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Coefficient estimates for a new subclasses of λ -pseudo biunivalent functions with respect to symmetrical points associated with the Horadam Polynomials

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Abstract: In the present article, we introduce two new subclasses of λ -pseudo biunivalent functions with respect to symmetrical points in the open unit disk \mathbb{U} defined by means of the Horadam polynomials. For functions belonging to these subclasses, estimates on the Taylor-Maclaurin coefficients a_2 and a_3 are obtained. Fekete-Szegő inequalities of functions belonging to these subclasses are also founded. Furthermore, we point out several new special cases of our results.

Key words: Analytic function, univalent and biunivalent functions, Fekete-Szegő problem, λ -pseudo biunivalent functions with respect to symmetrical points, Horadam polynomials, coefficient bounds, subordination

1. Introduction and preliminaries

Let \mathcal{A} denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, and denoted by \mathcal{A} . Let \mathcal{S} be class of all functions in \mathcal{A} which are univalent and normalized by the conditions

$$f(0) = 0 = f'(0) - 1$$

in \mathbb{U} . For two functions f and g , analytic in \mathbb{U} , we say that the function f is subordinate to g in \mathbb{U} , written as $f(z) \prec g(z)$, ($z \in \mathbb{U}$), provided that there exists an analytic function (that is, Schwarz function) $w(z)$ defined on \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad \text{for all } z \in \mathbb{U},$$

such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

Besides, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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It is well known that every univalent function f has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w + a_2 w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be biunivalent in \mathbb{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{U} , and denoted by Σ .

In 1967, the class Σ of biunivalent functions was first investigated by Lewin [12] and it was derived that $|a_2| < 1.51$. Brannan and Taha [6] also considered certain subclasses of biunivalent functions, and obtained estimates for the initial coefficients. In 2010, Srivastava et al. [17] revived the investigation of various classes of bi univalent functions. Moreover, many other authors (see [1-4, 7]) have introduced and investigated subclasses of biunivalent functions.

By $S^*(\varphi)$ and $K(\varphi)$ we denote the following classes of functions

$$S^*(\varphi) = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \quad z \in \mathbb{U},$$

and

$$K(\varphi) = \left\{ f : f \in \mathcal{A}, 1 + \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \quad z \in \mathbb{U},$$

where $S^*(\varphi)$ and $K(\varphi)$ are the class of starlike and convex functions, respectively, were defined and studied by Ma and Minda [14]. It is clear that if $f(z) \in K$, then $zf'(z) \in S^*$.

Sakaguchi [16] introduced the class S_s^* of functions starlike with respect to symmetric points, which consists of functions $f(z) \in S$ satisfying the condition

$$\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}.$$

Moreover, Wang et al. [18] introduced the class K_s of functions convex with respect to symmetric points, which consists of functions $f(z) \in S$ satisfying the condition

$$\Re \left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in \mathbb{U}.$$

It is easily seen that if $f(z) \in K_s$, then $zf'(z) \in S_s^*$. For such a function φ , Ravichandran [15] introduced the following subclasses: A function $f \in \mathcal{A}$ is in the class $S_s^*(\varphi)$ if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z), \quad z \in \mathbb{U},$$

and in the class $K_s(\varphi)$ if

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \varphi(z) \quad z \in \mathbb{U}.$$

Recently, Babalola [5] defined the class L_λ of λ -pseudo-starlike functions as follows: Let $f \in A$ and $\lambda \geq 1$ is real. Then $f(z)$ belongs to the class L_λ of λ -pseudo-starlike functions in the unit disc \mathbb{U} if and only if

$$\Re \left\{ \frac{z(f'(z))^\lambda}{f(z)} \right\} \geq 0, \quad z \in \mathbb{U}.$$

The Horadam polynomials $h_n(x)$ are given by the following recurrence relation (see [10])

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} \geq 2), \tag{1.3}$$

with $h_1 = a$, $h_2 = bx$, and $h_3 = pbx^2 + aq$ where (a, b, p, q) are some real constants).

The characteristic equation of recurrence relation (1.3) is

$$t^2 - pxt - q = 0. \tag{1.4}$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Note that, some particular cases of Horadam polynomials sequence are listed as follows:

- If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad F_1(x) = 1, \quad F_2(x) = x.$$

- If $a = 2, b = p = q = 1$, the Lucas polynomials sequence is obtained

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x), \quad L_0(x) = 2, \quad L_1(x) = x.$$

- If $a = q = 1, b = p = 2$, the Pell polynomials sequence is obtained

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x), \quad p_1(x) = 1, \quad P_2(x) = 2x.$$

- If $a = b = p = 2, q = 1$, the Pell-Lucas polynomials sequence is obtained

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x), \quad Q_0(x) = 2, \quad Q_1(x) = 2x.$$

- If $a = b = 1, p = 2, q = -1$, the Chebyshev polynomials of first kind sequence is obtained

$$T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x), \quad T_0(x) = 1, \quad T_1(x) = x.$$

- If $a = 1, b = p = 2, q = -1$, the Chebyshev polynomials of second kind sequence is obtained

$$U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x), U_0(x) = 1, U_1(x) = 2x.$$

- If $x = 1$, the Horadam numbers sequence is obtained

$$h_{n-1}(1) = ph_{n-2}(1) + qh_{n-3}(1), h_0(1) = a, h_1(1) = b.$$

For more information associated with these polynomials see [8], ([9, 11, 13]).

Remark 1.1 [9] Let $\Omega(x, z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then

$$\Omega(x, z) = \frac{a + (b - ap)xt}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x)z^{n-1}. \tag{1.5}$$

In this paper, we introduce two new subclasses of λ -pseudo biunivalent functions with respect to symmetrical points by using the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x, z)$ which are given by the recurrence relation (1.3) and (1.5), respectively. Furthermore, we find the initial coefficients and the Fekete–Szegő inequality for functions belonging to the classes $L\Sigma(\lambda, \alpha, x)$ and $M\Sigma(\lambda, \alpha, x)$. Also, several special cases to our results were obtained.

2. Coefficient bounds for the function class $L\Sigma(\lambda, \alpha, x)$

Definition 2.1 A function $f \in \Sigma$ given by (1.1) is said to be in the class $L\Sigma(\lambda, \alpha, x)$, if the following conditions are satisfied:

$$(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'} \prec \Omega(x, z) + 1 - \alpha \tag{2.1}$$

and

$$(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'} \prec \Omega(x, w) + 1 - \alpha \tag{2.2}$$

where the real constants a, b , and q are as in (1.3) and $g(w) = f^{-1}(z)$ is given by (1.2).

We first state and prove the following result.

Theorem 2.2 Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda, \alpha, x)$. Then

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{[(2\lambda^2 + \lambda - 1) + 2\alpha(3\lambda^2 - 1)]b - 4p\lambda^2(1 + \alpha)^2} |bx^2 - 4qa\lambda^2(1 + \alpha)^2|} \tag{2.3}$$

$$|a_3| \leq \frac{|bx|}{(3\lambda - 1)(1 + 2\alpha)} + \frac{(bx)^2}{4\lambda^2(1 + \alpha)^2}, \tag{2.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{(3\lambda - 1)(1 + 2\alpha)} & , |\eta - 1| \leq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} |A| \\ \frac{|bx|^3|1 - \eta|}{[(2\lambda^2 + \lambda - 1) + 2\alpha(3\lambda^2 - 1)][|bx|^2 - 4\lambda^2(1 + \alpha)^2(pb x^2 + qa)]} & , |\eta - 1| \geq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} |A| \end{cases} \tag{2.5}$$

where $A = (2\lambda^2 + \lambda - 1) + 2\alpha(3\lambda^2 - 1) - \frac{4\lambda^2(1 + \alpha)^2(pb x^2 + qa)}{b^2 x^2}$.

Proof Let $f \in \Sigma$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in \mathbb{U}$ and using Definition 2.1, we can write

$$(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'} = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'} = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \dots \quad (2.6)$$

and

$$(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'} = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \dots \quad (2.7)$$

From (2.6) and (2.7), we obtain

$$(1 - \alpha) \frac{2z[f'(z)]^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'} = 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \dots \quad (2.8)$$

and

$$(1 - \alpha) \frac{2w[g'(w)]^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'} = 1 + h_2(x)p_1w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \dots \quad (2.9)$$

Notice that if

$$|\Phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \dots| < 1 \quad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1w + q_2w^2 + q_3w^3 + \dots| < 1 \quad (w \in \mathbb{U}),$$

then

$$|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (2.8) and (2.9), we have

$$2\lambda(1 + \alpha)a_2 = h_2(x)p_1 \quad (2.10)$$

$$2\lambda(\lambda - 1)(1 + 3\alpha)a_2^2 + (3\lambda - 1)(1 + 2\alpha)a_3 = h_2(x)p_2 + h_3(x)p_1^2 \quad (2.11)$$

$$-2\lambda(1 + \alpha)a_2 = h_2(x)q_1 \quad (2.12)$$

$$[2(\lambda^2 + 2\lambda - 1) + 2\alpha(3\lambda^3 + 3\lambda - 2)]a_2^2 - (3\lambda - 1)(1 + 2\alpha)a_3 = h_2(x)q_2 + h_3(x)q_1^2. \quad (2.13)$$

From (2.10) and (2.12), we find that

$$p_1 = -q_1 \tag{2.14}$$

and

$$8\lambda^2(1 + \alpha)^2 a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{2.15}$$

Moreover, by using (2.13) and (2.11), we obtain

$$[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)]a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \tag{2.16}$$

By using (2.14) in (2.16), we get

$$\left[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1) - \frac{8\lambda^2(1 + \alpha)^2 h_3(x)}{[h_2(x)]^2} \right] a_2^2 = h_2(x)(p_2 + q_2). \tag{2.17}$$

From (1.3) and (2.17), we have the desired inequality (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11) and using (2.14) and (2.15) we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(1 + 2\alpha)} + \frac{h_2(x)[p_1^2 + q_1^2]}{8\lambda^2(1 + \alpha)^2}. \tag{2.18}$$

Hence, using (2.14) and applying (1.3), we get desired inequality (2.4).

Now, by using (2.16) and (2.18) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{[h_2(x)]^3(1 - \eta)(p_2 + q_2)}{[2(2\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)][h_2(x)]^2 - 8\lambda^2(1 + \alpha)^2 h_3(x)} + \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(1 + 2\alpha)} \\ &= h_2(x) \left[\left(\Theta(\eta, x) + \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \right) p_2 + \left(\Theta(\eta, x) - \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \right) q_2 \right], \end{aligned}$$

where

$$\Theta(\eta, x) = \frac{[h_2(x)]^2(1 - \eta)}{[2(\lambda^2 + \lambda - 1) + 4\alpha(3\lambda^2 - 1)][h_2(x)]^2 - 8\lambda^2(1 + \alpha)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{h_2(x)}{(3\lambda - 1)(1 + 2\alpha)} & , \quad |\Theta(\eta, x)| \leq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \\ 2|h_2(x)||\Theta(\eta, x)| & , \quad |\Theta(\eta, x)| \geq \frac{1}{2(3\lambda - 1)(1 + 2\alpha)} \end{cases}.$$

This proves Theorem 2.2. □

For $\alpha = 0$ the class $L\Sigma(\lambda, \alpha, x)$ reduced to the class of λ -pseudo bistarlike functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 2.3 *Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2\lambda^2 + \lambda - 1)b - 4p\lambda^2]bx^2 - 4qa\lambda^2|}} \tag{2.19}$$

$$|a_3| \leq \frac{|bx|}{3\lambda - 1} + \frac{(bx)^2}{4\lambda^2}, \tag{2.20}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3\lambda - 1} & , |\eta - 1| \leq \frac{1}{2(3\lambda - 1)} |A_0| \\ \frac{|bx|^3|1-\eta|}{|(2\lambda^2 + \lambda - 1)[bx]^2 - 4\lambda^2(pb x^2 + qa)|} & , |\eta - 1| \geq \frac{1}{2(3\lambda - 1)} |A_0| \end{cases} \tag{2.21}$$

where $A_0 = (2\lambda^2 + \lambda - 1) - \frac{4\lambda^2(pb x^2 + qa)}{b^2 x^2}$.

For $\alpha = 1$ the class $L\Sigma(\lambda, 1, x)$ reduced to the class of λ -pseudo biconvex functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 2.4 *Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda, 1, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1)]b - 16p\lambda^2]bx^2 - 16qa\lambda^2|}} \tag{2.22}$$

$$|a_3| \leq \frac{|bx|}{3(3\lambda - 1)} + \frac{(bx)^2}{16\lambda^2}, \tag{2.23}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3(3\lambda - 1)} & , |\eta - 1| \leq \frac{1}{6(3\lambda - 1)} |A_1| \\ \frac{|bx|^3|1-\eta|}{|[2(2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1)][bx]^2 - 16\lambda^2(pb x^2 + qa)|} & , |\eta - 1| \geq \frac{1}{6(3\lambda - 1)} |A_1| \end{cases} \tag{2.24}$$

where $A_1 = (2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1) - \frac{16\lambda^2(pb x^2 + qa)}{b^2 x^2}$.

For $\lambda = 1$ the class $L\Sigma(1, \alpha, x)$ reduced to the class of biunivalent functions with respect to symmetrical points. For functions belonging to this class we have the following corollary:

Corollary 2.5 *Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(1, \alpha, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[2(1 + 2\alpha)b - 4p(1 + \alpha)^2]bx^2 - 4qa(1 + \alpha)^2|}} \tag{2.25}$$

$$|a_3| \leq \frac{|bx|}{2(1 + 2\alpha)} + \frac{(bx)^2}{4(1 + \alpha)^2}, \tag{2.26}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{2(1 + 2\alpha)} & , |\eta - 1| \leq \frac{1}{4(1 + 2\alpha)} |B| \\ \frac{|bx|^3|1-\eta|}{|[2(1 + \alpha)[bx]^2 - 4(1 + \alpha)^2(pb x^2 + qa)|} & , |\eta - 1| \geq \frac{1}{4(1 + 2\alpha)} |B| \end{cases} \tag{2.27}$$

where $B = 2(1 + 2\alpha) - \frac{4(1 + \alpha)^2(pb x^2 + qa)}{b^2 x^2}$.

3. Coefficient bounds for the function class $M\Sigma(\lambda, \alpha, x)$

Definition 3.1 A function $f \in \Sigma$ given by (1.1) is said to be in the class $M\Sigma(\lambda, \alpha, x)$, if the following conditions are satisfied:

$$\left(\frac{2z[f'(z)]^\lambda}{f(z) - f(-z)}\right)^\alpha \left(\frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'}\right)^{1-\alpha} < \Omega(x, z) + 1 - \alpha \tag{3.1}$$

and

$$\left(\frac{2w[g'(w)]^\lambda}{g(w) - g(-w)}\right)^\alpha \left(\frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'}\right)^{1-\alpha} < \Omega(x, w) + 1 - \alpha \tag{3.2}$$

where the real constants $a, b,$ and q are as in (1.3) and $g(w) = f^{-1}(z)$ is given by (1.2).

Theorem 3.2 Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(\lambda, \alpha, x)$. Then

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3))b - 4p\lambda^2(\alpha - 2)^2]bx^2 - 4qa\lambda^2(\alpha - 2)^2|}} \tag{3.3}$$

$$|a_3| \leq \frac{|bx|}{(3\lambda - 1)(|3 - 2\alpha|)} + \frac{(bx)^2}{4\lambda^2(\alpha - 2)^2}, \tag{3.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{(3\lambda - 1)(3 - 2\alpha)} & , |\eta - 1| \leq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} |C| \\ \frac{|bx|^3|1 - \eta|}{|[2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3)][bx]^2 - 4\lambda^2(\alpha - 2)^2(pbx^2 + qa)|} & , |\eta - 1| \geq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} |C| \end{cases} \tag{3.5}$$

where $C = 2\lambda^2(\alpha - 2)^2 + (\lambda + 2\alpha - 3) - \frac{4\lambda^2(\alpha - 2)^2(pbx^2 + qa)}{b^2x^2}$.

Proof Let $f \in \Sigma$ be given by the Taylor–Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0, |\psi(z)| < 1$ and $|\Phi(w)| < 1, z, w \in \mathbb{U}$ and using Definition 3.1, we can write

$$\left(\frac{2z[f'(z)]^\lambda}{f(z) - f(-z)}\right)^\alpha \left(\frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'}\right)^{1-\alpha} = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$\left(\frac{2w[g'(w)]^\lambda}{g(w) - g(-w)}\right)^\alpha \left(\frac{2[(wg'(w))']^\lambda}{[g(w) - g(-w)]'}\right)^{1-\alpha} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$\left(\frac{2z[f'(z)]^\lambda}{f(z) - f(-z)}\right)^\alpha \left(\frac{2[(zf'(z))']^\lambda}{[f(z) - f(-z)]'}\right)^{1-\alpha} = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \dots \tag{3.6}$$

and

$$\left(\frac{2w[g'(w)]^\lambda}{g(w)-g(-w)}\right)^\alpha \left(\frac{2[(wg'(w))']^\lambda}{[g(w)-g(-w)]'}\right)^{1-\alpha} = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \dots \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$\left(\frac{2z[f'(z)]^\lambda}{f(z)-f(-z)}\right)^\alpha \left(\frac{2[(zf'(z))']^\lambda}{[f(z)-f(-z)]'}\right)^{1-\alpha} = 1 + h_2(x)p_1z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \dots \quad (3.8)$$

and

$$\left(\frac{2w[g'(w)]^\lambda}{g(w)-g(-w)}\right)^\alpha \left(\frac{2[(wg'(w))']^\lambda}{[g(w)-g(-w)]'}\right)^{1-\alpha} = 1 + h_2(x)p_1w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \dots \quad (3.9)$$

Notice that if

$$|\Phi(z)| = |p_1z + p_2z^2 + p_3z^3 + \dots| < 1 \quad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1w + q_2w^2 + q_3w^3 + \dots| < 1 \quad (w \in \mathbb{U}),$$

then

$$|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (3.8) and (3.9), we have

$$-2\lambda(\alpha - 2)a_2 = h_2(x)p_1 \quad (3.10)$$

$$[2\lambda^2(\alpha - 2)^2 + 2\lambda(3\alpha - 4)]a_2^2 + (3\lambda - 1)(3 - 2\alpha)a_3 = h_2(x)p_2 + h_3(x)p_1^2 \quad (3.11)$$

$$2\lambda(\alpha - 2)a_2 = h_2(x)q_1 \quad (3.12)$$

$$[2\lambda^2(\alpha - 2)^2 + 2\lambda(5 - 3\alpha) + 2(2\alpha - 3)]a_2^2 + (3\lambda - 1)(2\alpha - 3)a_3 = h_2(x)q_2 + h_3(x)q_1^2. \quad (3.13)$$

From (3.10) and (3.12), we find that

$$p_1 = -q_1 \quad (3.14)$$

and

$$8\lambda^2(\alpha - 2)^2a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \quad (3.15)$$

Moreover, by using (3.13) and (3.11), we obtain

$$[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)]a_2^2 = h_2(x)(p_2 + q_2) + h_3(x)(p_1^2 + q_1^2). \quad (3.16)$$

By using (3.14) in (3.16), we get

$$\left[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3) - \frac{8\lambda^2(\alpha - 2)^2h_3(x)}{[h_2(x)]^2}\right]a_2^2 = h_2(x)(p_2 + q_2). \quad (3.17)$$

From (1.3), and (3.17), we have the desired inequality (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.13) from (3.11) and using (3.14) and (3.15) we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(|3 - 2\alpha|)} + \frac{h_2(x)[p_1^2 + q_1^2]}{8\lambda^2(\alpha - 2)^2}. \tag{3.18}$$

Hence, using (3.14) and applying (1.3), we get desired inequality (3.4).

Now, by using (3.16) and (3.18) for some $\eta \in \mathbb{R}$, we get

$$\begin{aligned} a_3 - \eta a_2^2 &= \frac{[h_2(x)]^3(1 - \eta)(p_2 + q_2)}{[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)][h_2(x)]^2 - 8\lambda^2(\alpha - 2)^2 h_3(x)} + \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(3 - 2\alpha)} \\ &= h_2(x) \left[\left(\Theta(\eta, x) + \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \right) p_2 + \left(\Theta(\eta, x) - \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \right) q_2 \right], \end{aligned}$$

where

$$\Theta(\eta, x) = \frac{[h_2(x)]^2(1 - \eta)}{[4\lambda^2(\alpha - 2)^2 + 2(\lambda + 2\alpha - 3)][h_2(x)]^2 - 8\lambda^2(\alpha - 2)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{h_2(x)}{(3\lambda - 1)(3 - 2\alpha)} & , \quad |\Theta(\eta, x)| \leq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \\ 2|h_2(x)||\Theta(\eta, x)| & , \quad |\Theta(\eta, x)| \geq \frac{1}{2(3\lambda - 1)(3 - 2\alpha)} \end{cases}.$$

This proves Theorem 3.2. □

For $\lambda = 1$ the class $M\Sigma(\lambda, \alpha, x)$ reduced to the class of biunivalent functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 3.3 *Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(1, \alpha, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2(\alpha - 2)^2 + 2(\alpha - 1))b - 4p(\alpha - 2)^2]bx^2 - 4qa(\alpha - 2)^2|}} \tag{3.19}$$

$$|a_3| \leq \frac{|bx|}{2(|3 - 2\alpha|)} + \frac{(bx)^2}{4(\alpha - 2)^2}, \tag{3.20}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{2(3 - 2\alpha)} & , \quad |\eta - 1| \leq \frac{1}{4(3 - 2\alpha)} |C_1| \\ \frac{|bx|^3|1 - \eta|}{|[2(\alpha - 2)^2 + 2(\alpha - 1)][bx]^2 - 4(\alpha - 2)^2(pbx^2 + qa)|} & , \quad |\eta - 1| \geq \frac{1}{4(3 - 2\alpha)} |C_1| \end{cases} \tag{3.21}$$

where $C_1 = 2(\alpha - 2)^2 + 2(\alpha - 1) - \frac{4(\alpha - 2)^2(pbx^2 + qa)}{b^2 x^2}$.

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