup of $\mathrm{PSL}(2,p)$. Since $n(S_4) = 28$, $\alpha(\Delta(\mathrm{PSL}(2,p))) \geq 29$. If $p = 13$, then $\mathbb{Z}_{13} \rtimes \mathbb{Z}_6$ is a maximal subgroup of $\mathrm{PSL}(2,13)$ with 43 nontrivial subgroups, so $\alpha(\Delta(\mathrm{PSL}(2,13))) \geq 43$. If $p \geq 27$, then $D_{p+1} = D_{2n}$ is a maximal subgroup of $\mathrm{PSL}(2,p)$. We have $n \geq 14$, so $n(D_{2n}) \geq 27$. Thus, $\alpha(\Delta(\mathrm{PSL}(2,p))) > 15$.

(5) $\mathrm{Sz}(2^p)$, p an odd prime. If $p = 3$, then by [7], $\mathrm{Sz}(8)$ has a maximal subgroup M of order $2^6 \times 7$ which $n(M) > 15$. Thus, $\alpha(\Delta(\mathrm{Sz}(8))) > 15$. If $p \geq 5$, then by [14, Theorem 4.12], $D_{2(2^p-1)}$ is a subgroup of $\mathrm{Sz}(2^p)$. Therefore, $\alpha(\Delta(\mathrm{Sz}(2^p))) \geq n(D_{2(2^p-1)}) \geq 33$. ■

As an immediate corollary of Theorem 3.7, we obtain a characterization of A_5 by its join graph.

Corollary 3.8 *If G is a finite nonabelian simple group, then $\alpha(\Delta(G)) = 15$ if and only if $G \cong A_5$.*

The following Lemma plays a fundamental rule in the proof of Theorem B.

Lemma 3.9 *If G is a finite group with $n(G) \leq 7$, then G is supersolvable.*

Proof We may assume that G is nonnilpotent. Therefore, $|\mathscr{M}(G)| \geq 4$. If $|\mathscr{M}(G)| \geq 5$, then all maximal subgroups of G have at most 2 nontrivial proper subgroups. Therefore, all maximal subgroups of G are cyclic and by [17, Theorem 10.1.10], G is supersolvable. Thus, we may assume that $|\mathscr{M}(G)| = 4$. Then every maximal subgroup of G has at most 3 nontrivial proper subgroups. Thus, all maximal subgroups of G are supersolvable and by [17, Theorem 10.3.4], G is solvable. We claim that all maximal subgroups of G are cyclic. Suppose that M is a noncyclic maximal subgroup of G. Note that since M has at most 3 nontrivial proper subgroups and it is noncyclic, it follows that $|\mathscr{M}(M)| = 3 = n(M)$. Hence, M is a 2-group of order 4. By solvability of G, $|G:M|$ is a power of prime p. Let $|G:M| = p^n$. Thus, $|G| = 4p^n$, where $p \neq 2$. Since G is nonnilpotent, at least one of its Sylow subgroups is nonnormal. Let $P \in \mathrm{Syl}_p(G)$. If $|\mathrm{Syl}_2(G)| \neq 1$ and $|\mathrm{Syl}_p(G)| \neq 1$, then it is clear that $n(G) > 7$, a contradiction. If $|\mathrm{Syl}_2(G)| \neq 1$ and $|\mathrm{Syl}_p(G)| = 1$, then $|\mathrm{Syl}_2(G)| \geq 3$. Let H_1, H_2, H_3 be the subgroups of M. Since $P \trianglelefteq G$, $H_iP \leq G$, for all $1 \leq i \leq 3$. Thus, $n(G) > 7$, a contradiction. Now let $|\mathrm{Syl}_2(G)| = 1$ and $|\mathrm{Syl}_p(G)| \neq 1$. Hence, $|\mathrm{Syl}_p(G)| = 4$, so it is easy to see that $n(G) > 7$, a contradiction. Therefore, all maximal subgroups of G are cyclic and by [17, Theorem 10.1.10] (or Lemma 2.4), G is supersolvable. ■

Now we recall some well-known facts which are needed to prove Theorem B. A minimal nonsolvable group is a group which is nonsolvable but all of whose proper subgroups are solvable. If G is minimal nonsolvable, then $G/\Phi(G)$ is minimal simple. To see this, suppose that $G/\Phi(G)$ is not simple and let $N/\Phi(G)$ be a minimal normal subgroup of $G/\Phi(G)$. By [11, Theorem 4.1], $G/\Phi(G)$ is nonsolvable and it is clear that $G/\Phi(G)$ is minimal nonsolvable (see [11, Theorem 4.1]). Since $N \nleq \Phi(G)$, it follows that there exists a maximal subgroup M of G such that $N \nleq M$. Therefore, $G = MN$, so G is solvable, a contradiction (see [11, Theorem 4.1]). Hence, $G/\Phi(G)$ is minimal simple.

Recall that if G is a group and N is a normal subgroup of G such that $N \leq \Phi(G)$, then $\alpha(\Delta(G/N)) \leq \alpha(\Delta(G))$ (see [1, Lemma 2.6]).

Proof of Theorem B. First suppose that $\Phi(G) = 1$. Assume, on the contrary, that G is nonsolvable with $\alpha(\Delta(G)) \leq 14$. If all subgroups of G are solvable, then G is minimal nonsolvable. Therefore, G is minimal simple. By Theorem 3.7, $\alpha(\Delta(G)) \geq 15$, which is a contradiction. Hence, G has a maximal subgroup M which is nonsolvable. Thus, $k = |\mathscr{M}(M)| \geq 4$. By Lemma 3.9, $n(M) \geq 8$. Since $\alpha(\Delta(G)) \leq 14$, it follows that $n(M) \leq 13$. If $k \geq 6$, then all maximal subgroups of M have at most 7 nontrivial proper subgroups, so are supersolvable. Therefore, M is solvable, a contradiction. Thus, $k = 4$ or 5. Since M is nonsolvable, it follows that M has a nonsupersolvable maximal subgroup (see [17, Theorem 10.3.4]). Let H be a nonsupersolvable maximal subgroup of M. Thus, $|\mathscr{M}(H)| \geq 4$ and $n(H) = 8$ or 9. By Lemma 3.9, it is clear that all maximal subgroups of H are supersolvable, so H is solvable. Hence, M is minimal nonsolvable, so $M/\Phi(M)$ is minimal simple. By Theorem 3.7, $\alpha(\Delta(M/\Phi(M))) \geq 15$, contradicting to $\alpha(\Delta(M/\Phi(M))) \leq \alpha(\Delta(M)) < n(M) \leq 13$. Thus, if $\alpha(\Delta(G)) \leq 14$ and $\Phi(G) = 1$, then G is solvable.

Now suppose that $\Phi(G) \neq 1$. Let $\overline{G} = G/\Phi(G)$. We have $\alpha(\Delta(\overline{G})) \leq 14$ and $\Phi(\overline{G}) = 1$. Hence, the proof of Theorem for $\Phi(G) = 1$ implies that $\overline{G}$ is solvable, so G is solvable. This completes the proof. ■

Proof of Corollary C. First suppose that $G \cong A_5$. Then by Theorem 3.4, $\alpha(\Delta(G)) = 15$. Now to prove the converse let M be a nonsolvable maximal subgroup of G. It follows that $|\mathscr{M}(M)| \geq 4$. Since $\alpha(\Delta(G)) = 15$, $n(M) \leq 14$. A similar argument to that of the proof of Theorem B shows that all maximal subgroups of M are solvable. Therefore, $M/\Phi(M)$ is minimal simple. By Theorem 3.7, $\alpha(\Delta(M)) \geq 15$, contradicting to $\alpha(\Delta(M))) < n(M) = 14$. Hence, all maximal subgroups of G are solvable. Hence, G is minimal nonsolvable with $\Phi(G) = 1$. Thus, G is minimal simple with $\alpha(\Delta(G)) = 15$. Therefore, by Corollary 3.8, $G \cong A_5$. ■

Proof of Corollary D. Since G is nonsolvable, $\overline{G} = G/\Phi(G)$ is nonsolvable. Since $\alpha(\Delta(\overline{G})) \leq 15$, Theorem B implies that $\alpha(\Delta(\overline{G})) = 15$, and so by Corollary C, $\overline{G} \cong A_5$. ■

4. Finite Groups with independence number less than three

In this section we classify finite groups whose join graphs have an independence number less than or equal to 3. Since the independence number of a complete graph is 1, it follows that the independence number of the join graphs of groups $\mathbb{Z}_p \times \mathbb{Z}_p$ and Q_8 is 1. In the following Lemma we obtain the independence number of the join graph of a noncyclic group which is not of prime power order and all its maximal subgroups are cyclic.

Lemma 4.1 *If $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$, where p, q are distinct primes and $|\Phi(G)| = q^{n-1}$, then $\alpha(\Delta(G)) = n$.*

Proof First we show that if $M_i \cap M_j = \Phi(G)$, for every $M_i, M_j \in \mathscr{M}(G)$, then $\alpha(\Delta(G)) = \max\{\overline{n}(M_i) \mid 1 \leq i \leq k\}$, where k is the number of maximal subgroups of G. It is clear that each vertex of $\Delta(G)$ is contained in a unique maximal subgroup of G. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_n\}$ be an independent set. Thus, for all $1 \leq i, j \leq n$, H_i and H_j are not adjacent, so there exists a unique maximal subgroup M such that $\langle H_i, H_j \rangle \leq M$. Hence, for all $1 \leq i \leq n$, H_i is a subgroup of M. Therefore, $|\mathcal{A}| \leq \overline{n}(M)$, so $\alpha(\Delta(G)) = \max\{\overline{n}(M_i) \mid 1 \leq i \leq k\}$. It is easy to see that for $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^n}$ with $|\Phi(G)| = q^{n-1}$, we have $M_i \cap M_j = \Phi(G)$, for every $M_i, M_j \in \mathscr{M}(G)$. For all maximal subgroups M of order q^n, $\overline{n}(M) = 1$, and for maximal subgroup N of order pq^{n-1}, $\overline{n}(N) = n$. Thus, $\alpha(\Delta(G)) = n$. ■

Example 4.2 *The alternating group A_4 have 4 maximal subgroups isomorphic to $\mathbb{Z}_3$ and one maximal subgroup isomorphic to $\mathbb{Z}_2^2$. The Frattini subgroup of A_4 is trivial and it is clear that $M_i \cap M_j = 1$, for every $M_i, M_j \in \mathscr{M}(A_4)$. Thus, $\alpha(\Delta(A_4)) = \overline{n}(\mathbb{Z}_2^2) = 4$.*

It is clear that $\alpha(\Delta(G)) \geq \max\{\overline{n}(M) \mid M \in \mathscr{M}(G)\}$ and $\gamma(\Delta(G)) \leq \min\{\overline{n}(M) \mid M \in \mathscr{M}(G)\}$. Hence, $\gamma(\Delta(G)) \leq \alpha(\Delta(G))$. The following result is proved in [1, Theorem 2.5].

Proposition 4.3 *$\alpha(\Delta(G)) = 1$ if and only if $\Delta(G)$ is complete if and only if G is one of the following groups:*

(1) $\mathbb{Z}_{pq}$,

(2) $\mathbb{Z}_p \times \mathbb{Z}_p$,

(3) $\mathbb{Z}_p \rtimes \mathbb{Z}_q$,

(4) Q_8,

where p, q are distinct primes.

Proposition 4.4 *$\alpha(\Delta(G)) = 2$ if and only if G is isomorphic to one of the following groups:*

(1) *$G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^2}$ with $|\Phi(G)| = q$,*

(2) $\mathbb{Z}_{p_1 p_2^2}$,

(3) $\mathbb{Z}_{p_1^2 p_2^2}$,

where p, q, p_1 and p_2 are distinct primes.

Proof If G is one of the types (1)–(3), then according to the Figure 2 and Lemma 4.1, $\alpha(\Delta(G)) = 2$ (In all figures, we denoted the vertices by the orders of relative subgroups).

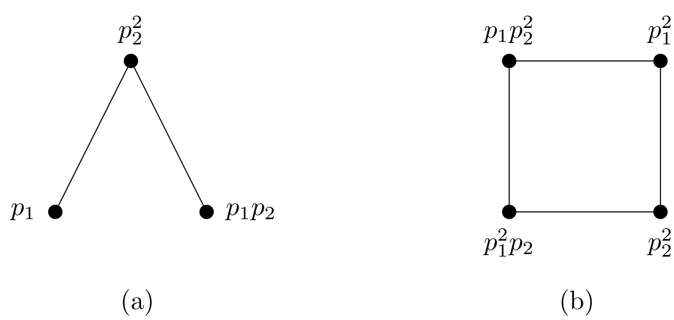

Figure 2. (a) $\Delta(\mathbb{Z}_{p_1 p_2^2})$ and (b) $\Delta(\mathbb{Z}_{p_1^2 p_2^2})$.

Conversely, suppose that $\alpha(\Delta(G)) = 2$. For all maximal subgroups M of G, $\overline{n}(M) \leq 2$. Hence, $|\mathscr{M}(M)| \leq 2$, so M is cyclic. Therefore, all maximal subgroups of G are cyclic. If G is noncyclic and $|\pi(G)| = 1$, then $\Delta(G)$ is complete, so $\alpha(\Delta(G)) = 1$. Now if G is noncyclic and $|\pi(G)| \geq 2$, then by Lemma

2.4, $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^m}$ with $|\Phi(G)| = q^{m-1}$. By Lemma 4.1, $\alpha(\Delta(G)) = m$. Hence, $m = 2$, so $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^2}$ with $|\Phi(G)| = q$. Thus, G is of type (1). Now suppose that G is cyclic of order $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$. Let $k \geq 3$. Then for every maximal subgroup M of G, we have $\overline{n}(M) > 2$, which is a contradiction. Thus, $k \leq 2$. Recall that all groups in the paper are distinct from a cyclic group of prime power order, so $k \neq 1$. Hence, we have $k = 2$ and G has two maximal subgroups M_1 and M_2 of orders $p_1^{n_1-1} p_2^{n_2}$ and $p_1^{n_1} p_2^{n_2-1}$, respectively. Since $\overline{n}(M) \leq 2$, it is easy to see that $n_1, n_2 \leq 2$. If $n_1 = n_2 = 1$, then $G \cong \mathbb{Z}_{p_1 p_2}$ and by Proposition 4.3, $\alpha(\Delta(G)) = 1$. Without loss of generality we can assume that $n_1 \leq n_2$. Therefore, $(n_1, n_2) \in \{(1,2),(2,2)\}$, so $G \cong \mathbb{Z}_{p_1^i p_2^2}$, where $i = 1, 2$. Thus, $\Delta(G)$ is the graph shown in Figure 2, and clearly $\alpha(\Delta(G)) = 2$. ■

Note that if $\Delta(G) = K_{n_1,n_2,\dots,n_k}$, then $\alpha(\Delta(G)) = \max\{n_1, n_2, \dots, n_k\}$.

Proposition 4.5 *$\alpha(\Delta(G)) = 3$ if and only if G is one of the following groups:*

(1) *$G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^3}$ with $|\Phi(G)| = q^2$, where p, q are distinct primes,*

(2) *G is a 2-group whose join graph is $K_{3,3,3}$,*

(3) $\mathbb{Z}_4 \times \mathbb{Z}_2$,

(4) D_8,

(5) Q_{16},

(6) *$\mathbb{Z}_{p_1 p_2 p_3}$, where p_1, p_2 and p_3 are distinct primes,*

(7) *$\mathbb{Z}_{p_1^n p_2^3}$, where p_1, p_2 are distinct primes and $1 \leq n \leq 3$.*

Proof If $\alpha(\Delta(G)) = 3$, then $\overline{n}(M) \leq 3$ for every $M \in \mathscr{M}(G)$. Hence, $|\mathscr{M}(M)| \leq 3$, so all maximal subgroups of G are supersolvable; thus, G is solvable (see [17, Theorem 10.3.4]). We consider two cases:

Case (i). G is noncyclic. If $|\pi(G)| = 1$ and all maximal subgroups are cyclic, then $\Delta(G)$ is complete, and clearly $\alpha(\Delta(G)) = 1$, a contradiction. If $|\pi(G)| \geq 2$ and all maximal subgroups are cyclic, then by Lemma 2.4, $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^m}$ with $|\Phi(G)| = q^{m-1}$ and by Lemma 4.1, $\alpha(\Delta(G)) = m$. Hence, $m = 3$, so $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{q^3}$ with $|\Phi(G)| = q^2$. Thus, G is of type (1).

Now suppose that G has a noncyclic maximal subgroup M, say. Since $|\mathscr{M}(M)| \leq 3$ and M is noncyclic, we conclude that $|\mathscr{M}(M)| = 3$. Therefore, M is a 2-group of order 2^n, where $n \geq 2$.
Since $|\mathscr{M}(M)| = \overline{n}(M) = 3$, $\Phi(G)$ is a maximal subgroup of M. Thus, $|\Phi(G)| = 2^{n-1}$. From $|\mathscr{M}(M)| = 3$ we have $|\Phi(M)| = 2^{n-2}$. By solvability of G, $|G : M|$ is a power of a prime. Thus, $|G| = 2^n p^m$ or 2^{n+1}, where p is a prime distinct from 2. We distinguish two subcases:

Subcase (i-a). $|G| = 2^n p^m$. Let $P \in \mathrm{Syl}_p(G)$ and M_1 be a maximal subgroup of G such that $P \leq M_1$. Since $|\Phi(G)| = 2^{n-1}$, $|M_1| = 2^{n-1} p^m$. Also since $|G : M_1| = 2$, $M_1 \trianglelefteq G$. If $m \geq 2$, then it is clear that $\overline{n}(M_1) > 3$, a contradiction. Therefore, $|G| = 2^n p$. We have $M_1 \trianglelefteq G$ and by hypothesis M_1 is cyclic, so $P \trianglelefteq G$. If G is nilpotent, then $\Phi(G) = \Phi(M) \times \Phi(P) = \Phi(M)$, so $|\Phi(G)| = 2^{n-2}$, a contradiction. Note that all maximal subgroups of G are nilpotent, so G is minimal nonnilpotent. Since $P \trianglelefteq G$, by Schmidt's Theorem [17, Theorem 9.1.9], we have $G = P \rtimes M$ such that M is cyclic, which is a contradiction. Thus, this case cannot happen.

Subcase (i-b). $|G| = 2^{n+1}$. Since $|\Phi(G)| = 2^{n-1}$, we have $|\mathscr{M}(G)| = 3$. Let $\mathscr{M}(G) = \{M_1, M_2, M_3\}$ and $\overline{n}(M_i) = n_i$, for $i \in \{1,2,3\}$. Hence, $\Delta(G) = K_{n_1,n_2,n_3}$. If all maximal subgroups of G are noncyclic, then $n_1 = n_2 = n_3 = 3$. Thus, $\Delta(G) = K_{3,3,3}$, so G is of type (2). Now suppose that G has a cyclic maximal subgroup. By [17, Theorem 5.3.4], G is one of the following types:
(a) $\mathbb{Z}_{2^{n+1}}$,
(b) $\mathbb{Z}_{2^n} \times \mathbb{Z}_2$,
(c) $\langle x, a \mid x^2 = 1 = a^{2^n}, a^x = a^{1+2^{n-1}} \rangle$, $n \geq 2$,
(d) $D_{2^{n+1}}$, $n \geq 2$,
(e) $Q_{2^{n+1}} = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, x^y = x^{-1} \rangle$, $n \geq 2$,
(f) $SD_{2^{n+1}} = \langle x, a \mid x^2 = 1 = a^{2^n}, a^x = a^{2^{n-1}-1} \rangle$, $n \geq 2$.
Recall that all groups in the paper are different from a cyclic group of prime power order, so case (a) cannot happen.

Suppose that (b) holds. Thus, $G = \langle a \rangle \times \langle b \rangle$, where $a^{2^n} = b^2 = 1$. Since $b \notin \Phi(G)$, $\langle b \rangle$ is a vertex of $\Delta(G)$. If $\langle b \rangle$ is a maximal subgroup of G, then $|G| = 4$, so $n = 1$. Therefore, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, so all subgroups of G are cyclic, a contradiction. Hence, there exists maximal subgroup M such that $\langle b \rangle < M$. Thus, M is not cyclic. Since $\overline{n}(M) = 3$ and $|\mathscr{M}(M)| = 3$, it follows that $\langle b \rangle$ is a maximal subgroup of M. Hence, $|M| = 4$, so $n = 2$. Therefore, $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \cong SD_8$. Since G has exactly 2 cyclic maximal subgroups $\langle a \rangle$ and $\langle ab \rangle$, we have $\Delta(G) = K_{1,1,3}$, so $\alpha(\Delta(G)) = 3$. Thus, G is of type (3).

Suppose that (c) holds. For subgroup $\langle x \rangle$, by a similar argument we have $n \leq 2$. Since $n \geq 2$, $n = 2$. Thus, $G \cong \langle x, a \mid x^2 = 1 = a^4, a^x = a^3 \rangle \cong D_8$. Since G has exactly one cyclic maximal subgroup $\langle a \rangle$, $\Delta(G) = K_{1,3,3}$, so $\alpha(\Delta(G)) = 3$. Thus, G is of type (4).

Similarly for types (d) and (f), we have $n = 2$. Therefore, G is isomorphic to D_8 and SD_8, respectively. Thus, G is of types (3) and (4), respectively.

Finally suppose that (e) holds. Note that G has a noncyclic maximal subgroup so we have $n \neq 2$. Since $\langle y \rangle$ is not contained in $\Phi(G)$, we conclude that $\langle y \rangle$ is a vertex. If $\langle y \rangle$ is maximal subgroup of G, then $|\langle y \rangle| = 2^n = 4$, so $n = 2$, a contradiction. Hence, there exists maximal subgroup M of G such that $\langle y \rangle < M$. Since $\overline{n}(M) = 3$ and $|\mathscr{M}(M)| = 3$, it follows that $\langle y \rangle$ is a maximal subgroup of M. Therefore, $|\langle y \rangle| = 2^{n-1} = 4$, so $n = 3$. Thus, $G \cong Q_{16}$. By an inspection it is easy to see that Q_{16} has 2 noncyclic maximal subgroups. Hence, $\Delta(G) = K_{1,3,3}$, so $\alpha(\Delta(G)) = 3$. Thus, G is of type (5).

Case (ii). G is cyclic of order $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$. It is clear to see that if $k \geq 4$, then $\overline{n}(M) > 3$, for every $M \in \mathscr{M}(G)$. Thus, $k \leq 3$. If $k = 3$, then $|G| = p_1^{n_1} p_2^{n_2} p_3^{n_3}$. Let M_i be a maximal subgroup of order $|G|/p_i$, for $i \in \{1,2,3\}$. If $n_i \geq 2$, then it easy to see $\overline{n}(M_i) > 3$, a contradiction. Therefore, $G \cong \mathbb{Z}_{p_1p_2p_3}$. Clearly by Figure 3, $\alpha(\Delta(\mathbb{Z}_{p_1p_2p_3})) = 3$. Thus, G is of type (6).

Now if $k = 2$, then $|G| = p_1^{n_1} p_2^{n_2}$. For $i = 1, 2$, if $n_i \geq 4$, then $\overline{n}(M_i) > 3$, a contradiction. Hence, $n_1, n_2 \leq 3$. Without loss of generality we can assume that $n_1 \leq n_2$, so

$$(n_1, n_2) \in \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}.$$

If G is isomorphic to $\mathbb{Z}_{p_1p_2}$, $\mathbb{Z}_{p_1p_2^2}$ or $\mathbb{Z}_{p_1^2p_2^2}$, then by Propositions 4.3 and 4.4, $\alpha(\Delta(G)) = 1$ or 2. Now suppose that G is isomorphic to $\mathbb{Z}_{p_1^{n_1}p_2^3}$, where $n_1 \in \{1,2,3\}$. According to the Figure 4, $\alpha(\Delta(G)) = 3$. Thus, G is of

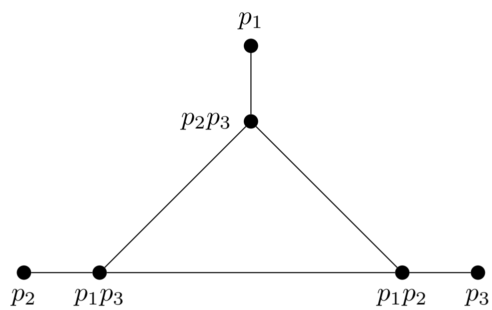

Figure 3. $\Delta(\mathbb{Z}_{p_1p_2p_3})$.

type (7).

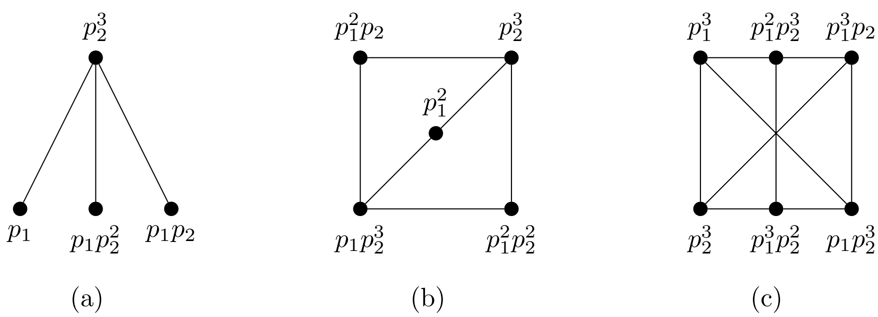

Figure 4. (a) $\Delta(\mathbb{Z}_{p_1p_2^3})$, (b) $\Delta(\mathbb{Z}_{p_1^2p_2^3})$ and (c) $\Delta(\mathbb{Z}_{p_1^3p_2^3})$.

Conversely, if G is a group of types (1)–(7), then it is easy to see that $\alpha(\Delta(G)) = 3$. ■

References

[1] Ahmadi H, Taeri B. A graph related to the join of subgroups of a finite group. Rendiconti del Seminario Matematico della Università di Padova. 2014; 131: 281-292. doi: 10.4171/rsmup/131-17

[2] Ahmadi H, Taeri B. On the planarity of a graph related to the join of subgroups of a finite group. Bulletin of the Iranian Mathematical Society 2014; 40 (6): 1413-1431.

[3] Ahmadi H, Taeri B. Finite groups with regular join graph of subgroups. Journal of Algebra and its Applications 2016; 15 (9): 1650170. doi: 10.1142/s021949881650170x

[4] Ashrafi AR. One the number of minimal and maximal subgroups of a finite group. Journal of Sciences University of Tehran International Edition 1998; 3 (1): 15-20.

[5] Barry MJJ, Ward MB. Simple groups contain minimal simple groups. Publicacions Matemàtiques 1997; 41 (2): 411-415. doi: 10.5565/publmat_41297_07

[6] Bosák J. The graphs of semigroups, Proceedings of the Symposium on theory of graphs, and its applications. 1963; 119-125.

[7] Connor T, Leemans D. An atlas of subgroup lattices of finite almost simple groups. Ars Mathematica Contemporanea 2015; 8: 259-266. doi: 10.26493/1855-3974.455.422

[8] Csákány, B. Pollák, G. The graph of subgroups of a finite group. Czechoslovak Mathematical Journal 1969; 19 (94): 241-247.

[9] Dickson LE. Linear groups: With an exposition of the Galois field theory. Teubner, Leipzig, 1901. doi: 10.5962/bhl.title.22174

[10] The GAP Group. Groups, Algorithms and Programing. Version 4.4; 2005; (http://www.gap-system.org).

[11] Gorenstein D. Finite Groups. New York, NY, USA: Harper and Row, 1968.

[12] Huppert B. Endliche Gruppen I. Berlin, Germany: Springer-Verlag, 1967 (in German).

[13] Kurzweil H, Stellmacher B. The theory of finite groups, an introduction. New York, NY, USA: Springer-Verlag, 2004.

[14] Lüneburg H. Die Suzukigruppen und ihre Geometrien. Lecture Notes in Mathematics. Berlin, Germany: Springer-Verlag, 1965 (in German).

[15] Miller GA, Moreno HC. Non-abelian groups in which every subgroup is abelian. Transactions of the American Mathematical Society 1903; 4(4) 398-404. doi: 10.2307/1986409

[16] Rédei L. Die endlichen einstufig nichtnilpotenten Gruppen. Publicationes Mathematicae Debrecen 1956; 4: 303-324 (in German).

[17] Robinson DJS. A Course in the Theory of Groups. New York, NY, USA: Springer-Verlag, 1996.

[18] Rose JS. A Course on Group Theory. Cambridge University Press, 1978.

[19] Shen R. Intersection graphs of subgroups of finite groups. Czechoslovak Mathematical Journal 2010; 60 (135): 945-950. doi: 10.1007/s10587-010-0085-4

[20] Thompson JG. Nonsolvable finite groups all of whose local subgroups are solvable. Bulletin of the American Mathematical Society 1968; 74: 383-437. doi: 10.1090/s0002-9904-1968-11953-6

[21] Zelinka B. Intersection graphs of finite abelian groups. Czechoslovak Mathematical Journal 1975; 25 (100): 171-174.