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Categorified groupoid-sets and their Burnside ring

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Abstract: We explore the category of internal categories in the usual category of (right) group-sets, whose objects are referred to as categorified group-sets. More precisely, we develop a new Burnside theory, where the equivalence relation between two categorified group-sets is given by a particular equivalence between the underlying categories. We also exhibit some of the differences between the old Burnside theory and the new one. Lastly, we briefly explain how to extend these new techniques and concepts to the context of groupoids, employing the categories of (right) groupoid-sets, aiming by this to give an alternative approach to the classical Burnside ring of groupoids.

Key words: Internal categories, categorification, group actions, Burnside ring, double categories, groupoid actions

1. Introduction

The Burnside theory for finite groups is a classical subject in the context of group representation theory and it was introduced for the first time in [5]. Afterwards, this topic was further developed in [7, 20] and it has found many applications in different fields such as homotopy theory (see [19]). Aiming to extend this theory to the framework of "finite" groupoids, we discovered (see El Kaoutit and Spinosa, "Burnside theory for groupoids", arXiv: 1807.04470v1, math.GR, July 2018) that the Burnside contravariant functor does not distinguish between a given groupoid and its bundle of isotropy groups. Specifically, it has been realized that, under appropriate finiteness conditions, the classical Burnside ring of a given groupoid is isomorphic to the product of the Burnside rings of its isotropy group types, although not in a canonical way. The crux is that the isomorphism relation between finite (and not finite) groupoid-sets leads only to the consideration of (right) cosets by subgroupoids with a single object, and this somehow obscures the whole structure of the handled groupoid. In other words, the classical Burnside ring of a (finite) groupoid does not codify, as in the classical case, information about the whole "lattice" of subgroupoids, since subgroupoids with several objects do not show up at all.

This paper is an attempt to give another approach to the Burnside ring of groupoids by considering the category (2-category, in fact) of internal categories inside the category of (right) groupoid-sets. The objects of this category (the 0-cells) are referred to as categorified groupoid-sets and, by abuse of terminology, the associated ring is called the categorified Burnside ring of the given groupoid. It is noteworthy to mention that, although in this way we get a commutative ring that strictly contains the classical one, we show that this new ring can also be decomposed, in a way that is not canonical, as a product of rings, which are the categorified

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Burnside rings of the isotropy group types of the groupoid. This makes manifest that the idea of employing the categorification of the notion of groupoid-sets also does not adjust to the groupoid structural characteristics. Nevertheless, in the case of groups (i.e. groupoids with only one object), this categorification approach leads to a new Burnside ring entering the picture (see Example 7.4).

The concept of categorification was explained extensively in [2, p. 495] and [3]. Roughly speaking, the idea behind it is to replace the underlying set of an algebraic structure (resp. the structure maps), like a group, with a certain category (resp. with functors), with the goal of obtaining a new structure that could help to understand the initial one. In the case of the category of groups, for example, the categorificaton process produces the notion of 2-group (see [3]), which has been proved to be equivalent to the concept of crossed module introduced by Whitehead in [21, 22]. To perform the categorification of a structure, the notions of internal category, internal functor, and internal natural transformation, introduced in [8–10], are crucial and they will be used extensively in this work.

The main idea of this paper is to categorify the notion of group action on a set to obtain a particular category with a group action on both the set of objects and of morphisms. Moreover, the source, target, identity, and composition maps of this category will have to be compatible with the group action. Regarding the usual right translation groupoid, it will be replaced by a right translation double category (an internal category in the category of small categories) to illustrate the new higher dimensional situation.

After this, we elaborate a new Burnside theory, based on a particular notion of weak equivalence between these new categories endowed with a group action. We refer the reader to [5, 20] for the classical Burnside theory of groups. We note that there are, in the literature, other generalizations of the classical Burnside theory: see, for instance, [6, 13, 14, 17].

In the last section we briefly explain how to extend this idea of categorification to the case of groupoid actions and how it can be reduced to the case of group actions. Regarding the theory of groupoid actions we refer the reader to [11, Section 2] (see also El Kaoutit and Spinosa, "Burnside theory for groupoids", arXiv: 1807.04470v1, math.GR, July 2018, for the classical Burnside theory applied to groupoids).

2. Preliminaries and basic definitions

In this section we recall the notion of internal categories in small categories with pull-backs,* and we use this notion to introduce what we will call the category of right categorified group-sets. This is a kind of a categorification of the usual notion of right group-set object (see [4]).

Given two functions $\alpha\colon A \longrightarrow D$ and $\beta\colon B \longrightarrow D$, we will use the following notation:

$$A\,{}_{\alpha}\!\times_{\beta} B = \{(a,b) \in A \times B \,|\, \alpha(a) = \beta(b)\}\,. \tag{2.1}$$

This set is well known as the fiber product of α and β and it is the pull-back of the maps α and β in the category of sets. This means that the commutative diagram in Figure 1 is Cartesian, where pr_1 and pr_2 are the canonical projections. This notion can also be adopted in a categorical setting replacing sets with objects in a given category with pull-backs. This, in particular, applies to the category of small categories and functors between them. Definition 2.1 is taken from [2, p. 495].

Definition 2.1 *Given a category with pull-back $\mathcal{C}$, we define an internal category $\mathcal{X}$ in $\mathcal{C}$ as a couple of objects $\mathcal{X}_0$ and $\mathcal{X}_1$ of $\mathcal{C}$ and morphisms as in Figure 2, where $\mathsf{s}_{\mathcal{X}}$ and $\mathsf{t}_{\mathcal{X}}$ are called the source and the target*

*A category is called small when it is a hom-set category [16], and its class of objects is actually a set.

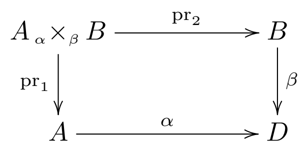

Figure 1. Fiber product.

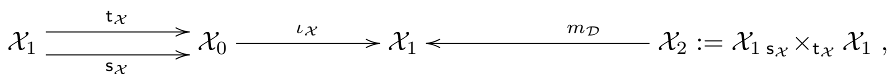

Figure 2. Definition of an internal category: morphisms.

morphisms, respectively; $\iota_{\mathcal{X}}$ *is called the identity morphism; and* $m_{\mathcal{X}}$ *is called the composition morphism or multiplication morphism, such that the diagrams in Figure 3 are commutative (note that we will use the notation* $\mathcal{X}_3 := \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1\,{}_{\mathsf{s}_{\mathcal{X}}}\times_{\mathsf{t}_{\mathcal{X}}} \mathcal{X}_1$*).*

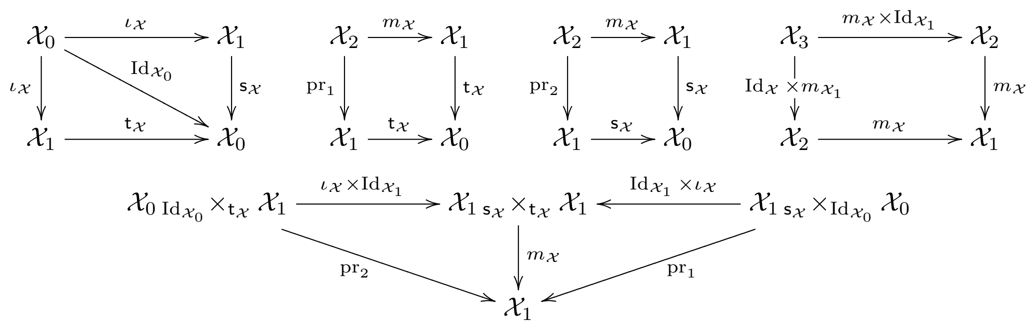

Figure 3. Definition of an internal category: conditions.

Internal categories in $\mathcal{C}$ can be viewed as 0-cells in a certain 2-category:

Definition 2.2 *Let* $\mathcal{C}$ *be a category with pull-back, and consider two internal categories* $\mathcal{X}$ *and* $\mathcal{Y}$ *in* $\mathcal{C}$*. We define an* $F\colon \mathcal{X} \longrightarrow \mathcal{Y}$ *in* $\mathcal{C}$ *as a couple of morphisms* $F_0\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_0$ *and* $F_1\colon \mathcal{X}_1 \longrightarrow \mathcal{Y}_1$ *such that the diagrams in Figure 4 are commutative, where we use the notation* $F_2 = F_1 \times F_1\colon \mathcal{X}_2 \longrightarrow \mathcal{Y}_2$*.*

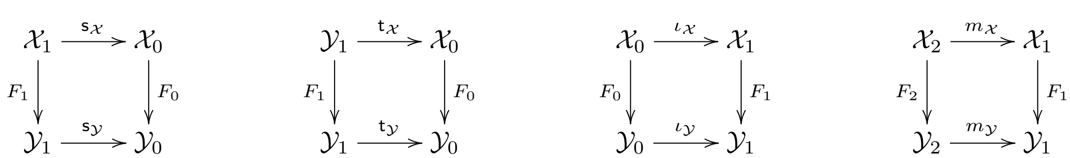

Figure 4. Definition of an internal functor.

Definition 2.3 *Given $\mathcal{C}$ as above, let us consider two internal categories $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{C}$, and two internal functors $F, G\colon \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathcal{C}$. We define an internal natural transformation $\alpha\colon F \longrightarrow G$ in $\mathcal{C}$ as a morphism $\alpha\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1$ in $\mathcal{C}$ such that the diagrams in Figure 5 are commutative, where Δ denotes the morphism given by the universal property of the pull-back in $\mathcal{C}$.*

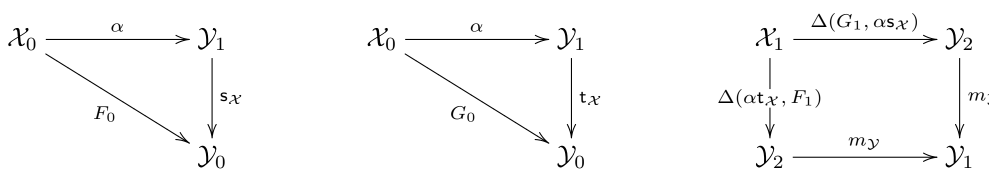

Figure 5. Definition of an internal natural transformation.

Internal natural transformations can be composed (horizontally or vertically) in a way to similar ordinary natural transformation and we refer to [2, p. 498] for a more detailed explanation.

We note that the experienced reader will not fail to see the similarities between the theory of internal categories and enriched category theory (see [15]). In this work, however, we chose to keep the internal categories approach already used in [2, 3].

For a given group G, we will denote by $\mathsf{Set}\text{-}G$ its category of right G-sets. Morphisms in this category are referred to as G-equivariant maps. The pull-backs in $\mathsf{Set}\text{-}G$ are given as follows: let us assume that we have a diagram of G-equivariant maps $\alpha : X \to Z \leftarrow Y : \beta$. Then the pull-back set $X\, {}_{\alpha}{\times}_{\beta}\, Y$ is a G-set via the action $(x, y)g = (xg, yg)$, with $x \in X$, $y \in Y$, and $g \in G$, such that the analogue diagram in Figure 1 becomes a Cartesian square of right G-sets.

Definition 2.4 *Given a group G, we define a right categorified G-set as an internal category in the category of right G-sets, a morphism of right categorified G-sets as an internal functor in the category of right G-sets, and a 2-morphism between morphisms of right categorified G-sets as an internal natural transformation in $\mathsf{Set}\text{-}G$. In this way, thanks to [8] and [2, Prop. 2.4], we obtain a 2-category that, by abuse of notations, we denote with $\mathsf{CSet}\text{-}G$. The category of left categorified G-sets is similarly defined and is clearly isomorphic to the right one. We will also employ the terminology "categorified right group-set" whenever the handled group is not relevant for the context. In all what follows, "categorified group-sets" refers to right ones.*

Remark 2.5 *In fact, the category of categorified G-sets is the category that "contains" $\mathsf{CSet}\text{-}G$ as a full subcategory. Such a category is defined as the category of functors $[\boldsymbol{\Delta}^{op}, \mathsf{Set}\text{-}G]$ from the opposite category of $\boldsymbol{\Delta}$ of finite sets $\Delta_n = \{0, 1, \cdots, n\}$, with increasing maps as arrows, to the category $\mathsf{Set}\text{-}G$ of right G-sets. This somehow justifies the employed terminology in Definition 2.4. The choice that we made in working with the category $\mathsf{CSet}\text{-}G$ instead of $[\boldsymbol{\Delta}^{op}, \mathsf{Set}\text{-}G]$ has its origin in some of the difficulties that the whole category of categorified right G-sets presents, especially in developing a certain kind of Burnside theory, as we will see in the sequel for $\mathsf{CSet}\text{-}G$. Nevertheless, it is noteworthy to mention that many of the results stated below for $\mathsf{CSet}\text{-}G$ might be directly extended to the whole category of categorified G-sets.*

Remark 2.6 *Following Definition 2.4, we have to note that categorified G-sets, morphisms of categorified G-sets, and the relative 2-morphisms constitute, respectively, categories, functors, and natural transformations* *in the usual sense. This means that many definitions of the usual category theory are valid in this setting. For example, given a G-set $\mathcal{X}$, an element $f \in \mathcal{X}_1$ is called an isomorphism if there is $h \in \mathcal{X}_1$ such that $hf = \iota_{\mathcal{X}}(\mathsf{s}_{\mathcal{X}}(f))$ and $fh = \iota_{\mathcal{X}}(\mathsf{t}_{\mathcal{X}}(f))$. In this way, we obtain a forgetful functor from the 2-category of internal categories in a category $\mathcal{C}$ to the category of ordinary small categories.*

Direct consequences of Definition 2.4 are as follows. Let $\mathcal{X}$ be a categorified G-set, given $a, b \in \mathcal{X}_0$, and we will use the notation

$$\mathcal{X}(a,b) = \Big\{ f \in \mathcal{X}_1 | \ \mathsf{s}_{\mathcal{X}}(f) = a \quad \text{and} \quad \mathsf{t}_{\mathcal{X}}(f) = b \Big\}. \tag{2.2}$$

As was mentioned above, the set $\mathcal{X}_2$ admits in a canonical way an action of the G-set, given by $(p,q).g = (pg, qg)$, for every $(p,q) \in \mathcal{X}_2$ and $g \in G$. In this way, the fact that the composition map $m_{\mathcal{X}}$ is a G-equivariant leads to the following equalities:

$$(p \circ q) \,.\, g \;=\; (pg) \circ (qg), \tag{2.3}$$

for every composable arrow p, q, and every element $g \in G$. In this direction, we have that a morphism $p \in \mathcal{X}_1$ is an isomorphism if and only if pg is an isomorphism for some $g \in G$. Furthermore, for any element $a \in \mathcal{X}_0$ and $g \in G$, we have that the map

$$\mathcal{X}(a,a) \longrightarrow \mathcal{X}(ag, ag), \quad \Big(\ell \longmapsto \ell\, g \Big) \tag{2.4}$$

is a morphism of monoids (or semigroups).

In the rest of the paper we will consider the category $\mathsf{CSet}\text{-}G$, defined in Definition 2.4, mainly as a category (the 2-category level will be used to define the concept of weak equivalence in Definition 4.3 below).

Remark 2.7 *Let $\mathcal{X} \in \mathsf{CSet}\text{-}G$: we consider the decomposition of $\mathcal{X}_0$ into orbits (i.e. transitive G-sets) $\mathcal{X}_0 = \biguplus_{\alpha \in A} [x_\alpha] G$ with $x_\alpha \in \mathcal{X}_0$ for each $\alpha \in A$, where A is a certain set of representative elements. Since $\iota_{\mathcal{X}}$ is a morphism of G-sets, for each $\alpha \in A$ we obtain $\iota_{\mathcal{X}}([x_\alpha] G) = [\iota_{\mathcal{X}}(x_\alpha)] G$. Therefore, we can state that*

$$\mathcal{X}_1 = \left(\biguplus_{\alpha \in A} [\iota_{\mathcal{X}}(x_\alpha)] \, G \right) \uplus \left(\biguplus_{\beta \in B} [y_\beta] \, G \right) \tag{2.5}$$

with $x_\alpha \in \mathcal{X}_0$ for each $\alpha \in A$ and $y_\beta \in \mathcal{X}_1 \setminus \iota_{\mathcal{X}}(\mathcal{X}_0)$ for each $\beta \in B$, with $B \cap A = \emptyset$. As a consequence we can state that $\mathcal{X}$ is a discrete category if and only if $\mathcal{X}_1 = \biguplus_{\alpha \in A} [\iota_{\mathcal{X}}(x_\alpha)] \, G$.

3. The symmetric monoidal structures of categorified group-sets

We describe the two symmetric monoidal structures underlying the category of categorified group-sets: one is given by the disjoint union, i.e. the coproduct $\biguplus$, and the other by the product $\times$. Moreover, there is a distributivity law, in an appropriate sense, between these two structures (see formula (3.6)). This fact will be mentioned nowhere below, however, and it will be implicitly used as long as needed.

Let us fix a group G; a categorified G-set will be denoted by $\mathcal{X} := (\mathcal{X}_0, \mathcal{X}_1, \mathsf{s}_{\mathcal{X}}, \mathsf{t}_{\mathcal{X}}, \iota_{\mathcal{X}}, m_{\mathcal{X}})$. In this way, it is evident that the object $\emptyset = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ is initial in $\mathsf{CSet}\text{-}G$, that the disjoint union $\biguplus$ is a coproduct in

$\mathsf{CSet}\text{-}G$, and that $\left(\mathsf{CSet}\text{-}G, \biguplus, \emptyset\right)$ is a strict monoidal category. Specifically, given objects of two categorified G-sets $\mathcal{X}, \mathcal{Y}$, we define

$$\mathsf{s}_{\mathcal{X}\uplus\mathcal{Y}} = \mathsf{s}_{\mathcal{X}} \biguplus \mathsf{s}_{\mathcal{Y}}, \quad \mathsf{t}_{\mathcal{X}\uplus\mathcal{Y}} = \mathsf{t}_{\mathcal{X}} \biguplus \mathsf{t}_{\mathcal{Y}}, \quad \iota_{\mathcal{X}\uplus\mathcal{Y}} = \iota_{\mathcal{X}} \biguplus \iota_{\mathcal{Y}}, \quad \text{and} \quad m_{\mathcal{X}\uplus\mathcal{Y}} = m_{\mathcal{X}} \biguplus m_{\mathcal{Y}}. \tag{3.1}$$

Moreover, given morphisms $\varphi\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $\psi\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathsf{CSet}\text{-}G$, we define the morphism $\varphi \biguplus \psi\colon \mathcal{X} \biguplus \mathcal{A} \longrightarrow \mathcal{Y} \biguplus \mathcal{B}$ as the couple of morphisms $\left(\varphi \biguplus \psi\right)_0 = \varphi_0 \biguplus \psi_0$ and $\left(\varphi \biguplus \psi\right)_1 = \varphi_1 \biguplus \psi_1$ in $\mathsf{Set}\text{-}G$.

Next, we construct a monoidal structure $(\mathsf{CSet}\text{-}G, \times, 1)$, where 1 is the group with a single element. Given $\mathcal{X}, \mathcal{Y} \in \mathsf{CSet}\text{-}G$ we can consider the Cartesian product of the underlying categories $\mathcal{X}$ and $\mathcal{Y}$ in the category of small categories. Thus, we define $(\mathcal{X} \times \mathcal{Y})_0 = \mathcal{X}_0 \times \mathcal{Y}_0$, $(\mathcal{X} \times \mathcal{Y})_1 = \mathcal{X}_1 \times \mathcal{Y}_1$, and

$$\mathcal{X} \times \mathcal{Y} = (\mathcal{X}_0 \times \mathcal{Y}_0,\, \mathcal{X}_1 \times \mathcal{Y}_1,\, \mathsf{s}_{\mathcal{X}\times\mathcal{Y}},\, \mathsf{t}_{\mathcal{X}\times\mathcal{Y}},\, \iota_{\mathcal{X}\times\mathcal{Y}},\, m_{\mathcal{X}\times\mathcal{Y}}), \tag{3.2}$$

where $\mathsf{t}_{\mathcal{X}\times\mathcal{Y}} = (\mathsf{t}_{\mathcal{X}}, \mathsf{t}_{\mathcal{Y}})$, $\mathsf{s}_{\mathcal{X}\times\mathcal{Y}} = (\mathsf{s}_{\mathcal{X}}, \mathsf{s}_{\mathcal{Y}})$, and $\iota_{\mathcal{X}\times\mathcal{Y}} = (\iota_{\mathcal{X}}, \iota_{\mathcal{Y}})$. Regarding the composition, we define

$$\begin{aligned} m_{\mathcal{X}\times\mathcal{Y}}\colon (\mathcal{X} \times \mathcal{Y})_2 &\longrightarrow (\mathcal{X} \times \mathcal{Y})_1 \\ \big((x, y), (a, b)\big) &\longrightarrow (m_{\mathcal{X}}(x, a), m_{\mathcal{Y}}(y, b)). \end{aligned} \tag{3.3}$$

It is immediate to verify that $\mathsf{s}_{\mathcal{X}\times\mathcal{Y}}$, $\mathsf{t}_{\mathcal{X}\times\mathcal{Y}}$, $\iota_{\mathcal{X}\times\mathcal{Y}} = (\iota_{\mathcal{X}}, \iota_{\mathcal{Y}})$, and $m_{\mathcal{X}\times\mathcal{Y}}$ are morphisms in $\mathsf{Set}\text{-}G$. We have to prove that the diagrams of Definition 2.1 about $\mathcal{X} \times \mathcal{Y}$ are commutative, but this is a direct verification and follows from the analogous diagrams about $\mathcal{X}$ and $\mathcal{Y}$.

Now, given morphisms $\varphi\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $\psi\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathsf{CSet}\text{-}G$, we define the morphism

$$\varphi \times \psi\colon \mathcal{X} \times \mathcal{A} \longrightarrow \mathcal{Y} \times \mathcal{B}$$

in $\mathsf{CSet}\text{-}G$ as the couple of morphisms $(\varphi \times \psi)_0 = \varphi_0 \times \psi_0$ and $(\varphi \times \psi)_1 = \varphi_1 \times \psi_1$ in $\mathsf{Set}\text{-}G$. We have to prove that the diagrams of Definition 2.2 about $\varphi \times \psi$ are commutative, but this is a direct verification and follows from the analogous diagrams about φ and ψ. It is now obvious that we have constructed a functor $(- \times -) : \mathsf{CSet}\text{-}G \times \mathsf{CSet}\text{-}G \longrightarrow \mathsf{CSet}\text{-}G$.

In order to complete the monoidal structure on $\mathsf{CSet}\text{-}G$, we need to construct natural isomorphisms

$$\Phi\colon\ (\mathrm{Id}_{\mathsf{CSet}\text{-}G} \times 1) \longrightarrow \mathrm{Id}_{\mathsf{CSet}\text{-}G}, \qquad \Psi\colon\ (1 \times \mathrm{Id}_{\mathsf{CSet}\text{-}G}) \longrightarrow \mathrm{Id}_{\mathsf{CSet}\text{-}G}$$

and the associator

$$((- \times -) \times -) \longrightarrow (- \times (- \times -)).$$

The associator is the identity, which is clearly a natural isomorphism and satisfies the pentagonal constraint. We will construct only Φ because Ψ can be realized in a similar way. Let $\mathcal{X}$ be a categorified G-set: for $i = 0, 1$ we have the isomorphisms of G-sets

$$\begin{aligned} \Phi(\mathcal{X})_i\colon \mathcal{X}_i \times 1 &\longrightarrow \mathcal{X}_i, \\ (a, 1) &\longrightarrow a. \end{aligned} \tag{3.4}$$

It is a direct verification to check that $\Phi(\mathcal{X})$ is a morphism of categorified G-sets. Now we just have to prove that

$$\Phi\colon\ (\mathrm{Id}_{\mathsf{CSet}\text{-}G} \times 1) \longrightarrow \mathrm{Id}_{\mathsf{CSet}\text{-}G}$$

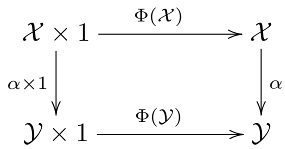

Figure 6. Naturality of Φ: first diagram.

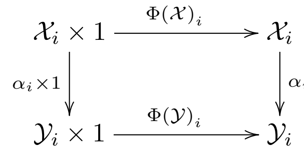

Figure 7. Naturality of Φ: second diagram.

is a natural transformation: let $\alpha \colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism in $\mathsf{CSet}\text{-}G$. We have to show that the diagram in Figure 6 is commutative, which is equivalent to saying that the diagram in Figure 7 is commutative for $i = 0, 1$, but this is immediate to check. Lastly, it is obvious that Φ and Ψ satisfy the triangular identities, and this completes the claimed constructions.

In the same direction, there is an "inclusion" functor from the category of usual right G-sets to the category of categorified G-sets. Specifically, we have a fully faithful functor

$$\begin{aligned} \mathcal{I}\text{-}G \colon \mathsf{Set}\text{-}G &\longrightarrow \mathsf{CSet}\text{-}G \\ X &\longrightarrow \Big(X, X, \mathsf{s}_X, \mathsf{t}_X, \iota_X, m_X\Big), \end{aligned} \tag{3.5}$$

where $\mathsf{s}_X = \mathsf{t}_X = \iota_X = \mathrm{Id}X$ and $m_X = \mathrm{pr}_1 \colon X_2 \longrightarrow X$; that is, the underlying category of the image of X is the discrete one. The behavior of $\mathcal{I}\text{-}G$ on morphisms is obvious. Basically the idea is that the image of $\mathcal{I}\text{-}G$ is given by discrete categories. Moreover, we will use the abuse of notation $1 = \mathcal{I}\text{-}G\,(1)$.

Summing up, we have proved the following result.

Proposition 3.1 *Given a group G, $(\mathsf{CSet}\text{-}G, \times, 1)$ is a symmetric monoidal category such that the functor $\mathcal{I}\text{-}\mathcal{G}$ becomes a strict symmetric monoidal functor.*

Lastly, let us discuss the distributivity between the two operations $\uplus$ and $\times$ in the category $\mathsf{CSet}\text{-}G$. We know that we have two monoidal structures $\big(\mathsf{CSet}\text{-}G, \biguplus, \emptyset\big)$ and $(\mathsf{CSet}\text{-}G, \times, 1)$. As was mentioned above, it is necessary to prove the distributivity of $\times$ over $\biguplus$. Let $\mathcal{X}, \mathcal{Y}, \mathcal{A} \in \mathsf{CSet}\text{-}G$: we have to construct a morphism

$$\lambda \colon \; \big[\mathcal{X} \biguplus \mathcal{Y}\big] \times \mathcal{A} \longrightarrow [\mathcal{X} \times \mathcal{A}] \biguplus [\mathcal{Y} \times \mathcal{A}] \tag{3.6}$$

in $\mathsf{CSet}\text{-}G$. We define it as the couple of morphisms in $\mathsf{Set}\text{-}\mathcal{G}$, for $i = 0, 1$,

$$\lambda_i \colon \; \big[\mathcal{X}_i \biguplus \mathcal{Y}_i\big] \times \mathcal{A}_i \longrightarrow [\mathcal{X}_i \times \mathcal{A}_i] \biguplus [\mathcal{Y}_i \times \mathcal{A}_i],$$

that send (a, b) to (a, b) both if $a \in \mathcal{X}_i$ and if $a \in \mathcal{Y}_i$. The proof of the commutativity of the diagrams of Definition 2.2 is now obvious.

4. Weak equivalences and 2-morphisms

We introduce the notion of weak equivalence in the category of categorified group-sets. Then we show that both operations $\biguplus$ and $\times$ are compatible with the weak equivalence relation, as well as with 2-morphisms. This will be crucial to build up the categorified Burnside functor in the forthcoming sections.

Fix a group G and consider its category $\mathsf{CSet}\text{-}G$ of categorified G-sets. Given morphisms in $\mathsf{CSet}\text{-}G$

$$\varphi, \psi \colon \mathcal{X} \longrightarrow \mathcal{Y} \qquad \text{and} \qquad \varepsilon, \eta \colon \mathcal{A} \longrightarrow \mathcal{B},$$

let us consider 2-morphisms in $\mathsf{CSet}\text{-}G$ $\alpha\colon \varphi \longrightarrow \psi$ and $\beta\colon \varepsilon \longrightarrow \eta$: we have

$$\varphi \biguplus \varepsilon,\, \psi \biguplus \eta \colon \mathcal{X} \biguplus \mathcal{A} \longrightarrow \mathcal{Y} \biguplus \mathcal{B}.$$

We define the 2-morphism $\alpha \biguplus \beta$ in $\mathsf{CSet}\text{-}G$ as the following morphism in $\mathsf{CSet}\text{-}G$:

$$\begin{array}{rrcl} \alpha \biguplus \beta\colon & \mathcal{X}_0 \biguplus \mathcal{A}_0 & \longrightarrow & \mathcal{Y}_1 \biguplus \mathcal{B}_1 \\ & \mathcal{X}_0 \ni x & \longrightarrow & \alpha(x) \in \mathcal{Y}_1 \\ & \mathcal{A}_0 \ni x & \longrightarrow & \beta(x) \in \mathcal{B}_1. \end{array} \tag{4.1}$$

The verification that $\alpha \biguplus \beta$ renders the diagrams of Definition 2.3 commutative is immediate and derives from the relative diagrams regarding α and β. Now we consider $\psi'\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $\eta'\colon \mathcal{A} \longrightarrow \mathcal{B}$, morphisms in $\mathsf{CSet}\text{-}G$, and $\alpha'\colon \psi \longrightarrow \psi'$ and $\beta'\colon \eta \longrightarrow \eta'$, a 2-morphism in $\mathsf{CSet}\text{-}G$.

Lemma 4.1 *We have*

$$\left(\alpha' \biguplus \beta'\right)\left(\alpha \biguplus \beta\right) = (\alpha'\alpha) \biguplus (\beta'\beta) : \varphi \biguplus \varepsilon \longrightarrow \psi \biguplus \eta \qquad \textit{and} \qquad \mathrm{Id}_{\varphi \uplus \psi} = \mathrm{Id}_{\varphi} \biguplus \mathrm{Id}_{\psi}\,. \tag{4.2}$$

Proof It is immediate. □

On the other hand, taking α, β as above, we have

$$\varphi \times \varepsilon,\, \psi \times \eta \colon\, (\mathcal{X}, \varsigma) \times (\mathcal{A}, \lambda) \longrightarrow (\mathcal{Y}, \theta) \times (\mathcal{B}, \mu)\,,$$

and we want to define a 2-morphism

$$\alpha \times \beta \colon \varphi \times \varepsilon \longrightarrow \psi \times \eta$$

in $\mathsf{CSet}\text{-}G$ as the morphism of G-sets

$$\begin{aligned} \alpha \times \beta \colon \mathcal{X}_0 \times \mathcal{A}_0 &\longrightarrow \mathcal{Y}_1 \times \mathcal{B}_1 \\ (x, a) &\longrightarrow (\alpha(x), \beta(a))\,. \end{aligned}$$

In this way, it is sufficient to check that the diagrams of Definition 2.3 in Figure 8 are commutative. Regarding the two triangular diagrams in Figure 8, the commutativity is obvious because $\mathsf{s}_{\mathcal{Y}\times\mathcal{B}} = \mathsf{s}_{\mathcal{Y}} \times \mathsf{s}_{\mathcal{B}}$ and $\mathsf{t}_{\mathcal{Y}\times\mathcal{B}} = \mathsf{t}_{\mathcal{Y}} \times \mathsf{t}_{\mathcal{B}}$. Regarding the commutativity of the rectangular diagram in Figure 8, we calculate, for each $(x, a) \in \mathcal{X}_1 \times \mathcal{A}_1$,

$$\begin{aligned} & m_{\mathcal{Y}\times\mathcal{B}} \Delta\left(\psi_1 \times \eta_1, (\alpha \times \beta)\, \mathsf{s}_{\mathcal{X}\times\mathcal{A}}\right)(x, a) = m_{\mathcal{Y}\times\mathcal{B}} \Big(\left(\psi_1(x), \eta_1(a)\right), \left(\alpha\mathsf{s}_{\mathcal{X}}(x), \beta\mathsf{s}_{\mathcal{A}}(a)\right) \Big) \\ &= \Big(m_{\mathcal{Y}}\left(\psi_1(x), \alpha\mathsf{s}_{\mathcal{X}}(x)\right), m_{\mathcal{B}}\left(\eta_1(a), \beta\mathsf{s}_{\mathcal{A}}(a)\right) \Big) = \Big(m_{\mathcal{Y}}\left(\alpha\mathsf{t}_{\mathcal{X}}(x), \varphi_1(x)\right), m_{\mathcal{B}}\left(\beta\mathsf{s}_{\mathcal{A}}(a), \varepsilon_1(a)\right) \Big) \\ &= m_{\mathcal{Y}\times\mathcal{B}} \Big(\left(\alpha\mathsf{t}_{\mathcal{X}}(x), \beta\mathsf{t}_{\mathcal{A}}(a)\right), \left(\varphi_1(a), \varepsilon_1(a)\right) \Big) = m_{\mathcal{Y}\times\mathcal{B}} \Delta\left((\alpha \times \beta)\, \mathsf{t}_{\mathcal{X}\times\mathcal{A}}, \varphi_1 \times \varepsilon_1\right)(x, a) \end{aligned}$$

and this finishes the proof that $\alpha \times \beta$ is a 2-morphism.

Lemma 4.2 *Let $\alpha, \alpha' : \varphi \to \psi$ and $\beta, \beta' : \varepsilon \to \eta$ as above. Then we have*

$$(\alpha' \times \beta')\,(\alpha \times \beta) = (\alpha'\alpha) \times (\beta'\beta) : \varphi \times \varepsilon \longrightarrow \psi \times \eta \qquad \textit{and} \qquad \mathrm{Id}_{\varphi\times\varepsilon} = \mathrm{Id}_{\varphi} \times \mathrm{Id}_{\varepsilon}\,. \tag{4.3}$$

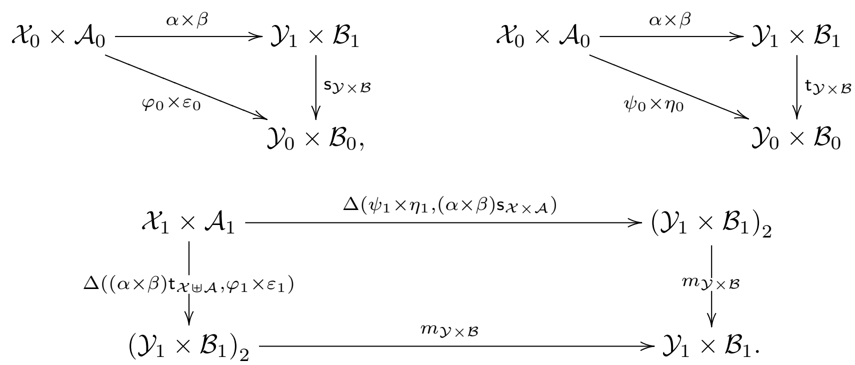

Figure 8. Product of two internal natural transformation.

Proof Take an object $(x, a) \in (\mathcal{X} \times \mathcal{A})_0 = \mathcal{X}_0 \times \mathcal{A}_0$, and then we have

$$(\mathrm{Id}_{\varphi} \times \mathrm{Id}_{\varepsilon})(x, a) = \mathrm{Id}_{\varphi}(x) \times \mathrm{Id}_{\varepsilon}(a) = \iota_{\varphi_0(x)} \times \iota_{\varepsilon_0(a)} = \iota_{\varphi_0(x) \times \varepsilon_0(a)} = \iota_{(\varphi_0 \times \varepsilon_0)(x,a)} = \mathrm{Id}_{\varphi \times \varepsilon}(x, a).$$

This gives the second stated equality. As for the first one, we have

$$\begin{aligned} (\alpha' \times \beta')\,(\alpha \times \beta)\,(x, a) &= m_{\mathcal{Y} \times \mathcal{B}} \Delta\,(\alpha' \times \beta', \alpha \times \beta)\,(x, a) \\ &= m_{\mathcal{Y} \times \mathcal{B}}\,((\alpha' \times \beta')\,(x, a), (\alpha \times \beta)\,(x, a)) = m_{\mathcal{Y} \times \mathcal{B}} \Big(\,(\alpha'(x), \beta'(a))\,, (\alpha(x), \beta(a))\,\Big) \\ &= \Big(m_{\mathcal{Y}}\,(\alpha'(x), \alpha(x))\,, m_{\mathcal{B}}\,(\beta'(a), \beta(a))\,\Big) = \Big(\,(\alpha'\alpha)\,(x), (\beta'\beta)\,(a)\Big) = ((\alpha'\alpha) \times (\beta'\beta))\,(x, a). \end{aligned}$$

Thus, $(\alpha' \times \beta')\,(\alpha \times \beta) = (\alpha'\alpha) \times (\beta'\beta)$ and this finishes the proof. □

Next, we give the main definition of this section.

Definition 4.3 *Let $\mathcal{X}, \mathcal{Y} \in \mathsf{CSet}\text{-}G$. We say that $\mathcal{X}$ and $\mathcal{Y}$ are* weakly equivalent *and we write $\mathcal{X} \sim_{\mathsf{we}} \mathcal{Y}$ if there are, in the category $\mathsf{CSet}\text{-}G$, morphisms $\varphi \colon \mathcal{X} \longrightarrow \mathcal{Y}$, $\psi \colon \mathcal{Y} \longrightarrow \mathcal{X}$ and 2-isomorphisms $\alpha \colon \psi\varphi \longrightarrow \mathrm{Id}_{\mathcal{X}}$, $\beta \colon \varphi\psi \longrightarrow \mathrm{Id}_{\mathcal{Y}}$, in the sense of Definition 2.4.*

The following lemma states that the disjoint union and the fiber product are compatible with the weak equivalence relation. A compatibility criterion will be used in the sequel.

Proposition 4.4 *Let $\mathcal{X}, \mathcal{Y}, \mathcal{A}$, and $\mathcal{B}$ be objects in the category $\mathsf{CSet}\text{-}G$ such that $\mathcal{X} \sim_{\mathsf{we}} \mathcal{Y}$ and $\mathcal{A} \sim_{\mathsf{we}} \mathcal{B}$. Then we have the following weak equivalence relations:*

$$\Big[\mathcal{X} \uplus \mathcal{A}\Big] \sim_{\mathsf{we}} \Big[\mathcal{Y} \uplus \mathcal{B}\Big] \qquad \textit{and} \qquad \Big[\mathcal{X} \times \mathcal{A}\Big] \sim_{\mathsf{we}} \Big[\mathcal{Y} \times \mathcal{B}\Big]. \tag{4.4}$$

Proof Attached to the stated weak equivalences relations, there are morphisms in $\mathsf{CSet}\text{-}G$

$$\varphi \colon \mathcal{X} \longrightarrow \mathcal{Y}, \qquad \psi \colon \mathcal{Y} \longrightarrow \mathcal{X}, \qquad \eta \colon \mathcal{A} \longrightarrow \mathcal{B}, \qquad \text{and} \qquad \varepsilon \colon \mathcal{B} \longrightarrow \mathcal{A}$$

such that there are 2-isomorphisms in $\mathsf{CSet}\text{-}G$

$$\alpha \colon \psi\varphi \longrightarrow \mathrm{Id}_{\mathcal{X}}, \qquad \beta \colon \varphi\psi \longrightarrow \mathrm{Id}_{\mathcal{Y}}, \qquad \gamma \colon \varepsilon\eta \longrightarrow \mathrm{Id}_{\mathcal{A}}, \qquad \text{and} \qquad \delta \colon \eta\varepsilon \longrightarrow \mathrm{Id}_{\mathcal{B}}\,.$$

Applying Lemma 4.1, we get

$$\left(\alpha^{-1} \biguplus \gamma^{-1}\right)\left(\alpha \biguplus \gamma\right) = \left(\alpha^{-1}\alpha\right) \biguplus \left(\gamma^{-1}\gamma\right) = \mathrm{Id}_{\psi\varphi} \biguplus \mathrm{Id}_{\varepsilon\eta} = \mathrm{Id}_{\psi\varphi \uplus \varepsilon\eta}$$

and

$$\left(\alpha \biguplus \gamma\right)\left(\alpha^{-1} \biguplus \gamma^{-1}\right) = \left(\alpha\alpha^{-1}\right) \biguplus \left(\gamma\gamma^{-1}\right) = \mathrm{Id}_{\mathrm{Id}_{\mathcal{X}} \uplus \mathrm{Id}_{\mathcal{Y}}} = \mathrm{Id}_{\mathrm{Id}_{\mathcal{X} \uplus \mathcal{Y}}}.$$

Thus,

$$\alpha \biguplus \gamma \colon \psi\varphi \biguplus \varepsilon\eta \longrightarrow \mathrm{Id}_{\mathcal{X} \uplus \mathcal{A}}$$

is a 2-isomorphism in $\mathsf{CSet}\text{-}G$. We can prove in the same way that $\beta \biguplus \delta \colon \varphi\psi \biguplus \eta\varepsilon \longrightarrow \mathrm{Id}_{\mathcal{Y} \uplus \mathcal{B}}$ is a 2-isomorphism. As a consequence, we obtain

$$\left[\mathcal{X} \biguplus \mathcal{A}\right] \sim_{\mathsf{we}} \left[\mathcal{Y} \biguplus \mathcal{B}\right].$$

Lastly, an analogue computation, this time using Lemma 4.2, shows the relation $\left[\mathcal{X} \times \mathcal{A}\right] \sim_{\mathsf{we}} \left[\mathcal{Y} \times \mathcal{B}\right]$, and this completes the proof. □

As already noted in Remark 2.6, many concepts of the ordinary category theory can be extended to categorified G-sets. With reference to [1, p. 51], we will now briefly explain how to extend the concept of a skeleton of a category. The proofs are essentially the same; thus, we will omit them.

Definition 4.5 *Let $\mathcal{X}$ and $\mathcal{Y}$ be categorified G-sets.*

1. *We say that $\mathcal{Y}$ is a categorified G-subset of $\mathcal{X}$ if the following two conditions are satisfied:*
   (a) *$\mathcal{Y}_0$ and $\mathcal{Y}_1$ are G-subsets respectively of $\mathcal{X}_0$ and $\mathcal{X}_1$;*
   (b) *the structure on $\mathcal{Y}$ is appropriately induced by that on $\mathcal{X}$ by restriction, in the usual sense for subcategories.*
2. *We say that $\mathcal{Y}$ is a full categorified G-subset of $\mathcal{X}$ if it is a categorified G-subset of $\mathcal{X}$ such that, for each $a, b \in \mathcal{Y}_0$, we have $\mathcal{Y}(a,b) = \mathcal{X}(a,b)$.*
3. *We say that $\mathcal{Y}$ is an isomorphism-dense categorified G-subset of $\mathcal{X}$ if it is a categorified G-subset of $\mathcal{X}$ such that, for each $a \in \mathcal{Y}_0$, there is $b \in \mathcal{X}_0$ such that there is an isomorphism $f \in \mathcal{X}_1$, with $\mathsf{s}_{\mathcal{X}}(f) = a$ and $\mathsf{t}_{\mathcal{X}}(f) = a$.*

Definition 4.6 *The skeleteon of a categorified G-set is a full, isomorphism-dense categorified G-subset in which no two distinct objects are isomorphic.*

Direct consequences of this definition are the following facts:

Proposition 4.7 *Let G be a group. Then:*

1. *Every categorified G-set has a skeleton.*
2. *Two skeletons of a categorified G-set $\mathcal{X}$ are isomorphic.*

3. Every skeleton of a categorified G-set $\mathcal{X}$ is weakly equivalent to $\mathcal{X}$.

In particular, two categorified G-sets $\mathcal{X}$ and $\mathcal{Y}$ are weakly equivalent if and only their skeletons are isomorphic as categorified G-sets.

Proof It is immediate. □

Remark 4.8 *Thanks to Proposition 4.7, given a categorified G-set $\mathcal{X}$, we can denote one of its skeletons by $\mathrm{Sk}(\mathcal{X})$, with the specification that $\mathrm{Sk}(\mathcal{X})$ is unique up to isomorphism. If there is a family $(\mathcal{X}_\alpha)_{\alpha\in A}$ of categorified G-subsets of $\mathcal{X}$ such that $\mathcal{X} = \biguplus_{\alpha\in A}\mathcal{X}_\alpha$, then we clearly have the following isomorphism of categorified G-set:*

$$\mathrm{Sk}(\mathcal{X}) = \mathrm{Sk}\left(\biguplus_{\alpha\in A} X_\alpha\right) \cong \biguplus_{\alpha\in A}\mathrm{Sk}(\mathcal{X}_\alpha). \tag{4.5}$$

Definition 4.9 *Given a categorified G-set $\mathcal{X}$, we denote with $\mathrm{d}(\mathcal{X})$ the disjoint union of the connected components of $\mathcal{X}$ whose skeleton is discrete; that is, it has only identities as arrows. On the other hand, we denote with $\mathrm{nd}(\mathcal{X})$ the disjoint union of the connected components of $\mathcal{X}$ whose skeleton is not discrete; that is, it has at least an arrow that is not an identity. As a consequence we have $\mathcal{X} = \mathrm{d}(\mathcal{X}) \biguplus \mathrm{nd}(\mathcal{X})$.*

5. Examples of weak equivalences

In this section we deal with a class of examples of categorified group-sets, and we give a certain criterion of the weak equivalence relation between the objects of this class. All group-sets are right ones.

Let $\mathcal{C}$ be a small category, G a group, and X a G-set. We set $\mathcal{X}_0 = \mathcal{C}_0 \times X$ and $\mathcal{X}_1 = \mathcal{C}_1 \times X$. The G-action on the $\mathcal{X}_i$s is defined as $(a,x)g := (a,xg)$ for each $(a,x)\in\mathcal{X}_i$ and $g\in G$, for $i=0,1$. We set

$$\begin{aligned} \mathsf{s}_{\mathcal{X}}\colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 & \mathsf{t}_{\mathcal{X}}\colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 & \iota_{\mathcal{X}}\colon \mathcal{X}_1 &\longrightarrow \mathcal{X}_0 \\ (a,x) &\longrightarrow (\mathsf{s}_{\mathcal{C}}(a),g), & (a,x) &\longrightarrow (\mathsf{t}_{\mathcal{C}}(a),g), & (a,x) &\longrightarrow (\iota_{\mathcal{C}}(a),g) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} m_{\mathcal{X}}\colon \mathcal{X}_2 &\longrightarrow \mathcal{X}_1 \\ \big((a,x),(b,y)\big) &\longrightarrow (m_{\mathcal{C}}(a,b),x) \end{aligned} \tag{5.2}$$

with $(\mathsf{s}_{\mathcal{C}}(a),x) = \mathsf{s}_{\mathcal{X}}(a,x) = \mathsf{t}_{\mathcal{X}}(b,y) = (\mathsf{t}_{\mathcal{C}}(b),y)$. It is evident that $\mathsf{s}_{\mathcal{X}}$, $\mathsf{t}_{\mathcal{X}}$, $\iota_{\mathcal{X}}$, and $m_{\mathcal{X}}$ are morphisms of G-sets and thus we just have to prove that the diagrams of Definition 2.1 about $\mathcal{X}$ are commutative, but this follows immediately by the analogous diagrams about $\mathcal{C}$ (recall that an ordinary small category, as defined in the footnote of page 2070, can be considered as an internal category in the category of sets).

Now let $\mathcal{C}$ and $\mathcal{D}$ be small categories, let X and Y be G-sets, and set $\mathcal{X}_0 = \mathcal{C}_0 \times X$, $\mathcal{X}_1 = \mathcal{C}_1 \times X$, $\mathcal{Y}_0 = \mathcal{D}_0 \times Y$, and $\mathcal{Y}_1 = \mathcal{D}_1 \times Y$. Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and $\varphi\colon X \longrightarrow Y$ a morphism in $\mathsf{Set}\text{-}G$. We want to define a morphism (F,φ) in $\mathsf{CSet}\text{-}G$ setting $(F,\varphi)_0 = (F_0,\varphi)$ and $(F,\varphi)_1 = (F_1,\varphi)$. It is evident that $(F,\varphi)_0$ and $(F,\varphi)_1$ are morphisms of G-sets; thus, we just have to prove that the diagrams of Definition 2.2 about (F,φ) are commutative, but this follows immediately by the analogous diagrams about F (an ordinary functor can be considered as an internal functor in the category of sets).

Given another functor $P\colon \mathcal{C} \longrightarrow \mathcal{D}$, we consider a natural transformation $\mu\colon F \longrightarrow P$. With the notations already introduced, we have a morphism $(P,\varphi)\colon \mathcal{X} \longrightarrow \mathcal{Y}$ in $\mathsf{CSet}\text{-}G$ such that $(P,\varphi)_0 = (P_0,\varphi)$ and $(P,\varphi)_1 = (P_1,\varphi)$. We want to define a 2-morphism $(\mu,\varphi) : (F,\varphi) \longrightarrow (P,\varphi)$ given by a morphism of right $\mathcal{G}$-sets

$$(\mu,\varphi)\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1$$
$$(a,x) \longrightarrow (\mu(a),\varphi(x))$$

and therefore we have to check that the diagrams in Figure 9 commute, but this is just a direct verification.

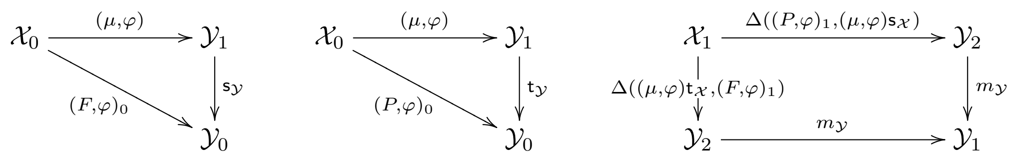

Figure 9. Internal natural transformation: example.

Remark 5.1 *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be small categories, $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $P\colon \mathcal{D} \longrightarrow \mathcal{E}$ functors, and $\varphi\colon X \longrightarrow Y$ and $\psi\colon Y \longrightarrow Z$ morphisms of right G-sets. Using the above notation, we set $\mathcal{X}_i = \mathcal{C}_i \times X$, $\mathcal{Y}_i = \mathcal{D}_i \times Y$ and $\mathcal{Z}_i = \mathcal{E}_i \times Z$ for $i = 0,1$. It is obvious that $\mathrm{Id}_{\mathcal{X}} = (\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_X) : \mathcal{X} \longrightarrow \mathcal{X}$ and $(PF, \psi\varphi) = (P,\psi)(F,\varphi)\colon \mathcal{X} \longrightarrow \mathcal{Z}$ are morphisms of categorified G-sets.*

Proposition 5.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, $X, Y \in \mathsf{Set}\text{-}G$, $F, P, Q\colon \mathcal{C} \longrightarrow \mathcal{D}$ functors, $\varphi\colon X \longrightarrow Y$ a morphism of right $\mathcal{G}$-sets, and $\mu\colon F \longrightarrow P$ and $\lambda\colon P \longrightarrow Q$ natural transformations. We define 2-morphisms*

$$(\mu,\varphi) : (F,\varphi) \longrightarrow (P,\varphi), \qquad (\lambda,\varphi) : (P,\varphi) \longrightarrow (Q,\varphi), \qquad \textit{and} \qquad (\lambda\mu,\varphi) : (F,\varphi) \longrightarrow (Q,\varphi)$$

in $\mathsf{CSet}\text{-}G$ given by morphisms of right G-sets

$$(\mu,\varphi)\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1 \qquad\qquad (\lambda,\varphi)\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1$$
$$(a,x) \longrightarrow (\mu(a),\varphi(x)), \qquad\qquad (a,x) \longrightarrow (\lambda(x),\varphi(x))$$

and

$$(\lambda\mu,\varphi)\colon (F,\varphi) \longrightarrow (Q,\varphi)$$
$$(a,x) \longrightarrow ((\lambda\mu)(a),\varphi(x)).$$

Then $(\lambda\mu,\varphi) = (\lambda,\varphi)\,(\mu,\varphi)$ and $(\mathrm{Id}_F,\varphi) = \mathrm{Id}_{(F,\varphi)}$.

Proof We consider the 2-morphism $(\lambda,\varphi)\,(\mu,\varphi) : (F,\varphi) \longrightarrow (Q,\varphi)$ given by a morphism in right G-set

$$r\colon \mathcal{X}_0 \longrightarrow \mathcal{Y}_1$$
$$(a,x) \longrightarrow r(a,x) = m_{\mathcal{Y}}\Big(\,(\lambda,\varphi)\,(a,x),(\mu,\varphi)\,(a,x)\Big).$$

For each $(a,x) \in \mathcal{X}_0$, we obtain

$$\begin{aligned} r(a,x) &= m_{\mathcal{Y}}\left((\lambda,\varphi)(a,x),(\mu,\varphi)(a,x)\right) = m_{\mathcal{Y}}\left((\lambda(a),\varphi(x)),(\mu(a),\varphi(x))\right) \\ &= \left(m_{\mathcal{C}}\left(\lambda(a),\mu(a)\right),\varphi(x)\right) = ((\lambda\mu)(a),\varphi(x)) = (\lambda\mu,\varphi)(a,x). \end{aligned}$$

The 2-morphism $\mathrm{Id}_{(F,\varphi)}$ is given by the morphism of right G-sets

$$\begin{aligned} z\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1 \\ (a,x) &\longrightarrow \iota_{\mathcal{Y}}\left(F(a),\varphi(x)\right) = (\iota_{\mathcal{D}}F(a),\varphi(x)) \end{aligned}$$

and (Id_F,φ) is given by

$$\begin{aligned} (\mathrm{Id}_F,\varphi)\colon \mathcal{X}_0 &\longrightarrow \mathcal{Y}_1 \\ (a,x) &\longrightarrow (\mathrm{Id}_F(a),\varphi(x)) = (\iota_{\mathcal{D}}(F(a)),\varphi(x)). \end{aligned}$$

Thus, we get $\mathrm{Id}_{(F,\varphi)} = (\mathrm{Id}_F,\varphi)$. □

Consider now $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $P\colon \mathcal{D} \longrightarrow \mathcal{C}$, two functors between small categories. Assume we have $\varepsilon\colon FP \longrightarrow \mathrm{Id}_{\mathcal{D}}$ and $\eta\colon PF \longrightarrow \mathrm{Id}_{\mathcal{C}}$, two natural isomorphisms, and $\varphi\colon X \longrightarrow Y$, an isomorphism of G-sets. Keeping the previous notations, we set $\mathcal{X}_i = \mathcal{C}_i \times X$ and $\mathcal{Y}_i = \mathcal{D}_i \times Y$ for $i=0,1$ and consider the morphisms of categorified G-sets $(F,\varphi)\colon \mathcal{X} \longrightarrow \mathcal{Y}$ and $(P,\varphi)\colon \mathcal{Y} \longrightarrow \mathcal{X}$. Henceforth, we have that

$$\left(P,\varphi^{-1}\right)(F,\varphi) = \left(PF,\varphi^{-1}\varphi\right) = (PF,\mathrm{Id}_X), \quad \mathrm{Id}_{\mathcal{X}} = (\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_X) : \mathcal{X} \longrightarrow \mathcal{X}$$

and

$$(F,\varphi)\left(P,\varphi^{-1}\right) = \left(FP,\varphi\varphi^{-1}\right) = (FP,\mathrm{Id}_Y), \quad \mathrm{Id}_{\mathcal{Y}} = (\mathrm{Id}_{\mathcal{D}},\mathrm{Id}_Y) : \mathcal{Y} \longrightarrow \mathcal{Y}.$$

We define 2-morphisms

$$(\varepsilon,\mathrm{Id}_X) : (P,\psi)(F,\varphi) \longrightarrow \mathrm{Id}_{\mathcal{X}} \qquad \text{and} \qquad (\eta,\mathrm{Id}_Y) : (F,\varphi)(P,\psi) \longrightarrow \mathrm{Id}_{\mathcal{Y}}$$

in $\mathsf{CSet}\text{-}G$ as the morphisms of G-sets

$$\begin{aligned} (\varepsilon,\mathrm{Id}_X)\colon \mathcal{X}_0 &\longrightarrow \mathcal{X}_1 \\ (a,x) &\longrightarrow (\varepsilon(a),x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} (\eta,\mathrm{Id}_Y)\colon \mathcal{Y}_0 &\longrightarrow \mathcal{Y}_1 \\ (\beta,y) &\longrightarrow (\eta(\beta),y), \end{aligned}$$

respectively. We have already proved that $(\varepsilon,\mathrm{Id}_X)$ and (η,Id_Y) are well defined. Considering $\varepsilon^{-1}\colon\ \mathrm{Id}_{\mathcal{D}} \longrightarrow FG$ and $\eta^{-1}\colon\ \mathrm{Id}_{\mathcal{C}} \longrightarrow GF$ we can construct the 2-morphisms

$$\left(\varepsilon^{-1},\mathrm{Id}_X\right) :\ \mathrm{Id}_{\mathcal{X}} \longrightarrow (P,\psi)(F,\varphi) \qquad \text{and} \qquad \left(\eta^{-1},\mathrm{Id}_Y\right) :\ \mathrm{Id}_{\mathcal{Y}} \longrightarrow (F,\varphi)(P,\psi)$$

in $\mathsf{CSet}\text{-}G$

$$\begin{aligned} \left(\varepsilon^{-1},\mathrm{Id}_X\right)\colon \mathcal{X}_0 &\longrightarrow \mathcal{X}_1 \\ (a,x) &\longrightarrow (\varepsilon^{-1}(a),x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \left(\eta^{-1},\mathrm{Id}_Y\right)\colon \mathcal{Y}_0 &\longrightarrow \mathcal{Y}_1 \\ (\beta,y) &\longrightarrow (\eta^{-1}(\beta),y). \end{aligned}$$

We calculate

$$\mathrm{Id}_{(FP,\mathrm{Id}_Y)} = (\mathrm{Id}_{FP},\mathrm{Id}_Y) = \left(\varepsilon^{-1}\varepsilon,\mathrm{Id}_Y\right) = \left(\varepsilon^{-1},\mathrm{Id}_Y\right)(\varepsilon,\mathrm{Id}_Y)$$

and

$$\mathrm{Id}_{(\mathrm{Id}_{\mathcal{D}},\mathrm{Id}_Y)} = (\mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}, \mathrm{Id}_Y) = \left(\varepsilon\varepsilon^{-1}, \mathrm{Id}_Y\right) = (\varepsilon, \mathrm{Id}_Y)\left(\varepsilon^{-1}, \mathrm{Id}_Y\right).$$

In the same way we prove that

$$\mathrm{Id}_{(PF,\mathrm{Id}_X)} = \left(\eta^{-1}, \mathrm{Id}_X\right)(\eta, \mathrm{Id}_X) \qquad \text{and} \qquad \mathrm{Id}_{(\mathrm{Id}_{\mathcal{C}},\mathrm{Id}_X)} = (\eta, \mathrm{Id}_X)\left(\eta^{-1}, \mathrm{Id}_X\right).$$

Therefore, we have that $\mathcal{X} \sim_{\mathsf{we}} \mathcal{Y}$.

Using the forgetful functor of Remark 2.6, we deduce, from the previous argumentations, that $\mathcal{X}$ and $\mathcal{Y}$ cannot be weakly equivalent if the categories $\mathcal{C}$ and $\mathcal{D}$ are not equivalent. Furthermore, we proved the following result.

Proposition 5.3 *Let G be any group. Then we have a functor*

$$\mathsf{SCat} \times \mathsf{Set}\text{-}G \longrightarrow \mathsf{CSet}\text{-}G, \quad \Big((\mathscr{C}, X) \longrightarrow \mathscr{C} \times X\Big), \tag{5.3}$$

where SCat denotes the category of small categories. Moreover, two categorified G-sets of the form $\mathscr{C} \times X$ and $\mathcal{D} \times Y$ are weakly equivalent if and only if X and Y are isomorphic two G-sets and $\mathscr{C}$, $\mathcal{D}$ are equivalent categories.

Proof Straightforward. □

6. The right double translation category of a right categorified group-set

Given a group G, let us consider a G-set X. It is well known (see, for instance, [12]) that the pair $(X \times G, X)$ admits in a natural way a structure of a groupoid[†], called a right translation groupoid $X \rtimes G$. Clearly, the nerve of the underlying category creates a categorified set, which is not a categorified group-set, except in some trivial cases. This translation groupoid illustrates, in fact, the orbits that the group G creates acting on X, and its isotropy groups coincide with the stabilizers. The aim of this section is to explore this construction for the new categorified G-sets. To this end, the notion of double category, introduced for the first time in [8], is essential.

Definition 6.1 *A double category is an internal category in the category of small categories.*

Given a double category $\mathcal{D}$, the relevant functors are illustrated in the diagram in Figure 10, where $\mathcal{D}_0$,

$$\mathcal{D}_1 \underset{\mathsf{S}_{\mathcal{D}}}{\overset{\mathsf{T}_{\mathcal{D}}}{\rightrightarrows}} \mathcal{D}_0 \xrightarrow{\ \mathsf{I}_{\mathcal{D}}\ } \mathcal{D}_1 \xleftarrow{\ \mathsf{M}_{\mathcal{D}}\ } \mathcal{D}_2$$

Figure 10. Defining functors of a double category.

$\mathcal{D}_1$, and $\mathcal{D}_2 = \mathcal{D}\,{}_{\mathsf{S}_{\mathcal{D}}}\!\times_{\mathsf{T}_{\mathcal{D}}} \mathcal{D}_1$ are categories (we remind the reader that the category of small categories has pull-backs).

[†]This is by definition a small category where each morphism is an isomorphism.

Definition 6.2 *Given a double category $\mathcal{D}$, the set $(\mathcal{D}_0)_0$ is called the set of objects, the set $(\mathcal{D}_0)_1$ is called the set of horizontal morphisms, the set $(\mathcal{D}_1)_0$ is called the set of vertical morphisms, and the set $(\mathcal{D}_1)_1$ is called the set of squares. Moreover, the category $\mathcal{D}_0$ is called the category of objects and $\mathcal{D}_1$ is called the category of morphisms.*

The reason behind Definition 6.2 will be manifest in the forthcoming diagrams of this section that will also illustrate how to operate with double categories.

Given a group G and a categorified G-set $\mathcal{X}$, we set $\mathcal{D}_i = \mathcal{X}_i \rtimes G$ for $i = 0, 1$. We are now going to construct a structure of double category $\mathcal{D}$ starting from the categories $\mathcal{D}_0$ and $\mathcal{D}_1$. We define the target functor $\mathsf{T}_{\mathcal{D}} \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ in Figure 11. We will now check that $\mathsf{T}_{\mathcal{D}}$ is a functor. Given a vertical morphism

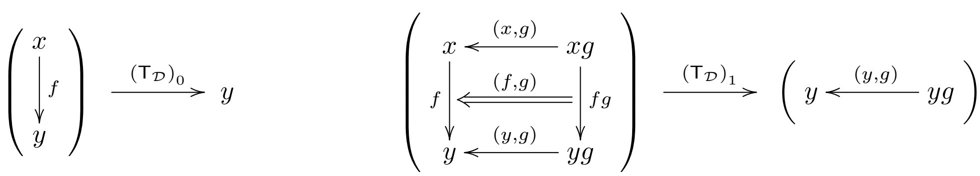

Figure 11. Definition of the target functor.

$f \colon x \longrightarrow y$ in $(\mathcal{D}_1)_0$ we check the identity condition in Figure 12 and the composition condition in Figure 13.

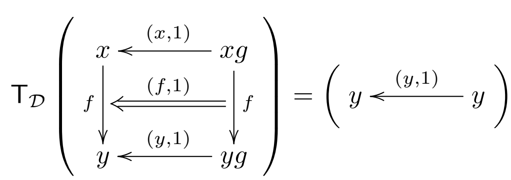

Figure 12. Target functor: identity condition.

We define the source functor $\mathsf{S}_{\mathcal{D}} \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_0$ in Figure 14, the identity functor $\mathsf{I}_{\mathcal{D}} \colon \mathcal{D}_0 \longrightarrow \mathcal{D}_1$ in Figure 15, and the composition functor (or multiplication functor) $\mathsf{M}_{\mathcal{D}} \colon \mathcal{D}_2 \longrightarrow \mathcal{D}_1$ in Figure 16. The proofs that $\mathsf{S}_{\mathcal{D}}$ and $\mathsf{I}_{\mathcal{D}}$ are functors are similar to the one for $\mathsf{T}_{\mathcal{D}}$. Regarding $\mathsf{M}_{\mathcal{D}}$, we prove that it is a functor in Figures 17 and 18, where $h \in G$ and $(lf)g = (lg)(fg)$ for every $g \in G$.

Definition 6.3 *The double category $\mathcal{D}$ just constructed is called the right translation double category of the right categorified G-set $\mathcal{X}$ and we denote it by $\mathcal{X} \rtimes G$.*

Example 6.4 *Given a small category $\mathcal{C}$ and a group G, let $\mathcal{X} = \mathcal{C} \times G$ be the right categorified G-set of Section 5. We want to describe the double translation category $\mathcal{D} = \mathcal{C} \times X$. For $i = 0, 1$ we have $(\mathcal{D}_0)_0 = (\mathcal{X}_0 \rtimes G)_0 = \mathcal{X}_0$, $(\mathcal{D}_0)_1 = (\mathcal{X}_0 \times G)_1 = \mathcal{X}_0 \times G$, $(\mathcal{D}_1)_0 = (\mathcal{X}_1 \rtimes G)_0 = \mathcal{X}_1$, and $(\mathcal{D}_1)_1 = (\mathcal{X}_1 \times G)_1 = \mathcal{X}_1 \times G$. The target, source, and identity functors are as follows:*

$$\mathsf{T}_{\mathcal{D}} = (\mathsf{t}_{\mathcal{C}} \times \mathrm{Id}_X, \mathsf{t}_{\mathcal{C}}), \qquad \mathsf{S}_{\mathcal{D}} = (\mathsf{s}_{\mathcal{C}} \times \mathrm{Id}_X, \mathsf{s}_{\mathcal{C}}), \qquad \textit{and} \qquad \mathsf{I}_{\mathcal{D}} = (\iota_{\mathcal{C}} \times \mathrm{Id}_X, \iota_{\mathcal{C}}). \tag{6.1}$$

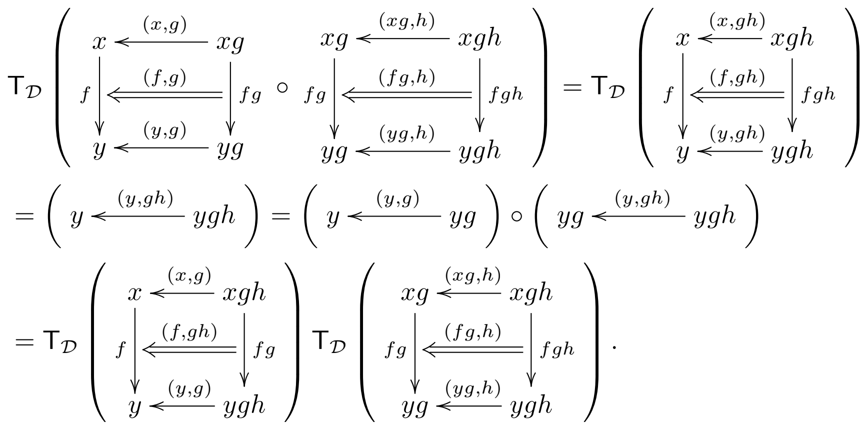

Figure 13. Target functor: composition condition.

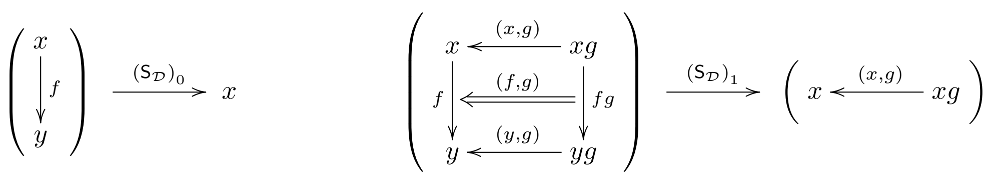

Figure 14. Definition of the source functor.

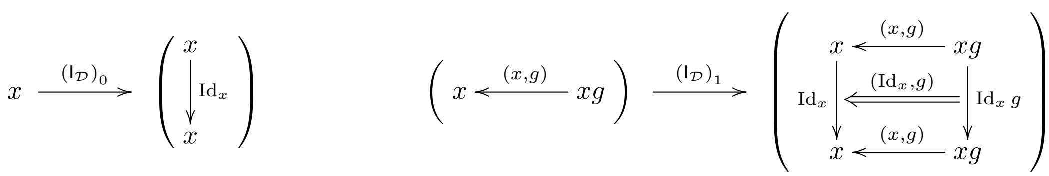

Figure 15. Definition of the identity functor.

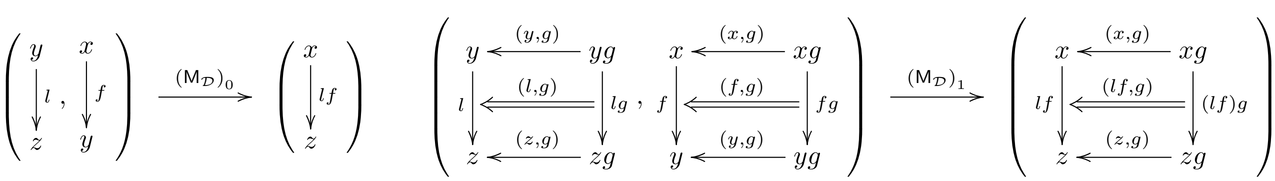

Figure 16. Definition of the composition functor.

Regarding the composition functor, we have:

$$\begin{aligned} \mathsf{M}_{\mathcal{D}} \colon \mathcal{D}_2 &\longrightarrow \mathcal{D}_1 \\ (\mathcal{D}_2)_0 \ni (h,f) &\longrightarrow m_{\mathcal{X}}(h,f) \\ (\mathcal{D}_2)_1 \ni ((h,g),(f,g)) &\longrightarrow (m_{\mathcal{X}}(h,f),g)\,. \end{aligned} \tag{6.2}$$

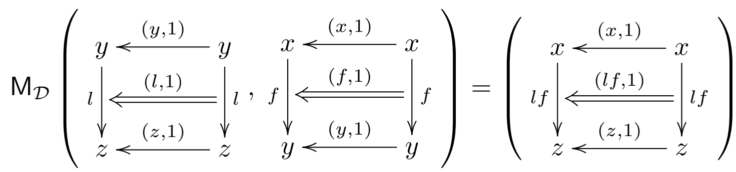

Figure 17. Composition functor: identity condition.

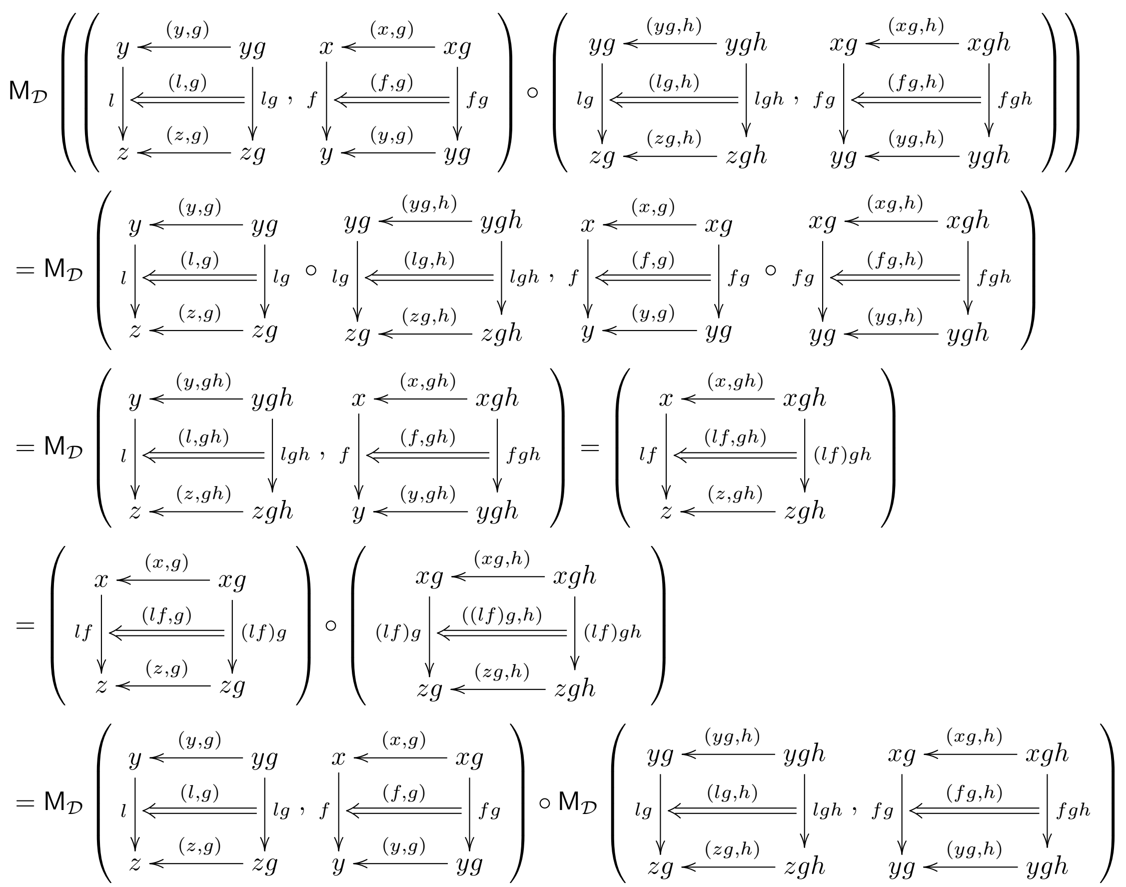

Figure 18. Composition functor: composition condition.

Definition 6.5 *Given a double category $\mathcal{D}$, let us consider a square $Q \in (\mathcal{D}_1)_1$. We define the vertices of the square Q as* $\mathsf{s}_{\mathcal{D}_0}(\mathsf{S}_{\mathcal{D}}(Q))$, $\mathsf{t}_{\mathcal{D}_0}(\mathsf{S}_{\mathcal{D}}(Q))$, $\mathsf{s}_{\mathcal{D}_0}(\mathsf{T}_{\mathcal{D}}(Q))$, *and* $\mathsf{t}_{\mathcal{D}_0}(\mathsf{T}_{\mathcal{D}}(Q))$.

Example 6.6 *Given $\mathcal{X} \in \mathsf{CSet}\text{-}G$, in the square in $\mathcal{X} \rtimes G$ in Figure 19, the vertices are x, xg, y, and yg.*

Definition 6.7 *Let $\mathcal{X} \in \mathsf{CSet}\text{-}G$: for each $a, b \in \mathcal{X}_0$ we define the relation $a \,\underline{\mathsf{sq}}\, b$ if and only if there is a square of $\mathcal{X} \rtimes G$ with a and b among its vertices.*

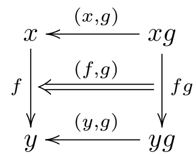

Figure 19. Vertices of a square.

Remark 6.8 *Given a morphism* $(f\colon a \longrightarrow b) \in \mathcal{X}_1$*, an object* $\alpha \in \mathcal{X}_0$*, and* $g \in G$*, we have the diagrams in Figure 20 and therefore* $a \, \underline{\mathsf{sq}} \, b$ *and* $\alpha \, \underline{\mathsf{sq}} \, \alpha g$*.*

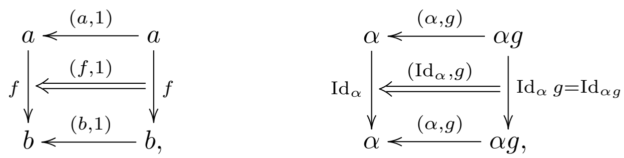

Figure 20. Relation $\underline{\mathsf{sq}}$: examples.

Remark 6.9 *The relation* $\underline{\mathsf{sq}}$ *is reflexive and symmetric but not transitive; it is enough to consider the following example with a trivial action:* $\mathcal{X}_0 = \{a, b, c\}$ *and only* $f\colon a \longrightarrow b$ *and* $h\colon a \longrightarrow c$ *as not isomorphism arrows. In this case it is clear that* $a \, \underline{\mathsf{sq}} \, b$ *and* $a \, \underline{\mathsf{sq}} \, c$ *but we do not have* $b \, \underline{\mathsf{sq}} \, c$*.*

This motivates us to give the next definition.

Definition 6.10 *Let* $\mathcal{X} \in \mathsf{CSet}\text{-}G$*: for each* $a, b \in \mathcal{X}_0$ *we define* $a \, \underline{\mathsf{Sq}} \, b$ *if and only if there are* $a_0, \ldots, a_n \in \mathcal{X}_0$*, with* $n \in \mathbb{N}^+$*, such that for each* $i \in \{0, \ldots, n-1\}$*,* $a_i \, \underline{\mathsf{sq}} \, a_{i+1}$*. This means that* $\underline{\mathsf{Sq}}$ *is the equivalence relations generated by* $\underline{\mathsf{sq}}$*. Given* $a \in \mathcal{X}_0$*, we denote with* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ *the full subcategory of* $\mathcal{X}$ *such that the set of objects of* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ *is the equivalence class of* a *with respect to* $\underline{\mathsf{Sq}}$*. We also set* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f) := \mathrm{Orb}_{\underline{\mathsf{Sq}}}(\mathsf{s}_{\mathcal{X}}(f)) = \mathrm{Orb}_{\underline{\mathsf{Sq}}}(\mathsf{t}_{\mathcal{X}}(f))$ *for every* $f \in \mathcal{X}_1$*. Moreover, we denote with* $\mathrm{rep}_{\underline{\mathsf{Sq}}}(\mathcal{X})$ *a set of objects of* $\mathcal{X}$ *that acts as a set of representative elements with respect to the relation* $\underline{\mathsf{Sq}}$*. Note that, for every* $f \in \mathcal{X}_1$ *(respectively, for each* $a \in \mathcal{X}_0$*),* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f)$ *(respectively,* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$*) contains both the* G*-orbit of* f *(respectively, of* a*) and the connected component of the category* $\mathcal{X}$ *(see [16, pp. 88 and 90]) that contains* f *(respectively,* a*). As a consequence, both* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(f)$ *and* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ *are right* G*-sets, can be decomposed in* G*-orbits, and are called the orbit categories of* f *and* a*, respectively.*

Remark 6.11 *Given a categorified* G*-set* $\mathcal{X}$*, let us consider a categorified* G*-subset* $\mathcal{Y}$ *of* $\mathcal{X}$*. If we consider a square like the one in Example 6.6, with* $(f\colon a \longrightarrow b) \in \mathcal{X}_1$ *and* $g \in G$*, then, thanks to Remark 6.8, the following statements are equivalent:*

1. $x \in \mathcal{Y}_0$*;*

2. $xg \in \mathcal{Y}_0$ *;*

3. $yg \in \mathcal{Y}_0$ *;*

4. $y \in \mathcal{Y}_0$ *.*

As a consequence, for each $\alpha \in \mathcal{Y}_0$ *,* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(\alpha)$ *is a categorified* G*-subset of* $\mathcal{Y}$ *.*

Proposition 6.12 *Let be* $\mathcal{X} \in \mathsf{CSet}\text{-}G$*: we have*

$$\mathcal{X} = \biguplus_{a \,\in\, \mathrm{rep}_{\underline{\mathsf{Sq}}}(\mathcal{X})} \mathrm{Orb}_{\underline{\mathsf{Sq}}}(a) \tag{6.3}$$

and, for every $a \in \mathrm{rep}_{\underline{\mathsf{Sq}}}(\mathcal{X})$*, there cannot be two categorified* G*-sets* $\mathcal{Y}$ *and* $\mathcal{Y}'$*, both not empty, such that* $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a) = \mathcal{Y} \uplus \mathcal{Y}'$*.*

Proof Every category is the disjoint union of its connected components and, for each $a \in \mathcal{X}_0$, $\mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ is the disjoint union of the connected components of $\mathcal{X}$ with objects in the $\underline{\mathsf{Sq}}$ relation with a.

Regarding the last statement, by contradiction, let us assume that there are two categorified G-sets $\mathcal{Y}$ and $\mathcal{Y}'$ such that $\mathcal{Y}_0 \cap \mathcal{Y}'_0 = \emptyset$ and $\mathcal{Y} \neq \emptyset \neq \mathcal{Y}'$. We consider $b \in \mathcal{Y}_0$ and $b' \in \mathcal{Y}'_0$: then $b, b' \in \mathrm{Orb}_{\underline{\mathsf{Sq}}}(a)$ and thus $b \,\underline{\mathsf{Sq}}\, a \,\underline{\mathsf{Sq}}\, b'$. Thanks to Remark 6.11 we obtain $b \in \left(\mathrm{Orb}_{\underline{\mathsf{Sq}}}(b')\right)_0 \subseteq \mathcal{Y}'_0$, which is absurd because $b \in \mathcal{Y}_0$.
□

7. Burnside ring functors of groups

We will give the main steps to construct the categorified Burnside ring of a given group, using its category of categorified group-sets, and we will compare this new ring with the classical one, providing a natural transformation between the two contravariant functors. Moreover, we will show that this natural transformation is injective but not surjective and we will conclude by illustrating, with an example, the difference between the classical case and the categorified one.

Given a morphism of groups $\varphi\colon H \longrightarrow G$, we define the induced functor, referred to as the induction functor (see [11, Section 2.3]):

$$\varphi^*\colon \mathsf{CSet}\text{-}G \longrightarrow \mathsf{CSet}\text{-}H, \tag{7.1}$$

which sends the categorified G-set $\mathcal{X}$ to the categorified H-set $\varphi^*(\mathcal{X})$ with the action $x \cdot h := x\varphi(h)$ for each $x \in \mathcal{X}_i$ and $h \in H$, for $i = 0, 1$. The target, source, identity, and multiplication maps remain the same but they will be morphisms in $\mathsf{Set}\text{-}H$ and not in $\mathsf{Set}\text{-}G$. Given a morphism of categorified G-set $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$, we define the morphism of categorified H-sets $\varphi^*(f)\colon \varphi^*(\mathcal{X}) \longrightarrow \varphi^*(\mathcal{Y})$ as the morphism f. In a similar way to how it has been done in the classical case, it is possible to prove that φ^* is monoidal with respect to both $\uplus$ and $\times$.

Let G be a group. We will now briefly develop a Burnside theory based on categorified G-sets, following the similar picture of the classical, noncategorified, case (see El Kaoutit and Spinosa, "Burnside theory for groupoids", arXiv: 1807.04470v1, math.GR, July 2018). We recall that by the word rig we mean a unital commutative ring that does not necessarily have additive inverses (another term often used is semiring). We denote by $\mathsf{csets}\text{-}G$ the 2-category of finite categorified G-sets (we say that $\mathcal{X} \in \mathsf{CSet}\text{-}G$ is finite if $\mathcal{X}_1$ is finite); however, as with $\mathsf{CSet}\text{-}G$, we will mainly consider it as a category. We denote by $\mathscr{L}_{\mathsf{C}}(G)$ the quotient set of $\mathsf{csets}\text{-}G$ by the equivalence relation $\sim_{\mathsf{we}}$. Given a morphism of groups $\varphi \colon H \longrightarrow G$, as in the classical case, we obtain a monomorphism of rigs

$$\begin{aligned} \mathscr{L}_{\mathsf{C}}(\varphi) \colon \mathscr{L}_{\mathsf{C}}(G) &\longrightarrow \mathscr{L}_{\mathsf{C}}(H) \\ [X] &\longrightarrow [\varphi^*(X)] . \end{aligned} \tag{7.2}$$

In this way we obtain a contravariant functor $\mathscr{L}_{\mathsf{C}} \colon \mathbf{Grp} \longrightarrow \mathbf{Rig}$, i.e. from the category of groups to the category of rigs.

Given a group G, using the functor $\mathcal{I}\text{-}G$ of Eq. (3.5), we can construct an injective morphism from the classical Burnside rig of G to the categorified Burnside rig of G in the following way:

$$\begin{aligned} \mathscr{L}_{\mathcal{I}}(G) \colon \mathscr{L}(G) &\longrightarrow \mathscr{L}_{\mathsf{C}}(G) \\ [X] &\longrightarrow [\mathcal{I}\text{-}G(X)] . \end{aligned} \tag{7.3}$$

In this way we obtain a natural transformation $\mathscr{L}_{\mathcal{I}} \colon \mathscr{L} \longrightarrow \mathscr{L}_{\mathsf{C}}$; that is, given a morphism of groups $\varphi \colon H \longrightarrow G$, the diagram in Figure 21 is commutative.

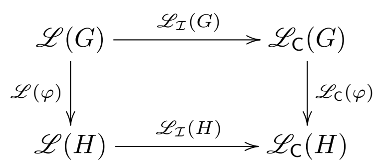

Figure 21. Naturality of $\mathscr{L}_{\mathcal{I}}$.

Remark 7.1 *In the classical case the cancellative property of the additive monoid $\mathscr{L}(G)$ is guaranteed by the Burnside theorem (see [4, Thm. 2.4.5]) but, in this context, we still do not know whether a similar theorem is true or not and, consequently, we cannot say whether the additive monoid $\mathscr{L}_{\mathsf{C}}(G)$ satisfies the cancellative property or not.*

Proposition 7.2 *The rig homomorphism $\mathscr{L}_{\mathcal{I}}(G)$ is injective but not surjective.*

Proof We will use the abuses of notations $\mathcal{I}\text{-}G = \mathcal{I}$ and $\mathscr{L}_{\mathcal{I}} = \mathscr{L}_{\mathcal{I}}(G)$. Given $X, Y \in \mathsf{Set}\text{-}\mathcal{G}$ such that $\mathscr{L}_{\mathcal{I}}([X]) = \mathscr{L}_{\mathcal{I}}([Y])$, we have $[\mathcal{I}(X)] = \mathscr{L}_{\mathcal{I}}([X]) = \mathscr{L}_{\mathcal{I}}([Y]) = [\mathcal{I}(Y)]$ and thus $\mathcal{I}(X) \sim_{\mathsf{we}} \mathcal{I}(Y)$. Therefore

$$\mathcal{I}(X) \cong \mathrm{Sk}(\mathcal{I}(X)) \cong \mathrm{Sk}(\mathcal{I}(Y)) \cong \mathcal{I}(Y)$$

and $[X] = [Y]$.

Now, by contradiction, let us assume that $\mathscr{L}_{\mathcal{I}}$ is surjective. We consider a categorified G-set $\mathcal{X}$ such that $\mathcal{X} \neq \mathrm{d}(\mathcal{X})$ (or, equivalently, such that $\mathrm{nd}(\mathcal{X}) \neq \emptyset$). Then there is a G-set Y such that $[\mathcal{X}] = \mathscr{L}_{\mathcal{I}}([Y]) = [\mathcal{I}(Y)]$ and thus $\mathcal{X} \sim_{\mathsf{we}} \mathcal{I}(Y)$. As a consequence,

$$\mathrm{Sk}(\mathcal{X}) \cong \mathrm{Sk}(\mathcal{I}(Y)) \cong \mathcal{I}(Y),$$

and thus $\mathrm{nd}(\mathrm{Sk}(\mathcal{X})) \cong \mathrm{nd}(\mathcal{I}(Y)) = \emptyset$, which is absurd. □

We define $\mathscr{B}_{\mathsf{C}} = \mathscr{G}\mathscr{L}_{\mathsf{C}}$ and $\mathscr{B}_{\mathcal{I}} = \mathscr{G}\mathscr{L}_{\mathcal{I}}$, where $\mathscr{G}$ is the Grothendieck functor (see [18, pp. 3–5]), i.e. a functor that associates to each rig an opportune ring with a specific universal property. In this way we obtain a contravariant functor $\mathscr{B}_{\mathsf{C}} \colon \mathbf{Grp} \longrightarrow \mathbf{CRing}$, i.e. from the category of groups to the category of commutative rings, and a natural transformation

$$\mathscr{B}_{\mathcal{I}} = \mathscr{G}\mathscr{L}_{\mathcal{I}} \colon \mathscr{B} = \mathscr{G}\mathscr{L} \longrightarrow \mathscr{B}_{\mathsf{C}} = \mathscr{G}\mathscr{L}_{\mathsf{C}}. \tag{7.4}$$

The commutative ring $\mathscr{B}_{\mathsf{C}}(G)$, for a given group G, is called the categorified Burnside ring of G.

Proposition 7.3 *The ring homomorphism $\mathscr{B}_{\mathcal{I}}(G)$ is injective but not surjective.*

Proof We will use the abuses of notations $\mathcal{I}\text{-}G = \mathcal{I}$, $\mathscr{L}_{\mathcal{I}} = \mathscr{L}_{\mathcal{I}}(G)$, and $\mathscr{B}_{\mathcal{I}} = \mathscr{B}_{\mathcal{I}}(G)$. We consider

$$\Big[[X], [Y] \Big], \; \Big[[A], [B] \Big] \in \mathscr{B}_{\mathsf{C}}(G)$$

such that

$$\Big[\mathscr{L}_{\mathcal{I}}([X]), \mathscr{L}_{\mathcal{I}}([Y]) \Big] = \mathscr{B}_{\mathcal{I}} \Big(\Big[[X], [Y] \Big] \Big) = \mathscr{B}_{\mathcal{I}} \Big(\Big[[A], [B] \Big] \Big) = \Big[\mathscr{L}_{\mathcal{I}}([A]), \mathscr{L}_{\mathcal{I}}([B]) \Big].$$

There is $[\mathcal{E}] \in \mathscr{L}_{\mathsf{C}}(G)$ such that

$$\mathscr{L}_{\mathcal{I}}([X]) + \mathscr{L}_{\mathcal{I}}([B]) + [\mathcal{E}] = \mathscr{L}_{\mathcal{I}}([A]) + \mathscr{L}_{\mathcal{I}}([Y]) + [\mathcal{E}],$$

and therefore

$$\mathcal{I}(X) \uplus \mathcal{I}(B) \uplus \mathcal{E} \sim_{\mathsf{we}} \mathcal{I}(A) \uplus \mathcal{I}(Y) \uplus \mathcal{E}.$$

As a consequence we obtain

$$\begin{gathered} \mathcal{I}(X) \uplus \mathcal{I}(B) \uplus \mathrm{Sk}(\mathcal{E}) \cong \mathrm{Sk}(\mathcal{I}(X)) \uplus \mathrm{Sk}(\mathcal{I}(B)) \uplus \mathrm{Sk}(\mathcal{E}) \\ \cong \mathrm{Sk}(\mathcal{I}(A)) \uplus \mathrm{Sk}(\mathcal{I}(Y)) \uplus \mathrm{Sk}(\mathcal{E}) \cong \mathcal{I}(A) \uplus \mathcal{I}(Y) \uplus \mathrm{Sk}(\mathcal{E}), \end{gathered}$$

and thus

$$\begin{gathered} \mathcal{I}(X) \uplus \mathcal{I}(B) \uplus \mathrm{d}(\mathrm{Sk}(\mathcal{E})) \cong \mathrm{d}(\mathcal{I}(X)) \uplus \mathrm{d}(\mathcal{I}(B)) \uplus \mathrm{d}(\mathrm{Sk}(\mathcal{E})) \\ \cong \mathrm{d}(\mathcal{I}(A)) \uplus \mathrm{d}(\mathcal{I}(Y)) \uplus \mathrm{d}(\mathrm{Sk}(\mathcal{E})) \cong \mathcal{I}(A) \uplus \mathcal{I}(Y) \uplus \mathrm{d}(\mathrm{Sk}(\mathcal{E})) \end{gathered}$$

so therefore, considering that $\mathrm{d}(\mathrm{Sk}(\mathcal{E}))$ is finite and can be considered a classical G-set, since the additive monoid $\mathscr{L}(G)$ satisfies the cancellative property (see Remark 7.1), we obtain

$$\mathcal{I}(X) \uplus \mathcal{I}(B) \cong \mathcal{I}(A) \uplus \mathcal{I}(Y).$$

This is equivalent to saying that $X \uplus B \cong A \uplus Y$; that is, $\Big[[X], [Y] \Big] = \Big[[A], [B] \Big]$.

Now, by contradiction, let us assume that $\mathscr{B}_{\mathcal{I}}$ is surjective. We consider a categorified G-set $\mathcal{X}$ such that $\mathcal{X} \neq \mathrm{d}(\mathcal{X})$ (or, equivalently, such that $\mathrm{nd}(\mathcal{X}) \neq \emptyset$). Then there are G-sets Y and Z such that

$$\Big[[\mathcal{X}], [\emptyset] \Big] = \mathscr{B}_{\mathcal{I}} \Big(\Big[[Y], [Z] \Big] \Big) = \Big[\mathscr{L}_{\mathcal{I}}([Y]), \mathscr{L}_{\mathcal{I}}([Z]) \Big],$$

and thus there is a categorified G-set $\mathcal{E}$ such that

$$[\mathcal{X}] + \mathscr{L}_{\mathcal{I}}([Z]) + [\mathcal{E}] = \mathscr{L}_{\mathcal{I}}([Y]) + [\emptyset] + [\mathcal{E}] .$$

As a consequence, $\mathcal{X} \biguplus \mathcal{I}(Z) \biguplus \mathcal{E} \sim_{\mathsf{we}} \mathcal{I}(Y) \biguplus \mathcal{E}$ and thus

$$\mathrm{Sk}(\mathcal{X}) \biguplus \mathrm{Sk}(\mathcal{I}(Z)) \biguplus \mathrm{Sk}(\mathcal{E}) \cong \mathrm{Sk}(\mathcal{I}(Y)) \biguplus \mathrm{Sk}(\mathcal{E}) .$$

Considering that $\mathrm{nd}(\mathrm{Sk}(\mathcal{I}(Z))) = \emptyset$ and $\mathrm{nd}(\mathrm{Sk}(\mathcal{I}(Y))) = \emptyset$, we obtain

$$\mathrm{nd}(\mathrm{Sk}(\mathcal{X})) \biguplus \mathrm{nd}(\mathrm{Sk}(\mathcal{E})) \cong \mathrm{nd}(\mathrm{Sk}(\mathcal{E})) .$$

Since both $\mathcal{X}$ and $\mathcal{E}$ are finite categorified G-sets ($\mathcal{X}_1$ and $\mathcal{E}_1$ are finite), examining the cardinalities, we have

$$|\mathrm{nd}(\mathrm{Sk}(\mathcal{X}))| + |\mathrm{nd}(\mathrm{Sk}(\mathcal{E}))| = |\mathrm{nd}(\mathrm{Sk}(\mathcal{E}))| ,$$

which implies $|\mathrm{nd}(\mathrm{Sk}(\mathcal{X}))| = 0$. This means that $\emptyset = \mathrm{nd}(\mathrm{Sk}(\mathcal{X})) \cong \mathrm{Sk}(\mathrm{nd}(\mathcal{X}))$; therefore, $\mathrm{nd}(\mathcal{X}) = \emptyset$, which is absurd. □

Example 7.4 *Given a group G, the categories in Figure 22, with only the identities as isomorphisms and with the actions opportunely defined, thanks to Proposition 4.7, are all examples of not weakly equivalent categorified*

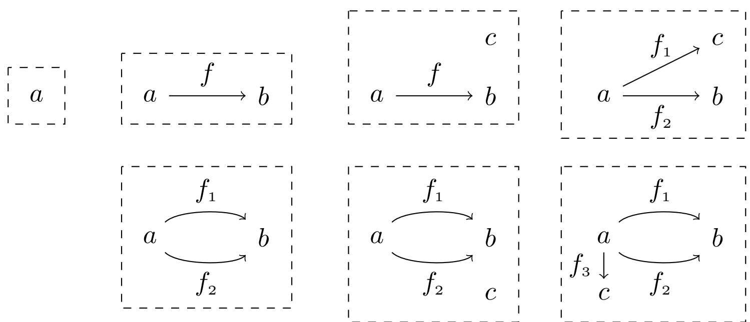

Figure 22. Examples of categorified G-sets.

G-sets; thus, they give rise to different elements in $\mathscr{L}_{\mathsf{C}}(G)$. For example, consider $G = 1$: in this case the G-action is trivial and thus $\mathscr{L}(G) = \mathbb{N}$ (we just have finite sets), but, regarding $\mathscr{L}_{\mathsf{C}}(G)$, we have to consider all the classes given by all the previous not weakly equivalent categorified G-sets. More specifically, for each $n \in \mathbb{N}^+$, in the case of $\mathscr{L}(1)$, we just have n points, but in the case of $\mathscr{L}_{\mathsf{C}}(G)$, we have to consider all the possible graphs with n vertices!

8. Categorified Burnside ring of a groupoid

In this section we will analyze the situation for a given groupoid. More precisely, we will introduce and examine the category of categorified $\mathcal{G}$-sets for a given groupoid $\mathcal{G}$ and prove analogous results concerning its associated categorified Burnside ring. We also provide a nonisomorphic map from the classical Burnside ring to the categorified one. The necessary definitions can be found in [11, 12] (see also El Kaoutit and Spinosa, "Burnside theory for groupoids", arXiv: 1807.04470v1, math.GR, July 2018). Here we will recall only the essential notions.

Definition 8.1 *A groupoid is a (small) category such that all its morphisms are invertible. Given a groupoid* $\mathcal{G}$*, a set* X*, and a map* $\varsigma\colon X \longrightarrow \mathcal{G}_0$*, we say that* (X,ς) *is a right* $\mathcal{G}$*-set, with a structure map* ς*, if there is a right action* $\rho\colon X\, {}_{\varsigma}\!\times_{\mathsf{t}} \mathcal{G}_1 \longrightarrow X$*, sending* (x,g) *to* xg*, and satisfying the following conditions:*

1. *for each* $x \in X$ *and* $g \in \mathcal{G}_1$ *such that* $\varsigma(x) = \mathsf{t}(g)$*, we have* $\mathsf{s}(g) = \varsigma(xg)$*;*
2. *for each* $x \in X$*, we have* $x\iota_{\varsigma(x)} = x$*;*
3. *for each* $x \in X$ *and* $g, h \in \mathcal{G}_1$ *such that* $\varsigma(x) = \mathsf{t}(g)$ *and* $\mathsf{s}(g) = \mathsf{t}(h)$*, we have* $(xg)h = x(gh)$*.*

The pair (X,ς) *is referred to as a right* $\mathcal{G}$*-set. If there is no confusion, then we will also say that* X *is a right groupoid-set.*

The main difference in the groupoid situation is that every object has a structure map that rules the groupoid action over itself; this can be directly seen once a groupoid-set is realized as a functor from the underlying category of $\mathcal{G}$ to the core of the category of sets[‡]. From now on all groupoid-sets are right ones.

Definition 8.2 *Given a groupoid* $\mathcal{G}$*, a morphism of* $\mathcal{G}$*-sets, also called a* $\mathcal{G}$*-equivariant map,* $F\colon (X,\varsigma) \longrightarrow (Y,\vartheta)$*, is a function* $F\colon X \longrightarrow Y$ *such that the diagrams in Figure 23 commute.*

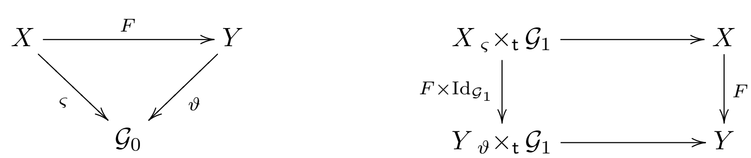

Figure 23. Definition of a groupoid-set.

The resulting category, denoted by $\mathsf{Set}\text{-}\mathcal{G}$, has two monoidal structures: the disjoint union $\biguplus$ and the fiber product, defined as follows. Given $(X,\varsigma),(Y,\vartheta) \in \mathsf{Set}\text{-}\mathcal{G}$, we set:

$$(X,\varsigma) \underset{\mathcal{G}_0}{\times} (Y,\vartheta) \;=\; \big(X \underset{\mathcal{G}_0}{\times} Y, \varsigma\vartheta\big), \tag{8.1}$$

where $X \underset{\mathcal{G}_0}{\times} Y = X\, {}_{\varsigma}\!\times_{\vartheta} Y$ and $\varsigma\vartheta\colon X \underset{\mathcal{G}_0}{\times} Y \longrightarrow \mathcal{G}_0$ sends (x,y) to $\varsigma(x)=\vartheta(y)$.

A categorified $\mathcal{G}$-set $(\mathcal{X},\varsigma)$ is defined as an internal category in $\mathsf{Set}\text{-}\mathcal{G}$ with set of objects $(\mathcal{X}_0,\varsigma_0)$ and set of morphisms $(\mathcal{X}_1,\varsigma_1)$. Furthermore, in the monoidal structure that realizes the multiplication of the Burnside rig, the Cartesian product $\times$ has to be replaced by the fiber product $\underset{\mathcal{G}_0}{\times}$.

Proposition 8.3 *The object*

$$\emptyset = \Big((\emptyset,\emptyset), (\emptyset,\emptyset), \emptyset, \emptyset, \emptyset, \emptyset \Big)$$

is initial in $\mathsf{CSet}\text{-}\mathcal{G}$*,* $\biguplus$ *is a coproduct in* $\mathsf{CSet}\text{-}\mathcal{G}$*, and* $\big(\mathsf{CSet}\text{-}\mathcal{G}, \biguplus, \emptyset\big)$ *is a strict monoidal category.*

[‡]The core category of a given category is the subcategory whose morphisms are all isomorphisms.

As in the group case, we have an "inclusion" functor from the category of usual $\mathcal{G}$-sets to the category of categorified $\mathcal{G}$-sets:

$$\begin{aligned} \mathcal{I}\text{-}\mathcal{G}\colon \mathsf{Set}\text{-}\mathcal{G} &\longrightarrow \mathsf{CSet}\text{-}\mathcal{G} \\ (X,\varsigma) &\longrightarrow \Big((X,\varsigma), (X,\varsigma), \mathsf{s}_X, \mathsf{t}_X, \iota_X, m_X \Big), \end{aligned} \tag{8.2}$$

where $\mathsf{s}_X = \mathsf{t}_X = \iota_X = \mathrm{Id}_X$ and

$$\begin{aligned} m_X = \mathrm{pr}_1\colon (X,\varsigma)_2 &\longrightarrow (X,\varsigma) \\ (a,b) &\longrightarrow a \end{aligned}$$

with $a = \mathrm{Id}_X(a) = \mathsf{s}_{\mathcal{X}}(a) = \mathsf{t}_X(b) = b$. The behavior of $\mathcal{I}\text{-}G$ on morphisms is obvious. Basically, the image of $\mathcal{I}\text{-}G$ is given by discrete categories. Moreover, we will use the abuse of notation $\mathcal{G}_0 = \mathcal{I}\text{-}G(\mathcal{G}_0)$.

Proposition 8.4 *We have that* $\left(\mathsf{CSet}\text{-}\mathcal{G}, \underset{\mathcal{G}_0}{\times}, \mathcal{G}_0\right)$ *is a monoidal category: the associator on* $\mathsf{CSet}\text{-}\mathcal{G}$

$$\left(\left(- \underset{\mathcal{G}_0}{\times} -\right) \underset{\mathcal{G}_0}{\times} -\right) \longrightarrow \left(- \underset{\mathcal{G}_0}{\times} \left(- \underset{\mathcal{G}_0}{\times} -\right)\right) \tag{8.3}$$

is the identity and the natural isomorphisms

$$\Phi\colon \left(\mathrm{Id}_{\mathsf{CSet}\text{-}\mathcal{G}} \underset{\mathcal{G}_0}{\times} \mathcal{G}_0\right) \longrightarrow \mathrm{Id}_{\mathsf{CSet}\text{-}\mathcal{G}}, \qquad \Psi\colon \left(\mathcal{G}_0 \underset{\mathcal{G}_0}{\times} \mathrm{Id}_{\mathsf{CSet}\text{-}\mathcal{G}} \longrightarrow \mathrm{Id}_{\mathsf{CSet}\text{-}\mathcal{G}}\right) \tag{8.4}$$

are defined as follows. Let $(\mathcal{X},\varsigma)$*: for* $i = 0,1$ *we set*

$$\begin{aligned} \Phi(\mathcal{X})_i\colon \mathcal{X}_i \underset{\mathcal{G}_0}{\times} \mathcal{G}_0 &\longrightarrow \mathcal{X}_i \\ (a,b) &\longrightarrow a \end{aligned} \quad \textit{and} \quad \begin{aligned} \Psi(\mathcal{X})_i\colon \mathcal{G}_0 \underset{\mathcal{G}_0}{\times} \mathcal{X}_i &\longrightarrow \mathcal{X}_i \\ (b,a) &\longrightarrow a \end{aligned} \tag{8.5}$$

where $\varsigma_i(a) = b$.

Now we have two monoidal structures $(\mathsf{CSet}\text{-}\mathcal{G}, \biguplus, \emptyset)$ and $\left(\mathsf{CSet}\text{-}\mathcal{G}, \underset{\mathcal{G}_0}{\times}, \mathcal{G}_0\right)$ and for our aims it is necessary to show the distributivity of $\underset{\mathcal{G}_0}{\times}$ over $\biguplus$. To this end, let us consider $(\mathcal{X},\varsigma), (\mathcal{Y},\vartheta), (\mathcal{A},\omega) \in \mathsf{CSet}\text{-}\mathcal{G}$: we have to construct a morphism

$$\lambda\colon \left[(\mathcal{X},\varsigma) \biguplus (\mathcal{Y},\vartheta)\right] \underset{\mathcal{G}_0}{\times} (\mathcal{A},\omega) \longrightarrow \left[(\mathcal{X},\varsigma) \underset{\mathcal{G}_0}{\times} (\mathcal{A},\omega)\right] \biguplus \left[(\mathcal{Y},\vartheta) \underset{\mathcal{G}_0}{\times} (\mathcal{A},\omega)\right] \tag{8.6}$$

in $\mathsf{CSet}\text{-}G$. We define it as the couple of morphisms in $\mathsf{Set}\text{-}\mathcal{G}$, for $i = 0,1$,

$$\lambda_i\colon \left[(\mathcal{X}_i,\varsigma_i) \biguplus (\mathcal{Y}_i,\vartheta_i)\right] \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i,\omega_i) \longrightarrow \left[(\mathcal{X}_i,\varsigma_i) \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i,\omega_i)\right] \biguplus \left[(\mathcal{Y}_i,\vartheta_i) \underset{\mathcal{G}_0}{\times} (\mathcal{A}_i,\omega_i)\right]$$

, that send (a,b) to (a,b) both if $\varsigma_i(a) = \omega_i(b)$ with $a \in \mathcal{X}_i$ and if $\vartheta_i(a) = \omega_i(b)$ with $a \in \mathcal{Y}_i$. The proof that λ is actually a morphism in $\mathsf{CSet}\text{-}\mathcal{G}$ is now obvious. In this direction, one can apply the general construction of the Burnside rig and Grothendieck functor (see Section 7).

Remark 8.5 *In general a groupoid $\mathcal{G}$ cannot be a right categorified $\mathcal{G}$-set once it is viewed as a pair of $\mathcal{G}$-sets via the $\mathcal{G}$-sets $(\mathcal{G}_1, \mathsf{t})$ and $(\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$. If this were the case, then the source map $\mathsf{s}\colon (\mathcal{G}_1, \mathsf{t}) \to (\mathcal{G}_0, \mathrm{Id}_{\mathcal{G}_0})$ should be $\mathcal{G}$-equivariant, which in particular implies $\mathsf{s} = \mathsf{t}$. That is, $\mathcal{G}$ should be a bundle of groups. Conversely, any group bundle with a fixed section leads to a groupoid whose source is equal to its target and thus to a categorified set of that form. Actually, a general groupoid $\mathcal{G}$ can be seen as a "twisted " and "asymmetrical" version of a right categorified $\mathcal{G}$-set: this idea will be explored in a forthcoming paper by the same authors.*

The following results (Proposition 8.6, Theorem 8.7, Proposition 8.8, and Theorem 8.9) are adaptations of the corresponding results about groupoid-sets to the new situation of categorified groupoid-sets (see Sections 3 and 6 of El Kaoutit and Spinosa, "Burnside theory for groupoids", arXiv: 1807.04470v1, math.GR, July 2018).

Proposition 8.6 *Given a groupoid $\mathcal{G}$, let $\mathcal{A}$ be a subgroupoid of $\mathcal{G}$. We define a functor*

$$F\colon \mathsf{CSet}\text{-}\mathcal{G} \longrightarrow \mathsf{CSet}\text{-}\mathcal{A}$$

in the following way: let $(\mathcal{X}, \varsigma) \in \mathsf{CSet}\text{-}\mathcal{G}$. We define $F((\mathcal{X}, \varsigma))$ as the internal category in $\mathsf{Set}\text{-}\mathcal{A}$ with set of objects $\left(\varsigma_0^{-1}(\mathcal{A}_0), \varsigma_0|_{\varsigma_0^{-1}(\mathcal{A}_0)}\right)$ and set of morphisms $\left(\varsigma_1^{-1}(\mathcal{A}_1), \varsigma_1|_{\varsigma_1^{-1}(\mathcal{A}_1)}\right)$. The source, target, identity, and composition maps of $F((\mathcal{X}, \varsigma))$ are opportune restrictions to $\varsigma_0^{-1}(\mathcal{A}_0)$ and $\varsigma_1^{-1}(\mathcal{A}_1)$ of the relative maps of $(\mathcal{X}, \varsigma)$. Then F is a strict monoidal functor with respect to both $\biguplus$ and the fiber product.

Proof It proceeds as in the classical situation of groupoid-sets. □

Given a groupoid $\mathcal{G}$ and a fixed object $x \in \mathcal{G}_0$, we denote with $\mathcal{G}^{(x)}$ the one-object subgroupoid with isotropy group $\mathcal{G}^x$. That is, viewed as a groupoid, $\mathcal{G}^{(x)}$ has $\{x\}$ as a set of objects and $\mathcal{G}^x = \{g \in \mathcal{G}_1 | \ \mathsf{s}(g) = \mathsf{t}(g) = x\}$ as a set of arrows. A groupoid $\mathcal{G}$ with only one connected component (i.e. for any pair of objects $x, y \in \mathcal{G}_1$, there is an arrow $g \in \mathcal{G}_1$ such that $\mathsf{s}(g) = x$ and $\mathsf{t}(g) = y$) is called a transitive groupoid.

Theorem 8.7 *Given a transitive and not empty groupoid $\mathcal{G}$, let $a \in \mathcal{G}_0$. Then there is an equivalence of monoidal categories with respect to both $\biguplus$ and the fiber product:*

$$\mathsf{CSet}\text{-}\mathcal{G} \simeq \mathsf{CSet}\text{-}\mathcal{G}^{(a)}.$$

Proof Let us set $\mathcal{A}_0 = \{a\}$. In one direction, we define the functor $F\colon \mathsf{CSet}\text{-}\mathcal{G} \longrightarrow \mathsf{CSet}\text{-}\mathcal{A}$, the one constructed in Proposition 8.6. In the other direction, the functor $G\colon \mathsf{CSet}\text{-}\mathcal{A} \longrightarrow \mathsf{CSet}\text{-}\mathcal{G}$ is defined as follows: given an object $(\mathcal{X}, \varsigma)$ in $\mathsf{CSet}\text{-}\mathcal{A}$, we set $G((\mathcal{X}, \varsigma))$ as the internal category in $\mathsf{Set}\text{-}\mathcal{G}$ with set of objects

$$\Big(\mathcal{Y}_0 = \mathcal{X}_0 \times \mathcal{G}_0,\, \vartheta_0 = \mathrm{pr}_2\colon \mathcal{Y}_0 = \mathcal{X}_0 \times \mathcal{G}_0 \longrightarrow \mathcal{G}_0\Big)$$

and object of morphisms

$$\Big(\mathcal{Y}_1 = \mathcal{X}_1 \times \mathcal{G}_0,\, \vartheta_1 = \mathrm{pr}_2\colon \mathcal{Y}_1 = \mathcal{X}_1 \times \mathcal{G}_0 \longrightarrow \mathcal{G}_0\Big).$$

The source, target, identity, and composition maps of $G((\mathcal{X}, \varsigma))$ are defined as $\mathsf{s}_{G((\mathcal{X},\varsigma))} = \mathsf{s}_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$, $\mathsf{t}_{G((\mathcal{X},\varsigma))} = \mathsf{t}_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$, $\iota_{G((\mathcal{X},\varsigma))} = \iota_{(\mathcal{X},\varsigma)} \times \mathcal{G}_0$, and $m_{G((\mathcal{X},\varsigma))}\Big((x,a),(y,b)\Big) = \Big(m_{(\mathcal{X},\varsigma)}(x,y), a\Big)$ for each $((x,a),(y,b)) \in \mathcal{X}_2$. The functors F and G are mutually inverse and induce the stated equivalence of symmetric monoidal categories. □

Proposition 8.8 *The Burnside rig functor $\mathscr{L}_{\mathsf{C}}$ sends coproduct to product. In particular, given a family of groupoids $(\mathcal{G}_j)_{j \in I}$, let $(i_j \colon \mathcal{G}_j \longrightarrow \mathcal{G})_{j \in I}$ be their coproduct in the category* **Grpd** *of groupoids. Then*

$$\left(\mathscr{L}_{\mathsf{C}}(i_j) : \mathscr{L}_{\mathsf{C}}(\mathcal{G}) \longrightarrow \mathscr{L}_{\mathsf{C}}(\mathcal{G}_j)\right)_{j \in I}$$

is the product of the family $(\mathscr{L}_{\mathsf{C}}(\mathcal{G}_j))_{j \in I}$ in the category **Rig***.*

Proof The idea of the proof is similar to the classical situation of groupoid-sets: the only difference to keep in mind is that we are dealing with particular small categories and not sets. □

Theorem 8.9 *Let $\mathcal{G}$ and $\mathcal{A}$ be groupoids such that there is a symmetric strong monoidal equivalence of categories $\mathsf{CSet}\text{-}\mathcal{G} \simeq \mathsf{CSet}\text{-}\mathcal{A}$ with respect to both $\biguplus$ and the fiber product. Then there is an isomorphism of commutative rings $\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \mathscr{B}_{\mathsf{C}}(\mathcal{A})$.*

Proof It proceeds as in the classical, not categorified, situation of groupoid-sets. □

Theorem 8.10 *Given a groupoid $\mathcal{G}$, fix a set of representative objects $\mathrm{rep}(\mathcal{G}_0)$ representing the set of connected components $\pi_0(\mathcal{G})$. For each $a \in \mathrm{rep}(\mathcal{G}_0)$, let $\mathcal{G}^{\langle a \rangle}$ be the connected component of $\mathcal{G}$ containing a, which we consider as a groupoid. Then we have the following isomorphism of rings:*

$$\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}_{\mathsf{C}}\left(\mathcal{G}^{\langle a \rangle}\right). \tag{8.7}$$

Proof It follows directly from Proposition 8.8. □

Corollary 8.11 *Given a groupoid $\mathcal{G}$, we have the following, noncanonical, isomorphism of rings:*

$$\mathscr{B}_{\mathsf{C}}(\mathcal{G}) \cong \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}_{\mathsf{C}}(\mathcal{G}^{a}), \tag{8.8}$$

where the right-hand side's term is the product of commutative rings.

Proof Immediate from Theorem 8.10, Theorem 8.9, and Theorem 8.7. □

Lastly, as in the case of groups, we have a monomorphism of commutative rings from the classical Burnside ring of $\mathcal{G}$ to its categorified Burnside ring. More precisely, the diagram of rings in Figure 24, where the vertical morphism of rings is deduced using the functors of equations (3.5) and (8.2), is commutative.

$$\begin{array}{ccc} \mathscr{B}(\mathcal{G}) & \longrightarrow & \mathscr{B}_{\mathsf{C}}(\mathcal{G}) \\ \cong\downarrow & & \downarrow\cong \\ \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}(\mathcal{G}^{a}) & \longrightarrow & \prod_{a \in \mathrm{rep}(\mathcal{G}_0)} \mathscr{B}_{\mathsf{C}}(\mathcal{G}^{a}) \end{array}$$

Figure 24. Morphisms between Burnside rings.

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