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Null scrolls as  $B$ -scrolls in Lorentz–Minkowski 3-spaceŽeljka MILIN ŠIPUŠ<sup>1</sup> , Ljiljana PRIMORAC GAJČIĆ<sup>2,\*</sup> , Ivana PROTRKA<sup>3</sup> <sup>1</sup>Department of Mathematics, Faculty of Science, University of Zagreb, Zagreb, Croatia<sup>2</sup>Department of Mathematics, University of Osijek, Osijek, Croatia<sup>3</sup>Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Zagreb, Croatia

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**Abstract:** Null scrolls, i.e. ruled surfaces whose base curve and rulings are both lightlike (null), are Lorentzian surfaces having no Euclidean counterparts. In this work we present reparametrization of nondegenerate null scroll as a  $B$ -scroll, i.e. as a ruled surface whose rulings correspond to the binormal vectors of a base curve. We prove that the curvature of a base curve, which determines the Gaussian and mean curvature of a null scroll, is invariant under such a reparametrization. We also determine a one-parameter family of null curves on null scroll which serve as base curves for this kind of reparametrization.

**Key words:** Null scroll,  $B$ -scroll, Lorentz–Minkowski space

## 1. Introduction

Lorentz–Minkowski space, due to its indefinite metrics, plays an important role in Einstein’s theory of relativity (e.g., [19]). Of special interest within the theory are lightlike particles which are described as null (lightlike) curves, interesting also from differential geometry point of view [8, 12]. Their tangent vectors cannot be normalized in the usual way, since their arc-length vanishes. In order to obtain Frenet-like formulas for null curves, different choices of parameters can be considered, such as parameters called distinguished and pseudo-arc parameters. From a side of the surface theory in Lorentz–Minkowski space, null curves appear only on timelike and lightlike surfaces. In particular, a null straight line moving along a null curve generates a special ruled surface called a null scroll. These surfaces are examples of Lorentzian surfaces having no Euclidean counterparts. Their properties can be found in, e.g., [6, 13–16]. In [12], authors state that every nondegenerate null scroll can be reparametrized as a certain type of a null scroll called a  $B$ -scroll, defined by Graves [10]. This type of surfaces plays an important role in the classification of flat surfaces [18], as well as in physics, e.g., [2–5].

The reparametrization of a null scroll as a  $B$ -scroll is given in [9], but starting from a conformal parametrization of a timelike ruled surface. In [15], the author considers such a reparametrization for null scrolls with constant curvatures and parametrized by a distinguished parameter. To the best of our knowledge, there are no other studies explicating such a reparametrization for a ruled surface with null rulings and a null base curve, but with no additional assumptions on parameters and curvatures. This motivated our study. In this work we present a method to obtain the required reparametrization and we show that it is not unique. In fact, we can determine a family of null curves on a null scroll which can be used as base curves of a  $B$ -scroll. We

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also show that the curvature  $k_3$  of a base curve is invariant under such a reparametrization, which necessarily needs to be ensured, since the curvatures of a null scroll, the Gaussian and mean curvature, depend on this curvature of a base curve only.

## 2. Preliminaries

The 3-dimensional Lorentz–Minkowski space, denoted by  $\mathbb{R}_1^3$ , is the smooth manifold  $\mathbb{R}^3$  with a flat Lorentzian pseudometric

$$g = dx_1^2 + dx_2^2 - dx_3^2,$$

where  $(x_1, x_2, x_3)$  is a system of canonical coordinates of  $\mathbb{R}_1^3$ . In  $\mathbb{R}_1^3$ , a vector  $x$  can be of the following causal character: spacelike if  $g(x, x) > 0$  or  $x = (0, 0, 0)$ , timelike if  $g(x, x) < 0$  or null (lightlike) if  $g(x, x) = 0$  and  $x \neq (0, 0, 0)$ . The causal character of a regular curve  $c : I \rightarrow \mathbb{R}_1^3$  is determined by the character of its velocity vector  $c'(u)$ .

In what follows we consider a null curve  $c : I \rightarrow \mathbb{R}_1^3$ ,  $I \subset \mathbb{R}$ , given by a local parametrization  $c(u)$ . We differentiate between two basic approaches regarding how to frame a curve and introduce a distinguished parameter. By the first approach, following [12], we start from an arbitrary null frame along a curve  $c = c(u)$  which is defined as ordered triple of vectors  $L(u) = (A(u), B(u), C(u))$  consisting of two null vectors  $A$  and  $B$  and a unit spacelike vector  $C$  satisfying orthogonality conditions, meaning  $g(A, B) = 1$ ,  $g(A, C) = g(B, C) = 0$ , whereby  $\det L = \pm 1$ . A null frame is called proper if its associated orthonormal frame defined by  $\mathcal{F}(L) = (\frac{1}{\sqrt{2}}(A - B), \frac{1}{\sqrt{2}}(A + B), C)$ , with the first vector timelike and other two spacelike, belongs to the special semiorthogonal group  $SO_1^+(3)$  consisting of regular matrices  $A$  preserving the pseudo-inner product of a Lorentz–Minkowski space and satisfying  $\det A = 1$ . Vectors in any such orthonormal frame  $\mathcal{F} = (x, y, z)$  satisfy  $x \times y = z$ ,  $y \times z = -x$ ,  $z \times x = y$ , where the Lorentzian cross-product of vectors  $x$  and  $y$  is defined by  $x \times y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_2y_1 - x_1y_2)$ . Therefore, for vector fields  $A, B, C$  of the null frame the following holds

$$A \times B = C, \quad A \times C = -A, \quad B \times C = B. \quad (2.1)$$

A null frame for a null curve  $c(u)$  is defined as a proper null frame field  $L(u) = (A(u), B(u), C(u))$  along  $c$ , where  $c'(u) = k_0(u) \cdot A(u)$ ,  $k_0(u) > 0$ . A parametrized null curve  $(c, L)$  with a proper null frame is called a framed null curve. A null curve can be framed in different ways; therefore, a curve and a proper null frame must be given together. Moreover, if  $L = (A, B, C)$  is a one such frame, then  $L_{m,n} := (mA, m^{-1}B - mn^2A/2 - nC, C + mnA)$ ,  $m, n \in \mathbb{R}^+$ , also frames  $c$ . Furthermore, vector fields  $A, B, C$  satisfy analogues of the Frenet formulas

$$A' = k_1A + k_2C, \quad B' = -k_1B + k_3C, \quad C' = -k_3A - k_2B, \quad (2.2)$$

where the functions  $k_i, i = 1, 2, 3$  are called the curvature functions of the curve  $c$  with respect to the frame  $L$ . By the fundamental theorem for null curves, for initial  $p \in \mathbb{R}_1^3$  and smooth functions  $k_i, i = 0, 1, 2, 3$ , there exists a unique framed null curve  $(c(u), L(u))$  such that  $c(0) = p$ ,  $c'(u) = k_0(u)A(u)$ .

For every null curve  $c$  there exists a parameter  $\bar{u}(u)$  such that  $k_1(\bar{u}) = 0$ . We refer to this parameter as a distinguished parameter of the curve  $c$ , and the associated frame as the Cartan frame. For a curve parametrized by a distinguished parameter, the following Frenet formulas hold

$$A' = k_2C, \quad B' = k_3C, \quad C' = -k_3A - k_2B. \quad (2.3)$$

In the second approach regarding how to introduce a distinguished parameter, a null frame of a null curve  $c = c(u)$  is constructed uniquely in such a way that the initial parameter  $u$  is the distinguished parameter of the framed curve. This construction is provided by Honda and Inoguchi, [11]. If we put  $A = c'(u)$ , since  $c'$  and  $c''$  are linearly independent (for nongeodesics) and orthogonal,  $c''$  needs to be spacelike. Therefore, there exists a unique section (vector field)  $B(u)$  of the orthogonal complement  $c''(u)^\perp$  of  $c''(u)$  such that  $g(A(u), B(u)) = 1$  and  $g(B(u), B(u)) = 0$ . The vector field  $C$  along  $c$  is defined by  $C(u) = A(u) \times B(u)$ . Then for the vector fields  $A, B, C$  holds  $g(A(u), C(u)) = g(B(u), C(u)) = 0$  and  $g(C(u), C(u)) = 1$ . Moreover, there exist functions  $k_2(u)$  and  $k_3(u)$  such that Frenet–Serret formulas (2.3) hold.

Besides, by a distinguished parameter, a null curve with  $k_2 \neq 0$  can also be reparametrized by a parameter called the pseudo-arc parameter, such that the vector fields  $A$  and  $C$  additionally satisfy  $A' = C$ . Although in our study we do not deal with the pseudo-arc parameter, it is worth mentioning, since it is considered a canonical parameter on null curves.

From the surface theory in Lorentz–Minkowski space, we need the following. A surface  $S$  in  $\mathbb{R}_1^3$  is said to be spacelike, timelike, or lightlike if the induced metric on  $S$  is positive definite, indefinite, or degenerate, respectively. For an immersed spacelike (resp. timelike) surface, given by a local parametrization  $f = f(u, v)$ , the unit normal vector field  $n = \frac{f_u \times f_v}{\|f_u \times f_v\|}$  is timelike (resp. spacelike) field, where  $f_u$  and  $f_v$  are derivatives of  $f$  with respect to parameters  $u$  and  $v$ , respectively.

A ruled surface is a surface obtained by moving a straight line along a curve. Therefore, it admits a parametrization of the form

$$f(u, v) = c(u) + vr(u), \quad u \in I \subset \mathbb{R}, v \in \mathbb{R}, \quad (2.4)$$

where the curve  $c$  is called the base curve and  $r$  are rulings of a ruled surface. In [1], authors showed that every ruled surface with null rulings can be reparametrized so the base curve is a null curve as well. These surfaces are called null scrolls. In [12] it is stated that every nondegenerate ( $g(c', r) \neq 0$ ) null scroll admits a parametrization as a  $B$ -scroll, i.e. as a ruled surface with null rulings and with a base curve  $c$  which is a null curve with a null frame whose vector fields satisfy Frenet formulas of the form (2.3) and rulings are determined by the field  $B$ . This reparametrization is given in [9], but starting from a conformal parametrization of a timelike ruled surface.

As already stated, a  $B$ -scroll is a ruled surface parametrized by

$$f(u, v) = c(u) + vB(u), \quad u \in I \subset \mathbb{R}, v \in \mathbb{R}, \quad (2.5)$$

where a base curve  $c$  is parametrized by the distinguished parameter and framed by a proper frame  $L(u) = (A(u), B(u), C(u))$ .  $B$ -scrolls are timelike surfaces with the first fundamental form  $I = k_3^2 v^2 du^2 + 2dudv$ . Their principal curvatures are real and identical; however, the Weingarten map (the shape operator) is not diagonalizable [7]. The Gaussian curvature  $K$  and the mean curvature  $H$  of  $B$ -scrolls are given by  $K = k_3^2$ ,  $H = k_3$ .

### 3. Parametrization of null scrolls

In this section, for a given null scroll we introduce a null frame for its base curve, as a natural frame connected to this ruled surface, such that the Frenet formulas (2.2) hold. In addition, we calculate its Gaussian and mean curvature in terms of the curvature functions of the base curve  $c$ .

**Proposition 3.1** *Let  $S$  be a nondegenerate null scroll given by the parametrization (2.4). The ordered triple  $L(u) = \{c', \frac{1}{g(c', r)}r, \frac{1}{g(c', r)}(c' \times r)\}$  is a proper null frame of the base curve  $c$  with the curvature functions given by*

$$k_1 = \frac{1}{g(c', r)}g(c'', r), \quad (3.1)$$

$$k_2 = \frac{1}{g(c', r)}\det(c', r, c''), \quad (3.2)$$

$$k_3 = \frac{1}{g(c', r)^2}\det(c', r, r'). \quad (3.3)$$

**Proof** Let us denote

$$A = c', \quad B = \frac{1}{g(c', r)}r, \quad C = \frac{1}{g(c', r)}(c' \times r). \quad (3.4)$$

Notice that  $C = A \times B$ . Obviously,  $g(A, A) = g(B, B) = 0$ , as well as  $g(A, B) = 1$ . Since  $C$  is a vector product of  $A$  and  $B$ , it holds  $g(A, C) = g(B, C) = 0$ . Furthermore, by using the Lagrange identity in  $\mathbb{R}_1^3$ , we obtain

$$g(C, C) = g(A \times B, A \times B) = g(A, B)^2 - g(A, A)g(B, B) = 1.$$

By (2.2), the curvature functions  $k_i$ ,  $i = 1, 2, 3$ , are determined from the relations  $k_1 = g(A', B)$ ,  $k_2 = g(A', C)$ ,  $k_3 = g(B', C)$ , which give the formulas (3.1)–(3.3).  $\square$

**Theorem 3.2** *Let  $S$  be a nondegenerate null scroll given by the parametrization (2.4) and let a base curve  $c$  be framed by  $L(u) = \{c', \frac{1}{g(c', r)}r, \frac{1}{g(c', r)}(c' \times r)\}$ . Then the mean curvature  $H$  and the Gaussian curvature  $K$  of the surface  $S$  depend on the curvature  $k_3$  of a base curve  $c$  only,  $H = \pm k_3$ ,  $K = k_3^2$ .*

**Proof** Again, let the fields  $(A, B, C)$  be defined by (3.4) and we denote  $b(u) = g(c'(u), r(u))$ . Hence, we can write  $r(u) = b(u)B(u)$ . By taking derivatives of  $f$  with respect to parameters  $u, v$  we get

$$\begin{aligned} f_u &= c'(u) + v \cdot r'(u) = A + v(-k_1 b(u) + b'(u))B + v \cdot k_3 b(u)C, \\ f_v &= r(u) = b(u)B. \end{aligned}$$

Straightforward computation gives the coefficients of the first fundamental form

$$g_{11} = 2v(-g(c'', r) + b'(u)) + v^2 k_3^2 b(u)^2, \quad g_{12} = b(u), \quad g_{22} = 0.$$

The normal vector field  $n$  of the surface  $S$  is a spacelike vector field

$$n(u) = \varepsilon(-vk_3 b(u)B + C),$$

where  $\varepsilon = \frac{b(u)}{|b(u)|}$ . The second derivatives of  $f$  with respect to  $u, v$  are given by

$$\begin{aligned} f_{uu} &= (k_1 - k_3^2 b(u))A + (v(-k_1' b(u) + b''(u) - 2k_1 b'(u) + k_1^2 b(u)) - k_2 k_3 b(u))B + \\ &+ (k_2 + vk_3(-k_1 b(u) + b'(u)) + (k_3 b(u))')C, \\ f_{uv} &= b(u)B, \quad f_{vv} = 0. \end{aligned}$$

The coefficients of the second fundamental form are therefore

$$\begin{aligned} h_{11} &= g(f_{uu}, n) = \varepsilon (vk_3(k_3^2 b(u)^2 - 2k_1 b(u) + b'(u)) + k_2 + (k_3 b'(u))), \\ h_{12} &= h_{21} = g(f_{uv}, n) = \varepsilon k_3 b(u), \\ h_{22} &= g(f_{vv}, n) = 0. \end{aligned}$$

Finally, the mean curvature  $H$  and the Gaussian curvature  $K$  are

$$\begin{aligned} H &= \frac{1}{2} \frac{h_{11}g_{22} - 2h_{12}g_{12} + h_{22}g_{11}}{g_{11}g_{22} - g_{12}^2} = \varepsilon k_3, \\ K &= \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = k_3^2. \end{aligned}$$

□

In addition, from the previous statements we conclude:

**Corollary 3.3** *Let  $S$  be a nondegenerate null scroll with a parametrization (2.4) and let a base curve  $c$  be framed by  $L(u) = \{c', \frac{1}{g(c', r)}r, \frac{1}{g(c', r)}(c' \times r)\}$ . Then  $S$  is a surface of constant curvatures if and only if*

$$\frac{1}{g(c', r)^2} \det(c', r, r') = \text{const.}$$

Now we provide the required reparametrization of a null scroll  $S$  as a  $B$ -scroll. In the case when  $k_1(u) = 0$ , that is  $g(c'', r) = 0$ , the parameter  $u$  is already distinguished.

**Theorem 3.4** *Let  $S$  be a nondegenerate null scroll with a parametrization (2.4) and let a base curve  $c$  be framed by  $L(u) = \{c', \frac{1}{g(c', r)}r, \frac{1}{g(c', r)}(c' \times r)\}$ . Then  $S$  admits a parametrization as a  $B$ -scroll of the form*

$$\bar{f}(\bar{u}, v) = \bar{c}(\bar{u}) + v\bar{B}(\bar{u}),$$

where

$$\bar{u}(u) = c_1 \int_{u_0}^u e^{\int_{u_0}^u \frac{g(c'', r)}{g(c', r)} du} du + c_2, \quad c_1 > 0, \quad c_2 \in \mathbb{R}, \tag{3.5}$$

is a distinguished parameter of a base curve  $c$ ,  $\bar{A}$  its tangent vector field,  $\bar{B}$  a binormal, and  $\bar{C}$  a normal vector field of the curve  $c$  with respect to  $\bar{u}$ , whereby the following holds

$$\bar{A}(\bar{u}(u)) = A(u) \frac{du}{d\bar{u}},$$

$$\bar{B}(\bar{u}(u)) = B(u) \frac{d\bar{u}}{du},$$

$$\bar{C}(\bar{u}(u)) = C(u).$$

The curvature functions satisfy  $\bar{k}_1(\bar{u}) = 0$ , and

$$k_2(u) = \bar{k}_2(\bar{u}) \left( \frac{d\bar{u}}{du} \right)^2 \quad (3.6)$$

$$k_3(u) = \bar{k}_3(\bar{u}). \quad (3.7)$$

**Proof** For the frame  $(A, B, C)$  defined by (3.4), the formula (3.5) introduces a distinguished parameter  $\bar{u} = \bar{u}(u)$  for the curve  $c$  given as a solution of ordinary differential equation

$$k_1 \frac{du}{d\bar{u}} - \left( \frac{du}{d\bar{u}} \right)^2 \frac{d^2\bar{u}}{du^2} = 0,$$

obtained from condition  $\bar{k}_1 = 0$ , (see [12]). Moreover, by differentiating both sides of  $\bar{c}(\bar{u}) = c(u)$  with respect to a parameter  $u$ , we obtain  $\bar{A}(\bar{u}) \frac{d\bar{u}}{du} = A(u)$ , i.e.  $\bar{A}(\bar{u}) = \frac{du}{d\bar{u}} A(u)$ . Since  $\bar{A}$  and  $\bar{B}$  have to satisfy  $g(\bar{A}, \bar{B}) = 1$ , the vector field  $\bar{B}$  is defined by  $\bar{B}(\bar{u}) = \frac{d\bar{u}}{du} B(u)$ , while  $\bar{C}(\bar{u}) = C(u)$ . Then the  $B$ -scroll has the parametrization of the form

$$\bar{f}(\bar{u}, v) = \bar{c}(\bar{u}) + v\bar{B}(\bar{u}),$$

where  $\bar{c}(\bar{u}) = \int_{\bar{u}_0}^{\bar{u}} \bar{A}(\bar{u}) d\bar{u}$ .

Furthermore, we establish relations between curvature functions  $k_i$  and  $\bar{k}_i$ ,  $i = 1, 2, 3$ . From  $\bar{A}(\bar{u}) \frac{d\bar{u}}{du} = A(u)$ , by differentiating both sides with respect to parameter  $u$  and multiplying by  $B(u) = \frac{du}{d\bar{u}} \bar{B}(\bar{u})$ , we obtain

$$\bar{k}_1 \frac{d\bar{u}}{du} + \frac{du}{d\bar{u}} \frac{d^2\bar{u}}{du^2} = k_1, \text{ i.e. } \bar{k}_1 = k_1 \frac{du}{d\bar{u}} - \left( \frac{du}{d\bar{u}} \right)^2 \frac{d^2\bar{u}}{du^2}.$$

By using (3.5) we have  $\bar{k}_1 = 0$ , as already stated. By differentiating both sides of  $\bar{A}(\bar{u}) \frac{d\bar{u}}{du} = A(u)$ , with respect to a parameter  $u$ , and multiplying by  $C = \bar{C}$ , we obtain (3.6). Finally, by differentiating both sides of  $\bar{B}(\bar{u}) = \frac{d\bar{u}}{du} B(u)$ , with respect to a parameter  $u$  and multiplying by  $\bar{C} = C$ , we obtain (3.7).  $\square$

#### 4. Introducing a new null base curve

Now we consider a reparametrization of the given null scroll by another null base curve. We can obtain its proper null frame of the form (2.3) such that vector field  $B$  is collinear to rulings of the initial null scroll. Such

curves can be considered to be deformations of the initial base curve preserving torsion, which are studied in [11] for curves that are already parametrized by a distinguished parameter.

A curve  $\hat{c}$  lying on the initial null scroll can be written as

$$\hat{c}(u) = c(u) + v(u)r(u),$$

with no assumptions on a parameter  $u$ . Therefore,

$$\hat{c}'(u) = c'(u) + v'(u)r(u) + v(u)r'(u). \tag{4.1}$$

From  $g(\hat{c}', \hat{c}') = 0$ , we obtain ordinary differential equation for  $v = v(u)$  of Bernoulli type

$$2g(c', r)v'(u) + 2g(c', r')v(u) + g(r', r')v^2(u) = 0, \tag{4.2}$$

whose family of solutions provides a family of null curves lying on the initial null scroll. The solution of (4.2) is given by

$$v(u) = \frac{1}{(e^{\int \frac{g(c', r')}{g(c', r)} du} + c_1) \cdot \int e^{-\int \frac{g(c', r')}{g(c', r)} du} \cdot \frac{g(r', r')}{2g(c', r)} du},$$

and therefore every null curve of the family can be parametrized as

$$\hat{c}(u) = c(u) + \frac{1}{(e^{\int \frac{g(c', r')}{g(c', r)} du} + c_1) \cdot \int e^{-\int \frac{g(c', r')}{g(c', r)} du} \cdot \frac{g(r', r')}{2g(c', r)} du} r(u), \quad c_1 \in \mathbb{R}. \tag{4.3}$$

Notice that (4.1) and (4.2) can be written in terms of vector fields frame  $\{A, B, C\}$ , respectively  $\{\hat{A}, \hat{B}, \hat{C}\}$ , of the form (3.4) using the formulas (2.2) and (2.1) as follows. Using the notation  $b(u) = g(c'(u), r(u))$  (since  $g(c'(u), r(u)) \neq 0$ ), we obtain  $r(u) = b(u)B(u)$ . Analogously, we obtain  $r(u) = \hat{b}(u)\hat{B}(u)$ . Furthermore, from (4.1) we can deduce

$$\hat{b}(u) = g(c'(u) + v'(u)r(u) + v(u)r'(u), r(u)) = b(u).$$

Therefore,

$$\hat{B}(u) = \frac{1}{\hat{b}(u)}r(u) = \frac{1}{b(u)}r(u) = B(u). \tag{4.4}$$

From (4.1), we obtain

$$\begin{aligned} \hat{A}(u) &= A(u) + v'(u)b(u)B(u) + v(u)(b(u)B(u))' = \\ &= A(u) + (v'(u)b(u) + v(u)b'(u) - v(u)b(u)k_1(u))B(u) + \\ &+ v(u)k_3(u)b(u)C(u), \end{aligned} \tag{4.5}$$

where  $v = v(u)$  satisfies (4.2).

Equations (4.1), (4.2), and (2.1) imply that

$$\begin{aligned} \hat{C}(u) &= \hat{A}(u) \times \hat{B}(u) = \hat{A}(u) \times B(u) = C + v(u)k_3(u)b(u)(C(u) \times B(u)) = \\ &= C(u) - v(u)b(u)k_3(u)B(u). \end{aligned} \tag{4.6}$$

Obviously,  $\hat{C}(u)$  is a unit spacelike vector. Therefore, we have proved the following statement.



**Proposition 4.1** Let  $S$  be a null scroll with parametrization (2.4) and  $\hat{c}$  a null curve lying on  $S$ . The vector fields  $\hat{A}(u), \hat{B}(u), \hat{C}(u)$  of the proper null frame of a curve  $\hat{c}$  satisfy

$$\begin{aligned}\hat{A}(u) &= A(u) + (v'(u)b(u) + v(u)b'(u) - v(u)b(u)k_1(u))B(u) + v(u)k_3(u)b(u)C(u), \\ \hat{B}(u) &= B(u), \\ \hat{C}(u) &= C(u) - v(u)b(u)k_3(u)B(u),\end{aligned}$$

where  $u$  is an arbitrary parameter,  $A(u), B(u), C(u)$  are vector fields of the proper null-frame of  $c(u)$ ,  $k_1(u)$  and  $k_3(u)$  corresponding curvatures and  $b(u) = g(c'(u), r(u))$ .

For the curvatures of curves  $c$  and  $\hat{c}$ , we give the following relations:

**Proposition 4.2** Let  $c$  and  $\hat{c}$  be two null curves lying on a null scroll  $S$ , parametrized by same arbitrary parameter  $u$  and framed by null proper frames  $\{A, B, C\}$ , respectively  $\{\hat{A}, \hat{B}, \hat{C}\}$ . Then for the curvature functions  $k_1(u), k_2(u), k_3(u)$   $\hat{k}_1(u), \hat{k}_2(u), \hat{k}_3(u)$  of curves  $c$  and  $\hat{c}$  respectively, the following relations hold

$$\begin{aligned}\hat{k}_1(u) &= k_1(u) - k_3^2(u)b(u)v(u), \\ \hat{k}_2(u) &= k_2(u) + (v(u)b(u)k_3(u))' - 2v(u)b(u)k_1(u)k_3(u) + (v(u)b(u))'k_3(u) \\ &\quad - (v(u)b(u))^2k_3^3(u), \\ \hat{k}_3(u) &= k_3(u).\end{aligned}\tag{4.7}$$

**Proof** For easier notation, we denote  $\beta(u) = v'(u)b(u) + v(u)b'(u) - v(u)b(u)k_1(u)$ . By straightforward computation, we obtain

$$\begin{aligned}\hat{k}_1 &= g(\hat{A}', B) = -g(\hat{A}, \hat{B}') = -g(\hat{A}, B') = -g(A + \beta B + vk_3bC, -k_1B + k_3C) \\ &= k_1g(A, B) - vk_3b \cdot k_3g(C, C) = k_1 - vbk_3^2, \\ \hat{k}_2 &= g(\hat{A}', \hat{C}) - g(\hat{A}, \hat{C}') = g(A + \beta B + vk_3bC, (C - vbk_3B)') = \\ &= k_2 + (vbk_3)' - vbk_1k_3 + \beta k_3 - (vbk_3)^2k_3 = \\ &= k_2 + (vbk_3)' - 2vbk_1k_3 + (vb)'k_3 - (vbk_3)^2k_3, \\ \hat{k}_3 &= g(\hat{B}, \hat{C}) = g(B', \hat{C}) = g(-k_1B + k_3C, C - vbk_3B) = k_3g(C, C) = k_3.\end{aligned}$$

□

**Remark 4.3** By the fundamental theorem for null curves, curves  $c$  and  $\hat{c}$  are in general not congruent ([12]). Distinguished parameters on  $c$ , respectively  $\hat{c}$  are determined from  $k_1(u) = 0$ ,  $\hat{k}_1(u) = 0$  respectively. They coincide if and only if  $k_1(u) = \hat{k}_1(u) = 0$  which is fulfilled only when  $k_3(u) = 0$ , that is, the initial null scroll is a cylindrical surface. In that case, curvatures  $k_2$  and  $\hat{k}_2$  also coincide; therefore, curves  $c$  and  $\hat{c}$  are congruent.

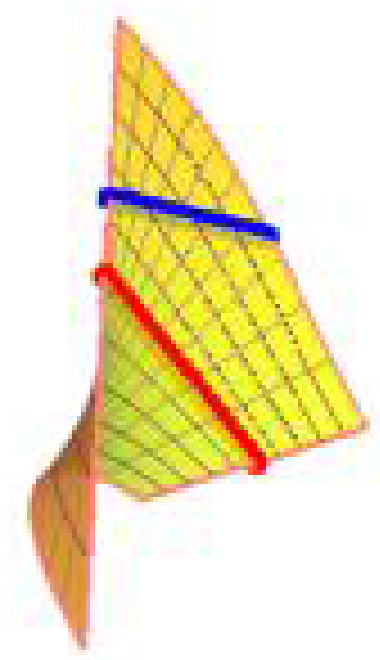
For every null curve lying on a null scroll, we can carry out the proposed method for the reparametrization. Therefore, we conclude the following:

**Theorem 4.4** *Every null scroll admits a one-parameter family of null curves which serve as base curves and which provide its reparametrization as a B-scroll.*

**Example 4.5** *Consider the null scroll given by*

$$f(u, v) = \left(\frac{1}{u}, 2, -\frac{1}{u}\right) + v\left(\frac{1}{2}(1 - u^2), u, \frac{1}{2}(-1 - u^2)\right) \tag{4.8}$$

where  $u > 0, v \in \mathbb{R}$ , Figure 1.



**Figure 1.** Null scroll with parametrization (4.8) with base curves  $c$  (red) and  $\hat{c}$  (blue).

The  $(A, B, C)$  frame for  $c$  is given by (3.4)

$$\begin{aligned} A(u) &= \left(-\frac{1}{u^2}, 0, \frac{1}{u^2}\right), \\ B(u) &= \left(\frac{1}{2}(1 - u^2), u, \frac{1}{2}(-1 - u^2)\right), \\ C(u) &= \left(-\frac{1}{u}, -1, \frac{1}{u}\right). \end{aligned}$$

For the first curvature  $k_1$  we have  $k_1(u) = -2/u$ . The change of parameters (3.5), which yields the required parametrization, is defined by

$$\bar{u} = -c_1 \frac{1}{u} + c_2, \quad c_1 > 0, \quad c_2 \in \mathbb{R}.$$

Without loss of generality with  $c_1 = 1, c_2 = 0$ , the considered null scroll can be reparametrized as a B-scroll of the form

$$\bar{f}(\bar{u}, v) = (-\bar{u}, 2, \bar{u}) + v \left( \frac{1}{2} \left(1 - \frac{1}{\bar{u}^2}\right), -\frac{1}{\bar{u}}, -\frac{1}{2} \left(1 + \frac{1}{\bar{u}^2}\right) \right).$$

It is a  $B$ -scroll of constant Gaussian and mean curvatures  $K = H = 1$ . Another parametrization can be obtained by using the curve  $\hat{c}(u) = (u, 0, u)$ . The  $(\hat{A}, \hat{B}, \hat{C})$  frame for  $\hat{c}$  is given by

$$\begin{aligned}\hat{A}(u) &= (1, 0, 1), \\ \hat{B}(u) &= \left(\frac{1}{2}(1 - u^2), u, \frac{1}{2}(-1 - u^2)\right), \\ \hat{C}(u) &= (-u, 1, -u).\end{aligned}$$

The first curvature  $\hat{k}_1$  is zero, so the parameter  $u$  is the distinguished parameter of a curve  $\hat{c}$ . Therefore, parametrization as a  $B$ -scroll with respect to curve  $\hat{c}$  has a form

$$\hat{f}(u, v) = (u, 0, u) + v\left(\frac{1}{2}(1 - u^2), u, \frac{1}{2}(-1 - u^2)\right).$$

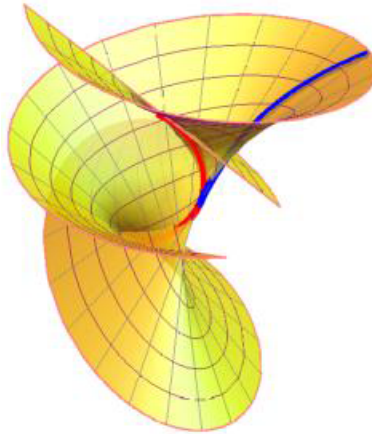
Another null curve laying on considered null scroll is of the form  $\hat{c}(u) = c(u) + v(u)r(u)$ , where

$$v(u) = \frac{2}{u(2u - 1)}.$$

**Example 4.6** Consider the null scroll given by

$$f(u, v) = (\cos u, \sin u, u) + v(\sin u, \cos u, 1) \tag{4.9}$$

where  $u, v \in \mathbb{R}$ , Figure 2. It is a null scroll of constant Gaussian and mean curvatures  $K = H = 1$ .



**Figure 2.** Null scroll with parametrization (4.9) with base curves  $c$  (red) and  $\hat{c}$  (blue).

The  $(A, B, C)$  frame for  $c$  is given by (3.4)

$$\begin{aligned}A(u) &= (-\sin u, \cos u, 1), \\ B(u) &= \frac{-1}{2 \sin^2 u}(\sin u, \cos u, 1), \\ C(u) &= \frac{-1}{\sin u}(0, 1, \cos u).\end{aligned}$$

For the first curvature  $k_1$  we have  $k_1(u) = \cot u$ . The change of parameters (3.5), which yields the required parametrization, is defined by

$$\bar{u} = -c_1 \cos u + c_2, \quad c_1 > 0, \quad c_2 \in \mathbb{R}.$$

Without loss of generality with  $c_1 = 1$ ,  $c_2 = 0$ , the considered null scroll can be reparametrized as a B-scroll of the form

$$\bar{f}(\bar{u}, v) = (-\bar{u}, \sqrt{1 - \bar{u}^2}, \arccos(-\bar{u})) + v \left( -\frac{1}{2}, \frac{\bar{u}}{2\sqrt{1 - \bar{u}^2}}, \frac{-1}{2\sqrt{1 - \bar{u}^2}} \right).$$

Another null curve on the null scroll is given by  $\hat{c}(u) = c(u) + v(u)r(u)$ , where

$$v(u) = \frac{32}{-4 \sin u \log \left( \sin \frac{u}{2} \right) + \sin u \csc^2 \left( \frac{u}{2} \right) - \sin u \sec^2 \left( \frac{u}{2} \right) + 4 \sin u \log \left( \cos \frac{u}{2} \right)}.$$

**Example 4.7** Consider the null scroll given by

$$\begin{aligned} f(u, v) &= \left( u + \frac{4u}{1 + k^2u^2} - \frac{4}{k} \arctan(ku), -\frac{4}{k(1 + k^2u^2)} - 2 \frac{\log(1 + k^2u^2)}{k}, u \right) + \\ &+ v(\cos(\varphi(u)), \sin(\varphi(u)), 1), \end{aligned} \tag{4.10}$$

where  $k$  is a constant,  $u, v \in \mathbb{R}$ , Figure 3, and  $\varphi(u) = \operatorname{arccot}(ku)$ , [17]. It is a surface of constant Gaussian and mean curvature,  $K = k^2$ ,  $H = -k$ . The distinguished parameter  $\bar{u}$  is defined by (3.5) as

$$\frac{d\bar{u}}{du} = \frac{1}{(1 + k^2u^2)^2}$$

that is,

$$\bar{u}(u) = c_1 \left( \frac{u}{2(1 + k^2u^2)} + \frac{\arctan(ku)}{2k} \right) + c_2,$$

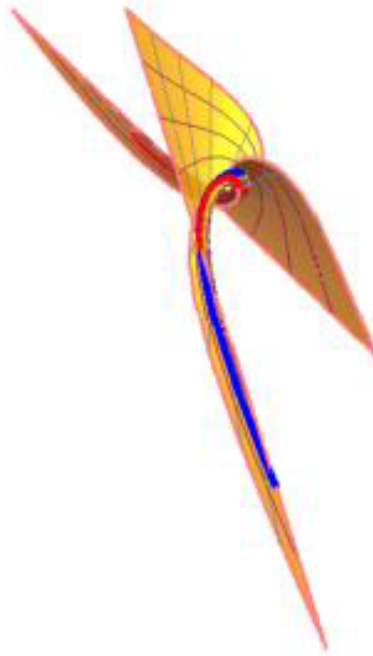
with  $c_1 > 0$ ,  $c_2 \in \mathbb{R}$ . Without loss of generality with  $c_1 = 1$ ,  $c_2 = 0$ , the vector fields  $\bar{A}, \bar{B}, \bar{C}$  of a curve  $\bar{c}$  are obtained as

$$\begin{aligned} \bar{A}(\bar{u}) &= \left( 1 - 6k^2u^2 + k^4u^4, -4ku(-1 + k^2u^2), (1 + k^2u^2)^2 \right), \\ \bar{B}(\bar{u}) &= \left( \frac{1 - k^2u^2}{2(1 + k^2u^2)^2}, \frac{ku}{(1 + k^2u^2)^2}, -\frac{1}{2(1 + k^2u^2)} \right), \\ \bar{C}(\bar{u}) &= \left( ku \left( 1 - \frac{4}{1 + k^2u^2} \right), -3 + \frac{4}{1 + k^2u^2}, -ku \right), \end{aligned}$$

where  $u = u(\bar{u})$ . However, the explicit formula for the inverse change of parameters  $u = u(\bar{u})$  cannot be obtained.

Another null curve laying on considered null scroll is given by  $\hat{c}(u) = c(u) + v(u)r(u)$ , where

$$v(u) = -\frac{2}{u(k^3u^2 \tan^{-1}(ku) - 2k^2u^2 + k^2u + k \tan^{-1}(ku) - 2)}.$$



**Figure 3.** Null scroll with parametrization (4.10) with base curves  $c$  (red) and  $\hat{c}$  (blue).

### References

- [1] Alias LJ, Ferrandez A, Lucas P, Merono MA. On the Gauss map of  $B$ -scroll. *Tsukuba Journal of Mathematics* 1998; 22 (2): 371-377. doi: 10.21099/tkbjm/1496163588
- [2] Barros M, Ferrandez A. Null scrolls as fluctuating surfaces: A new simple way to construct extrinsic string solutions. *Journal of High Energy Physics* 2012; 68 (5). doi: 10.1007/JHEP05(2012)068
- [3] Barros M, Ferrandez A. Null scrolls as solutions of a sigma model. *Journal of Physics A: Mathematical and Theoretical* 2012; 45 (14). doi: 10.1088/1751-8113/45/14/145203
- [4] Barros M, Ferrandez A. How big is the family of stationary null scrolls? *Journal of Geometry and Physics* 2013; 64: 54-60. doi: 10.1016/j.geomphys.2012.10.009
- [5] Barros M, Ferrandez A, Lucas P, Merono MA. Solutions of the Betchow-DaRios soliton equations: a Lorentzian approach. *Journal of Geometry and Physics* 1999; 31 (2-3): 217-228. doi: 10.1016/S0393-0440(99)00005-4
- [6] Choi SM, Ki UH, Suh YJ. On the Gauss map of null scrolls. *Tsukuba Journal of Mathematics* 1998; 22 (1): 272-279
- [7] Clelland JN. Totally quasi-umbilic timelike surfaces in  $\mathbb{R}^{1,2}$ . *Asian Journal of Mathematics* 2012; 16 (2): 189-208.
- [8] Duggal KL, Bejancu A. *Lightlike Submanifolds of semi-Riemannian Manifolds and Applications, Mathematics and its Applications*. Dordrecht, the Netherlands: Kluwer Academic Publishers, 1996.
- [9] Fujioka A, Inoguchi JI. Timelike Bonnet surfaces in Lorentzian space forms. *Differential Geometry and its Applications* 2003; 18 (1): 103-111. doi: 10.1016/S0926-2245(02)00141-9
- [10] Graves LK. Codimension one isometric immersions between Lorentz spaces, *Transactions of the American Mathematical Society* 1979; 252: 367-392. doi: 10.2307/1998094
- [11] Honda K, Inoguchi JI. Deformations of Cartan framed null curves preserving the torsion. *Differential Geometry - Dynamical Systems* 2003; 5(1): 31-37.
- [12] Inoguchi JI, Lee S. Null curves in Minkowski 3-space. *International Electronic Journal of Geometry* 2008; 1(2): 40-83.

- [13] Kılıçoğlu Ş, Hacısalihoğlu H, Şenyurt S. On the fundamental forms of the B-scroll with null directrix and Cartan frame in Minkowskian 3 -Space. Applied Mathematical Sciences 2015; 9(80): 3957-3965. doi: 10.12988/ams.2015.53230
- [14] Kılıçoğlu Ş. On the generalized B-Scrolls with P Th degree in n-dimensional Minkowski spaces and l (Central) Spaces. SAÜ, Fen Bilimleri Dergisi 2008, 10(2): 15-29.
- [15] Liu H. Ruled surfaces with lightlike ruling in 3-Minkowski space. Journal of Geometry and Physics 2009; 59 (1): 74-78. doi: 10.1016/j.geomphys.2008.10.003
- [16] Liu H. Structures and properties of null scroll in Minkowski 3-space. International Journal of Geometric Methods in Modern Physics, 14(5): 11 pages. doi: 10.1142/S0219887817500669
- [17] Lopez R, Milin Šipuš Ž, Primorac Gajčić LJ, Protrka I. Harmonic Evolutes Of B-Scrolls with Constant Mean Curvature in Lorentz-Minkowski Space. International Journal of Geometric Methods in Modern Physics; 16(5): 15 pages. doi: 10.1142/S0219887819500762
- [18] McNertney L. One-parameter families of surfaces with constant curvature in Lorentz 3-space. PhD, Brown University, USA, 1980.
- [19] Nersessian A, Ramos E. Massive spinning particles and the geometry of null curves. Physics Letters B 1998; 445 (1-2): 123-128. doi: 10.1016/S0370-2693(98)01408-7