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## On composition factors in modules over some group rings

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**Abstract:** The aim of this paper is to prove the following result: Let  $G$  be an FC-hypercentral group and let  $A$  have a finite  $FG$ -composition series. Then  $A$  contains two  $FG$ -submodules  $B, C$  such that  $A = B \oplus C$ , where each  $FG$ -composition factor of  $B$  has finite  $F$ -dimension and each  $FG$ -composition factor of  $C$  has infinite  $F$ -dimension. This has consequences for  $FG$ -modules whose proper submodules all have finite  $F$ -dimension and for those  $FG$ -modules whose proper quotients all have finite  $F$ -dimension.

**Key words:** FC-hypercentral, module,  $FG$ -composition series, simple  $FG$ -module, quasifinite module, just infinite dimensional module

### 1. Introduction

Let  $R$  be a ring, let  $G$  be a group, and let  $A$  be a (left) module over the group ring  $RG$ , notation that we shall use throughout with little further mention. We shall also let  $F$  denote a field. Suppose that  $A$  has a finite  $RG$ -composition series. Some factors in this series may have similar properties (for example, some may be  $RG$ -central); however, their relative positions in the composition series may be quite diverse. Therefore, it is natural to consider questions regarding the positioning of  $RG$ -composition factors having similar properties. The roots of this problem date to the famous Fitting lemma, a slight generalization of which can be stated as follows:

Let  $G$  be a finite nilpotent group. If  $A$  has finite composition length, then  $A = Z \oplus E$ , where each  $RG$ -composition factor  $U/V$  of  $Z$  (respectively of  $E$ ) satisfies the condition  $G = C_G(U/V)$  (respectively  $G \neq C_G(U/V)$ ).

This result served as a starting point for many generalizations that have found applications in various fields of group theory (see [1, Chapters 1, 7, and 8], for example). In turn, these results were also generalized to factors associated with various formations, a topic that will not be pursued further here, but as references see the book [5, Chapter 10] and the survey [2]. In this paper, we consider a different side of this issue.

Suppose that  $A$  has a finite  $FG$ -composition series. The factors of this series are simple  $FG$ -modules, some of which will have finite  $F$ -dimension and some infinite  $F$ -dimension. The question naturally arises whether all the finite dimensional factors can be made consecutive and whether all infinite dimensional factors can be made consecutive. Clearly, the answer will generally depend on the properties of the group  $G$ .

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If a simple  $FG$ -factor  $U/V$  has finite  $F$ -dimension  $k$ , we can consider  $G/C_G(U/V)$  as an irreducible subgroup of the finite dimensional linear group  $GL_k(F)$ . There is a lot of information concerning such groups. On the other hand, if the simple  $FG$ -factor  $U/V$  has infinite  $F$ -dimension, then information concerned with the structure of  $G/C_G(U/V)$  is rather limited. Such information is available in the book [4, Chapters 2 and 3], where the structure of irreducible hypercentral and FC-hypercentral groups was given.

It is appropriate here to recall the following notation and definitions.

For the group  $G$ , if  $x \in G$ , then let

$$x^G = \{x^g = g^{-1}xg \mid g \in G\} \text{ and } \mathbf{FC}(G) = \{x \in G \mid x^G \text{ is finite}\},$$

this latter characteristic subgroup of  $G$  being the FC-center of  $G$ .

Beginning with the FC-center, we may construct the upper FC-central series of the group  $G$ , the series of characteristic subgroups of  $G$ ,

$$1 = C_0 \leq C_1 \leq \dots C_\alpha \leq C_{\alpha+1} \leq \dots C_\gamma,$$

defined by

$$\begin{aligned} C_1 &= \mathbf{FC}(G), \\ C_{\alpha+1}/C_\alpha &= \mathbf{FC}(G/C_\alpha) \text{ for ordinals } \alpha, \\ C_\lambda &= \bigcup_{\beta < \lambda} C_\beta \text{ for limit ordinals } \lambda. \end{aligned}$$

The last term  $C_\gamma$  of this series is called the upper FC-hypercenter of the group  $G$ , and if  $G = C_\gamma$ , then  $G$  is called FC-hypercentral. In the case when  $\gamma$  is finite and  $G = C_\gamma$  the group  $G$  is called  $FC$ -nilpotent.

The main result of this paper is the following:

**Theorem A** *Let  $G$  be an FC-hypercentral group and let  $A$  have a finite  $FG$ -composition series. Then  $A$  contains two  $FG$ -submodules  $B, C$  satisfying the following conditions:*

- (i) *each  $FG$ -composition factor of  $B$  has finite  $F$ -dimension;*
- (ii) *each  $FG$ -composition factor of  $C$  has infinite  $F$ -dimension;*
- (iii)  $A = B \oplus C$ .

We note some consequences of this theorem. Suppose that the module  $A$  has the property that  $A$  is not a finitely generated  $RG$ -module but every proper  $RG$ -submodule of  $A$  is finitely generated as an  $RG$ -submodule.

In this case, one of the following situations arises:

- (i)  $A$  is the ascending union of its proper  $RG$ -submodules.
- (ii)  $A$  contains a proper  $RG$ -submodule  $B$  such that  $A/B$  is a simple  $RG$ -module.

An  $RG$ -module satisfying condition (i) is called a quasifinite  $RG$ -module. Quasifinite  $FG$ -modules have been considered in the papers [7, 8] of the last two authors of this paper and these results were generalized by Kurdachenko [3] to the case of  $RG$ -modules when  $R$  is a Dedekind domain.

**Corollary A** *Let  $G$  be an FC-hypercentral group and let  $A$  be an FG-module having infinite  $F$ -dimension. If every proper FG-submodule of  $A$  has finite  $F$ -dimension, then either  $A$  is a quasifinite FG-module or  $A$  is a simple FG-module.*

We may also consider the dual situation. Suppose that  $A$  has infinite  $F$ -dimension but every proper FG-quotient module of  $A$  has finite  $F$ -dimension. (The quotient module  $A/B$  is proper if the submodule  $B$  is nonzero.)

In this case, one of the following situations arises:

- (i) The intersection of all nonzero FG-submodules of  $A$  is zero, so  $A$  is a nonmonolithic FG-module.
- (ii)  $A$  contains a proper nonzero simple FG-submodule  $B$  (the FG-monolith of  $A$ ) such that the factor module  $A/B$  has finite  $F$ -dimension.

The FG-module satisfying condition (i) is called a just-infinite dimensional FG-module. Just-infinite dimensional FG-modules were discussed in the papers [9, 10] of the last two authors. It is also worth mentioning that an RG-module  $A$  is called just-non-Artinian if  $A$  is not Artinian but each proper factor of  $A$  is. Such modules were discussed in the papers [11, 12] again in the case where  $R$  is Dedekind.

**Corollary B** *Let  $G$  be an FC-hypercentral group and let  $A$  be an FG-module having infinite  $F$ -dimension. If every proper FG-quotient module of  $A$  has finite  $F$ -dimension, then either  $A$  is a just-infinite dimensional FG-module or  $A$  is a simple FG-module.*

## 2. Extensions of finite dimensional modules by infinite dimensional simple modules

We begin with the following result.

**Lemma 2.1** *Let  $A$  be an FG-module such that  $C_G(A) = 1$ . Suppose that  $A$  contains an FG-submodule  $B$  such that  $B$  and  $A/B$  are simple FG-modules,  $B$  has finite  $F$ -dimension, and  $A/B$  has infinite dimension. If  $C_G(B) \cap \mathbf{FC}(G) \neq 1$ , then there is an FG-submodule  $L$  of  $A$  such that  $A = B \oplus L$ .*

**Proof** Let  $z$  be a nontrivial element of  $C_G(B) \cap \mathbf{FC}(G)$ . Since the conjugacy class of  $z$  is finite the normal subgroup  $V = C_G(\langle z \rangle^G)$  has finite index in  $G$ . Let  $\varphi : A \rightarrow A$  be the mapping defined by  $\varphi(a) = (z - 1)a = [z, a]$  for all  $a \in A$ . If  $v \in V$ , then

$$v(z - 1)a = (z - 1)va = (z - 1)(va) \in [z, A],$$

which shows that  $[z, A] = \mathbf{Im}(\varphi)$  is an FV-module. Likewise,  $\mathbf{ker}(\varphi) = C_A(z)$  is an FV-module. Clearly  $B \leq C_A(z)$  and  $C_A(z) \neq A$  since  $C_G(A) = 1$ . Since  $A/B$  is a simple FG-module, it follows from [5, Theorem 5.5] that  $A/B$  is a direct sum of finitely many FV-submodules, each of which has infinite  $F$ -dimension. Since  $A/B$  is a semisimple FV-module its FV-submodule  $C_A(z)/B$  has an FV-complement  $E/B$  (see [5, Corollary 4.3]). Again by [5, Corollary 4.3] both  $E/B$  and  $C_A(z)/B$  are semisimple FV-modules, each of which is a direct sum of finitely many simple factors having infinite  $F$ -dimension. However,

$$E/B \cong_{FV} (A/B)/(C_A(z)/B) \cong_{FV} A/C_A(z) \cong_{FV} [z, A],$$

so  $[z, A]$  is a semisimple  $FV$ -module, each of whose simple  $FV$ -factors has infinite  $F$ -dimension. Let  $Y$  be one of these infinite-dimensional simple  $FV$ -submodules of  $[z, A]$ . Let  $\{g_1, \dots, g_l\}$  be a transversal to  $V$  in  $G$  and let  $L = g_1Y + \dots + g_lY$ . Then  $L$  is a semisimple  $FV$ -module, by [5, Corollary 4.3] again, whose  $FV$ -factors are infinite-dimensional. Thus,  $L$  is an  $FG$ -submodule of infinite  $F$ -dimension. Since  $A/B$  is a simple  $FG$ -module and  $B$  has finite  $F$ -dimension it follows that  $A = L + B$ . Moreover, if  $L \cap B \neq 0$ , then  $B \leq L$  since  $B$  is simple, but then  $L = A$  is a semisimple  $FV$ -module whose  $FV$ -factors have infinite  $F$ -dimension. This contradicts the fact that  $B$  is finite-dimensional. Hence,  $L \cap B = 0$  and  $A = B \oplus L$ .  $\square$

**Lemma 2.2** *Let  $G$  be an FC-hypercentral group and let  $A$  be a simple  $FG$ -module such that  $C_G(A) = 1$ . If  $A$  has finite  $F$ -dimension, then  $G$  is abelian-by-finite.*

**Proof** It follows from [13, Theorem 2] that  $G$  is nilpotent-by-finite and then that  $G$  is abelian-by-finite follows from [1, Theorem 1.4.11], for example.  $\square$

We shall require the following technical result. As usual we let the annihilator of a subset  $Y$  of  $A$  be

$$\mathbf{Ann}_R(Y) = \{r \in R \mid ry = 0 \text{ for all } y \in Y\}.$$

Assume now that  $A$  is an  $FG$ -module and that  $C_G(A) = 1$ . For each  $1 \neq g \in \zeta(G)$  we let  $X(g) = \langle x(g) \rangle$  denote an infinite cyclic group and let  $D(g) = FX(g)$  denote the corresponding group algebra. We note that  $D(g)$  is a principal ideal domain. We can define an action of  $x(g)$  on  $A$  by setting

$$x(g)a = ga \text{ for all } a \in A.$$

This action can be extended in a natural way to an action of  $D(g)$  on  $A$  and hence we may consider  $A$  as a left  $D(g)G$ -module.

Let  $R$  be an integral domain and let  $A$  be an  $R$ -module. Let

$$\mathbf{Tor}_R(A) = \{a \in A \mid \mathbf{Ann}_R(a) \neq 0\}.$$

It is easy to see that  $\mathbf{Tor}_R(A)$  is an  $R$ -submodule of  $A$  called the  $R$ -periodic part of  $A$ . We say that  $A$  is periodic as an  $R$ -module or, simply, that  $A$  is  $R$ -periodic, if  $\mathbf{Tor}_R(A) = A$  so that  $\mathbf{Ann}_R(a) \neq 0$ , for each element  $a \in A$ . We say that  $A$  is  $R$ -torsion-free if  $\mathbf{Tor}_R(A) = 0$ .

Analogous to the annihilator of a subset of  $A$  is the notion of the annihilator of a subset  $X$  of  $R$ . In this case, we have

$$\mathbf{Ann}_A(X) = \{a \in A \mid xa = 0 \text{ for all } x \in X\}.$$

We define the  $R$ -assassinator of  $A$  to be the set

$$\mathbf{Ass}_R(A) = \{P \mid P \text{ is a prime ideal of } R \text{ such that } \mathbf{Ann}_A(P) \neq 0\}.$$

If  $U$  is an ideal of  $R$ , then we set

$$A_U = \{a \in A \mid U^n a = 0 \text{ for some natural number } n\}.$$

It is easy to see that  $A_U$  is an  $R$ -submodule of  $A$  called the  $U$ -component of  $A$ . If  $A = A_U$ , then  $A$  is called a  $U$ -module. Furthermore, let

$$\Omega_{U,k}(A) = \{a \in A \mid U^k a = 0 \text{ for this fixed } k\}.$$

It is easy to see that  $\Omega_{U,k}(A)$  is an  $R$ -submodule, that

$$\Omega_{U,1}(A) \leq \Omega_{U,2}(A) \leq \dots \leq \Omega_{U,k}(A) \leq \dots, \text{ and that}$$

$$A_U = \bigcup_{k \in \mathbb{N}} \Omega_{U,k}(A).$$

We refer the reader to [6], which has further details.

**Lemma 2.3** *Let  $G$  be an abelian group and  $A$  an  $FG$ -module. Suppose that  $A$  contains an  $FG$ -submodule  $B$  such that  $B$  and  $A/B$  are simple. If  $B$  has finite  $F$ -dimension and  $A/B$  has infinite  $F$ -dimension, then  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Without loss of generality we may assume that  $C_G(A) = 1$ . Note that if  $A$  contains a nonzero  $FG$ -submodule  $C$  such that  $C \cap B = 0$ , then  $(C + B)/B$  is a nonzero  $FG$ -submodule of  $A/B$ . Since  $A/B$  is simple it follows that  $A = B + C = B \oplus C$ .

We may therefore suppose that every nonzero  $FG$ -submodule of  $A$  contains  $B$ , so  $B$  is then the  $FG$ -monolith of  $A$ . Using the notation introduced above,  $A$  is a left  $D(g)$ -module for each element  $g \in G$ .

Let  $b \in B$  so that  $D(g)b \leq B$ . We note that  $D(g)$  has infinite  $F$ -dimension. Since  $D(g)b \cong D(g)/\mathbf{Ann}_{D(g)}(b)$  and  $\mathbf{dim}_F(B)$  is finite it follows that  $b$  has nonzero annihilator in the ring  $D(g)$ . Hence, the  $D(g)$ -periodic part of  $A$  is nonzero also. Let  $\mathbf{Tor}_{D(g)}(A) = T$  so that  $B \leq T$ . Also,  $\mathbf{Ann}_{D(g)}(b) = \mathbf{Ann}_{D(g)}(B) = P$  is a maximal ideal of  $D(g)$ . Since  $G$  is abelian,  $T$  is an  $FG$ -submodule and since  $A/B$  is simple we have  $T = B$  or  $T = A$ .

Suppose first that  $T = B$ . Then there is a  $D(g)$ -submodule  $U$  of  $A$  such that  $A = B \oplus U$  by [6, Proposition 8.9]. Then  $U$  is  $D(g)$ -torsion-free so that  $PU \neq 0$ . Since  $PA = PU \leq U$  we have

$$PA \cap B = PA \cap T = 0.$$

However,  $G$  is abelian so  $PA$  is an  $FG$ -submodule and hence  $B \leq PA$ , a contradiction, which shows that  $T = A$ . Therefore,  $A$  is a periodic  $D(g)$ -module.

In this case  $A = \bigoplus_{Q \in \pi} A_Q$  where  $A_Q$  is a non-zero  $Q$ -component of  $A$  and  $\pi = \mathbf{Ass}_{D(g)}(A)$  (see [6, Corollary 3.8], for example). Since  $G$  is abelian,  $A_Q$  is an  $FG$ -submodule of  $A$  and since  $A$  is  $FG$ -monolithic with  $B \leq A_P$  we have  $A = A_P$ .

Suppose that  $\Omega_{P,1}(A) \neq A$ . Then  $\Omega_{P,1}(A) = B$  since  $\Omega_{P,1}(A)$  is an  $FG$ -module. Since  $\Omega_{P,2}(A)$  is also an  $FG$ -module it then follows that  $\Omega_{P,2}(A) = A$ . Since  $D(g)$  is a principal ideal domain there exists  $y \in P$  such that  $P = yD(g)$  so  $y^2a = 0$ , since  $\Omega_{P,2}(A) = A$ , for each element  $a \in A$ . Therefore,  $yA \leq B$  so  $yA$  has finite dimension. Since  $G$  is abelian the map  $\rho : A \rightarrow A$  defined by  $\rho(a) = ya$  for  $a \in A$  is an  $FG$ -endomorphism of  $A$  and  $yA = \mathbf{Im}(\rho) \leq B$ . Clearly  $\mathbf{ker}(\rho) = \Omega_{P,1}(A) = B$  so  $yA \cong A/B$ , a contradiction since  $yA$  has finite dimension whereas  $A/B$  has infinite  $F$ -dimension. Therefore,  $\Omega_{P,1}(A) = A$ . Hence,  $\mathbf{Ann}_{D(g)}(A) = P$  is a maximal ideal of  $D(g)$ ; this is true for all  $g \in G$ .

Let  $d \in A \setminus B$  and let  $D = (FG)d$ , the  $FG$ -submodule generated by  $d$ . Clearly  $B \leq D$  since  $A$  is  $FG$ -monolithic so there exists  $u \in FG$  such that  $0 \neq ud \in B$ . Let

$$u = \alpha_1 g_1 + \dots + \alpha_m g_m$$

for certain  $\alpha_i \in F, g_i \in G, 1 \leq i \leq m$ . From what we proved above,  $\mathbf{Ann}_{D(g_1)}(A)$  is a maximal ideal of  $D(g_1)$ . Let  $F_1$  be the field  $D(g_1)/\mathbf{Ann}_{D(g_1)}(A)$ , so we can consider  $A$  as an  $F_1G$ -module. This module is again  $F_1G$ -monolithic and  $B$  is its  $F_1G$ -monolith. By the arguments used above  $A$  is annihilated by some maximal ideal of the group ring  $F_1\langle g_2 \rangle$  and it follows that  $\mathbf{Ann}_{F\langle g_1, g_2 \rangle}(A)$  is a maximal ideal of the ring  $F\langle g_1, g_2 \rangle$ . Using these arguments we see that after finitely many steps  $\mathbf{Ann}_{F\langle g_1, \dots, g_m \rangle}(A)$  is a maximal ideal of the ring  $F\langle g_1, \dots, g_m \rangle$ . Then  $F\langle g_1, \dots, g_m \rangle/\mathbf{Ann}_{F\langle g_1, \dots, g_m \rangle}(A)$  is simple, so  $F\langle g_1, \dots, g_m \rangle d$  is simple. As  $B$  is an  $FG$ -module it is an  $F\langle g_1, \dots, g_m \rangle$ -module and hence  $(F\langle g_1, \dots, g_m \rangle d) \cap B = 0$ , since  $d \notin B$ . On the other hand,

$$0 \neq ud \in (F\langle g_1, \dots, g_m \rangle d) \cap B,$$

and we obtain a contradiction, which proves the lemma. □

We next extend this result to the abelian-by-finite case.

**Lemma 2.4** *Let  $G$  be an abelian-by-finite group and  $A$  an  $FG$ -module. Suppose that  $A$  contains a simple  $FG$ -submodule  $B$  such that  $A/B$  is also simple. If  $B$  has finite  $F$ -dimension and  $A/B$  has infinite  $F$ -dimension, then  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** As in the proof of Lemma 2.3 we may assume that  $C_G(A) = 1$  and that every nonzero  $FG$ -submodule of  $A$  contains  $B$ . We shall obtain a contradiction in this case.

Let  $H$  be a normal abelian subgroup of  $G$  of finite index. Then both  $B$  and  $A/B$  contain simple  $FH$ -submodules, denoted by  $C$  and  $D/B$ , respectively, such that  $B = \bigoplus_{1 \leq j \leq n} x_j C$  and  $A/B = \bigoplus_{1 \leq m \leq k} y_m D/B$  for certain  $x_j, y_j \in A$  (see [5, Theorem 5.5], for example). We note that  $C$  has finite  $F$ -dimension and that  $D/B$  has infinite  $F$ -dimension. It follows that  $A$  has a finite series of  $FH$ -submodules:

$$0 = B_0 \leq C = B_1 \leq \dots \leq B_n = B \leq D = B_{n+1} \leq \dots \leq B_{n+k} = A,$$

whose factors are simple  $FH$ -modules, where  $B_j/B_{j-1}$  has finite dimension for  $1 \leq j \leq n$  and  $B_m/B_{m-1}$  has infinite  $F$ -dimension for  $n+1 \leq m \leq n+k$ . Let  $E = E_0 = B_{n-1}$ .

By Lemma 2.3  $B_{n+1}/E$  contains an  $FH$ -submodule  $E_1/E$  such that  $B_{n+1}/E = B_n/E \oplus E_1/E$ . In particular,  $B_{n+1}/E_1$  is a simple  $FH$ -module of finite  $F$ -dimension. Suppose, inductively, that for some  $r \geq 0$  we have constructed a proper  $FH$ -submodule  $E_r$  such that  $B_{n+r}/E_r$  is a simple  $FH$ -module of finite  $F$ -dimension. Then  $B_{n+r+1}/E_r$  satisfies the hypotheses of Lemma 2.3 and we see that there is an  $FH$ -submodule  $E_{r+1}/E_r$  such that  $B_{n+r+1}/E_r = B_{n+r}/E_r \oplus E_{r+1}/E_r$ . Then  $B_{n+r+1}/E_{r+1}$  is a simple  $FH$ -module of finite  $F$ -dimension. This argument implies the existence of a proper  $FH$ -submodule  $E_k$  such that  $A/E_k$  has finite  $F$ -dimension.

Let  $\{g_1, \dots, g_t\}$  be a transversal to  $H$  in  $G$ . Since  $A/E_k \cong_F A/g_j E_k$  for each  $j$  with  $1 \leq j \leq t$ , it follows that  $A/g_j E_k$  has finite  $F$ -dimension. Hence, the  $FG$ -submodule  $L = \bigcap_{1 \leq j \leq t} g_j E_k$  has finite codimension. Since  $L \neq 0$  we have  $B \leq L$  and  $L/B$  is a proper  $FG$ -submodule of  $A/B$ , a contradiction since  $A/B$  is simple. This proves the result. □

**Proposition 2.5** *Let  $G$  be an FC-hypercentral group and  $A$  an  $FG$ -module. Suppose that  $A$  contains an  $FG$ -submodule  $B$  such that  $B$  and  $A/B$  are simple  $FG$ -modules. If  $B$  has finite  $F$ -dimension and  $A/B$  has infinite  $F$ -dimension, then  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Without loss of generality we may suppose that  $C_G(A)$  is trivial. If  $C_G(B) \neq 1$ , then  $C_G(B) \cap \mathbf{FC}(G) \neq 1$  also (see [5, Corollary 3.16], for example), in which case the result follows from Lemma 2.1. On the other hand, if  $C_G(B) = 1$ , then we may think of  $G$  as a subgroup of  $GL_n(F)$ , where  $n$  is the  $F$ -dimension of  $B$ . By Lemma 2.2  $G$  is then abelian-by-finite and Lemma 2.4 gives the result.  $\square$

**Corollary 2.6** *Let  $G$  be FC-hypercentral and let  $A$  be an FG-module. Suppose that  $A$  contains a finite dimensional FG-submodule  $B$  such that  $A/B$  is an infinite dimensional simple FG-module. Then  $A$  contains an FG-submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Since  $B$  has finite  $F$ -dimension it has a finite series

$$0 = B_0 \leq B_1 \leq \dots \leq B_n = B$$

whose factors are simple  $FG$ -modules. We use induction on  $n$ , the case  $n = 1$  being covered by Proposition 2.5.

Suppose that  $n > 1$  and that the result is true for  $A/B_1$ . Then  $A/B_1$  contains an  $FG$ -submodule  $D/B_1$  such that  $A/B_1 = B/B_1 \oplus D/B_1$ . Then  $D/B_1$  is a simple infinite-dimensional  $FG$ -module and since  $B_1$  is a simple finite-dimensional  $FG$ -module, Proposition 2.5 implies that  $D$  contains an  $FG$ -submodule  $C$  such that  $D = B_1 \oplus C$ . Then we have

$$A = B + D = B + B_1 + C = B + C$$

and  $B \cap C = 0$  since  $D \cap B = B_1$ . Hence,  $A = B \oplus C$ , as required.  $\square$

To conclude this section we note that Corollary 2.6 is no longer true when we step outside the realm of FC-hypercentral groups even in the case when  $G$  is soluble. In his paper [14], Zaitsev gave an example of an  $\mathbb{F}_2G$ -module  $A$  over the Charin  $\{2, 3\}$ -group  $G$ , which has a finite  $\mathbb{F}_2G$ -submodule  $B$  such that  $A/B$  is an infinite simple  $\mathbb{F}_2G$ -module and  $B$  has no direct  $\mathbb{F}_2G$ -complement in  $A$ .

### 3. Extensions of infinite-dimensional simple modules by finite-dimensional modules

In this section we consider the dual situation to that occurring in Section 2; many of the results in this section are analogous to those occurring in that last section.

**Lemma 3.1** *Let  $A$  be an FG-module such that  $C_G(A) = 1$ . Suppose that  $A$  contains an FG-submodule  $B$  such that  $B$  and  $A/B$  are simple FG-modules,  $B$  has infinite  $F$ -dimension, and  $A/B$  has finite dimension. If  $C_G(A/B) \cap \mathbf{FC}(G) \neq 1$ , then there is an FG-submodule  $L$  of  $A$  such that  $A = B \oplus L$ .*

**Proof** Let  $1 \neq z \in C_G(A/B) \cap \mathbf{FC}(G)$ . The conjugacy class of  $z$  is finite so  $V = C_G(\langle z \rangle^G)$  has finite index in  $G$ . Let  $\varphi : A \rightarrow A$  be defined by  $\varphi(a) = (z-1)a = [z, a]$  for all  $a \in A$ , as in Lemma 2.1. Then  $[z, A] = \mathbf{Im}(\varphi)$  and  $\mathbf{ker}(\varphi) = C_A(z)$  are  $FV$ -submodules and  $[z, A] \leq B$ . Since  $C_G(A) = 1$ ,  $[z, A] \neq 0$ . It follows from [5, Theorem 5.5] that the simple  $FG$ -module  $B$  has a simple  $FV$ -submodule  $C$  such that  $B = \bigoplus_{1 \leq j \leq n} x_j C$  is a direct sum of finitely many simple  $FV$ -submodules each having infinite  $F$ -dimension. Since  $[z, A] \leq B$ , [5, Corollary 4.4] implies that  $[z, A]$  is also a semisimple  $FV$ -submodule such that its simple  $FV$ -factors have infinite  $F$ -dimension. However,

$$[z, A] = \mathbf{Im}(\varphi) \cong A/\mathbf{ker}(\varphi) = A/C_A(z)$$



by the first isomorphism theorem so  $A$  has a proper  $FV$ -submodule  $Y$  such that  $A/Y$  is a simple  $FV$ -module of infinite  $F$ -dimension. Let  $\{g_1, \dots, g_k\}$  be a transversal to  $V$  in  $G$  and let  $L = g_1Y \cap \dots \cap g_kY$ , an  $FG$ -module. By Remak's theorem we obtain an embedding:

$$A/L \longrightarrow \bigoplus_{1 \leq j \leq k} A/g_jY.$$

Since  $A/g_jY = g_jA/g_jY \cong A/Y$  for each  $j$  it follows that  $A/g_jY$  is a simple  $FV$ -module of infinite  $F$ -dimension. Thus,  $A/L$  is a semisimple  $FV$ -module and each of its  $FV$ -factors has infinite  $F$ -dimension (see [5, Corollary 4.4]). Furthermore, if  $L = 0$ , then  $A$  is a semisimple  $FV$ -module each of whose  $FV$ -factors have infinite  $F$ -dimension. This implies that  $A/B$  cannot be a semisimple  $FV$ -module of finite  $F$ -dimension. Hence,  $L \neq 0$ .

Since  $B$  is a simple  $FG$ -submodule, either  $B \cap L = 0$  or  $B \leq L$ . Suppose that  $B \leq L$ . Since  $A/B$  is a simple  $FG$ -module and  $L$  is a proper  $FG$ -submodule of  $A$  we deduce that  $L = B$ . However,  $B$  has finite codimension and  $L$  has infinite codimension, a contradiction that proves that  $B \cap L = 0$ . Since  $A/B$  is a simple  $FG$ -module we have  $A = B + L = B \oplus L$ , as required.  $\square$

**Lemma 3.2** *Let  $G$  be an abelian group and let  $A$  be an  $FG$ -module. Suppose that  $B$  is an  $FG$ -submodule of  $A$  such that  $B$  and  $A/B$  are simple  $FG$ -modules,  $B$  has infinite  $F$ -dimension, and  $A/B$  has finite dimension. Then there is an  $FG$ -submodule  $L$  of  $A$  such that  $A = B \oplus L$ .*

**Proof** As in the proof of Lemma 2.3 we may assume that  $C_G(A) = 1$  and that every nonzero  $FG$ -submodule of  $A$  contains  $B$ . We shall obtain a contradiction in this case.

We may therefore suppose that every nonzero  $FG$ -submodule of  $A$  contains  $B$  so that  $B$  is the  $FG$ -monolith of  $A$ . We again consider  $A$  as a  $D(g)$ -module for an arbitrary element  $g \in G$ .

Let  $a \in A \setminus B$  and note that  $D(g)$  has infinite  $F$ -dimension. Then  $D(g)(a + B) \cong D(g)/\mathbf{Ann}_{D(g)}(a + B)$  and since  $\mathbf{dim}_F(A/B)$  is finite it follows that  $a + B$  has nonzero annihilator in the ring  $D(g)$ . Thus,  $A/B$  is  $D(g)$ -periodic and  $A/B$  has finite dimension, so it is a periodic finitely generated module over the principal ideal domain  $D(g)$ . Hence, there is an element  $y \in D(g)$  such that  $yA \leq B$ .

Suppose that  $A$  is  $D(g)$ -torsion-free. Since  $G$  is abelian, the map  $\rho : A \rightarrow A$  defined by  $\rho(a) = ya$  is an  $FG$ -endomorphism of  $A$  and  $\mathbf{Im}(\rho) = yA$  is an  $FG$ -submodule of  $A$ . Since  $A$  is  $D(g)$ -torsion-free, we have  $\mathbf{ker}(\rho) = 0$ , so  $yA = B$  since  $B$  is simple. However,

$$yA = \mathbf{Im}(\rho) \cong A/\mathbf{ker}(\rho) \cong A,$$

a contradiction since  $yA$  is simple.

This shows that  $A$  is not  $D(g)$ -torsion-free so that  $T = \mathbf{Tor}_{D(g)}(A) \neq 0$  and since  $B$  is the  $FG$ -monolith of  $A$  we have  $B \leq T$ . Since  $G$  is abelian  $T$  is an  $FG$ -submodule and since  $A/B$  is simple either  $T = B$  or  $T = A$ . However, if  $T = B$ , then we have  $yA \leq T$ , so that  $A$  is a  $D(g)$ -periodic module in this case and hence  $A = T$ . Hence, we may suppose that  $A$  is  $D(g)$ -periodic.

Then  $A = \bigoplus_{Q \in \pi} A_Q$ , where  $A_Q$  is the nonzero  $Q$ -component of  $A$  and  $\pi = \mathbf{Ass}_{D(g)}(A)$  (see [6, Corollary 3.8], for example). Each  $Q$ -component  $A_Q$  is an  $FG$ -submodule of  $A$  so  $B = A_P$  for some maximal ideal  $P$  of  $D(g)$ , since  $A$  is  $FG$ -monolithic. Also,  $\Omega_{P,1}(A)$  is a nonzero  $FG$ -submodule and if  $\Omega_{P,1}(A) \neq A$ , then  $\Omega_{P,1}(A) = B$ . Since  $\Omega_{P,2}(A)$  is an  $FG$ -submodule, it then follows that  $\Omega_{P,2}(A) = A$ . We know that  $D(g)$

is a principal ideal domain so it contains an element  $z$  such that  $P = zD(g)$  and we deduce that  $z^2a = 0$  for each element  $a \in A$ . Therefore, defining  $\rho$  in a similar manner to that above we see that  $\mathbf{Im}(\rho) = zA \leq B$ , but in this case  $\ker(\rho) = \Omega_{P,1}(A)$  so that

$$zA \cong A/\Omega_{P,1}(A).$$

Thus,  $zA$  has finite  $F$ -dimension so  $zA \neq B$ . On the other hand,  $A \neq \Omega_{P,1}(A)$  so  $zA$  is nonzero, contradicting the fact that  $B$  is simple. It therefore follows that  $\Omega_{P,1}(A) = A$  and hence  $\mathbf{Ann}_{D(g)}(A) = P$  is a maximal ideal of  $D(g)$ , which is true for all  $g \in G$ .

The last part of the proof now follows almost verbatim the last part of the proof of Lemma 2.3, so we omit it. □

We now extend this to the abelian-by-finite case.

**Lemma 3.3** *Let  $G$  be an abelian-by-finite group. Suppose that  $B$  is an  $FG$ -submodule of  $A$  such that  $B$  and  $A/B$  are simple. If  $B$  has infinite  $F$ -dimension and  $A/B$  has finite  $F$ -dimension, then  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Let  $H$  be a normal abelian subgroup of  $G$  of finite index. Then both  $B$  and  $A/B$  contain minimal  $H$ -invariant subspaces, denoted by  $C$  and  $D/B$ , respectively, such that  $B = \bigoplus_{1 \leq j \leq n} x_j C$  and  $A/B = \bigoplus_{1 \leq m \leq k} y_m D/B$  (see [5, Theorem 5.5], for example) for certain  $x_j, y_j \in A$ . We note that  $C$  has infinite  $F$ -dimension and that  $D/B$  has finite  $F$ -dimension. It follows that  $A$  has a finite series

$$0 = B_0 \leq C = B_1 \leq \dots \leq B_n = B \leq D = B_{n+1} \leq \dots \leq B_{n+k} = A$$

of  $FH$ -submodules whose factors are simple, the factors  $B_j/B_{j-1}$  have infinite dimension for  $1 \leq j \leq n$ , and the factors  $B_m/B_{m-1}$  have finite dimension for  $n+1 \leq m \leq n+k$ .

Let  $E = E_0 = B_{n-1}$  and consider  $D/E$ . Using Lemma 3.2 we see that  $D/E$  contains an  $FH$ -submodule  $E_1/E$  such that  $D/E = B/E \oplus E_1/E$ . Thus,  $E_1/E \cong_{FH} D/B$  is a simple  $FH$ -module of finite  $F$ -dimension.

Suppose, inductively, that we have constructed for some  $r \geq 0$  an  $FH$ -submodule  $E_r$  such that  $E_r/B_{n-r}$  is a simple  $FH$ -module of finite  $F$ -dimension. Then, using Lemma 3.2, we see that  $E_r/B_{n-r-1}$  contains an  $FH$ -submodule  $E_{r+1}/B_{n-r-1}$  such that  $E_r/B_{n-r-1} = E_{r+1}/B_{n-r-1} \oplus B_{n-r}/B_{n-r-1}$  so  $E_{r+1}/B_{n-r-1}$  is simple of finite  $F$ -dimension. We obtain a nonzero  $FH$ -submodule  $E_k$ , which has finite  $F$ -dimension. Let  $\{g_1, \dots, g_t\}$  be a transversal to  $H$  in  $G$ . Then  $E_k \cong_F g_j E_k$  for each  $j$  with  $1 \leq j \leq t$  and hence  $g_j E_k$  is finite-dimensional. Hence,  $L = \sum_{1 \leq j \leq t} g_j E_k$  is a finite-dimensional  $FG$ -module. Since  $E_k \neq 0$ , we have  $L \neq 0$ . However,  $B$  is simple and has infinite  $F$ -dimension, so  $L \cap B = 0$ . Since  $A/B$  is a simple  $FG$ -module,  $A/B = (L+B)/B$ , so  $A = L+B = L \oplus B$  as required. □

**Proposition 3.4** *Let  $G$  be an  $FC$ -hypercentral group and let  $A$  be an  $FG$ -module. Suppose that  $B$  is an  $FG$ -submodule of  $A$  such that  $B$  and  $A/B$  are simple  $FG$ -modules. If  $B$  has infinite  $F$ -dimension and  $A/B$  has finite  $F$ -dimension, then  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Without loss of generality we may assume that  $C_G(A)$  is trivial. Suppose first that  $C_G(A/B) \neq 0$ . Then  $C_G(A/B) \cap \mathbf{FC}(G) \neq 0$  also (see [5, Corollary 3.16], for example) and the result follows from Lemma 3.1.

Suppose that  $C_G(A/B) = 1$ . In this case we may think of  $G$  as a subgroup of  $GL_n(F)$ , where  $n = \dim_F(A/B)$ . Using Lemma 2.2 we deduce that  $G$  is abelian-by-finite and then the result follows from Lemma 3.3.  $\square$

**Corollary 3.5** *Let  $G$  be an FC-hypercentral group and let  $A$  be an FG-module. Suppose that  $B$  is a simple FG-submodule of  $A$  of infinite  $F$ -dimension such that  $A/B$  has finite  $F$ -dimension. Then  $A$  contains an FG-submodule  $C$  such that  $A = B \oplus C$ .*

**Proof** Since  $B$  has finite codimension over  $F$  it follows that  $A$  has a finite series

$$0 = B_0 \leq B_1 = B \leq B_2 \leq \dots \leq B_{n+1} = A$$

whose factors are simple FG-modules and the factors  $B_j/B_{j-1}$  have finite  $F$ -dimension for  $2 \leq j \leq n + 1$ . We use induction on  $n$ . If  $n = 1$ , the result follows from Proposition 3.4.

Suppose that  $n > 1$  and that the result is true for  $B_n$ . Thus,  $B_n$  contains an FG-submodule  $D$  such that  $B_n = B \oplus D$ . Since  $D \cong B_n/B$  it follows that  $D$  has finite  $F$ -dimension. The simple FG-module  $B_n/D \cong B$  has infinite dimension and  $A/B_n$  is a simple FG-module of finite  $F$ -dimension. By Proposition 3.4 we deduce that  $A/D$  contains an FG-submodule  $C/D$  of finite  $F$ -dimension such that  $A/D = B_n/D \oplus C/D$ . We have

$$A = B_n + C = (B + D) + C = B + C.$$

Since  $C$  has finite dimension and  $B$  is a simple FG-submodule,  $C \cap B = 0$ , so  $A = B \oplus C$ .  $\square$

We note that just as with Corollary 2.6, Corollary 3.5 is no longer true for arbitrary groups  $G$ , even for soluble groups; a counterexample appears in the paper [14] of Zaitsev.

**Proof of Theorem A**

Let

$$0 = A_0 \leq A_1 \leq \dots \leq A_{n-1} \leq A_n = A$$

be an FG-composition series of  $A$ . We remark that the number  $n$  is an invariant of the module  $A$ .

We use induction on the length  $n$  of the FG-composition series of  $A$ . If  $n = 1$ , then there is nothing to prove. Suppose that  $n > 1$  and that our statement has been proved for modules having a composition series of length strictly less than  $n$ . We apply the induction hypothesis to  $A/A_1$  so  $A/A_1 = D/A_1 \oplus E/A_1$  where  $D, E$  are FG-submodules of  $A$ , each FG-composition factor of  $D/A_1$  has finite  $F$ -dimension, and each FG-composition factor of  $E/A_1$  has infinite  $F$ -dimension.

Suppose first that  $A_1$  has infinite  $F$ -dimension. If  $E = A_1$ , the result follows from Corollary 3.5. Therefore, suppose that  $E/A_1 \neq 0$ . Then the FG-composition length of  $D$  is less than the FG-composition length of  $A$ . By induction hypothesis  $D$  contains an FG-submodule  $B$  such that  $D = B \oplus A_1$  and every FG-composition factor of  $B$  has finite dimension. We have

$$A = D + E = B + A_1 + E = B + E.$$

The choice of  $B$  shows that

$$B \cap E \leq B \cap D = A_1, \text{ so that } B \cap E \leq B \cap A_1 = 0,$$

which shows that  $A = B \oplus E$ .

Suppose now that  $A_1$  has finite  $F$ -dimension. If  $E = A_1$ , then each  $FG$ -composition factor of  $A$  has finite  $F$ -dimension and the result follows. Therefore, suppose that  $E/A_1 \neq 0$ . Then  $A/A_1$  contains a simple  $FG$ -submodule  $S/A_1$  of infinite  $F$ -dimension. By Proposition 2.5  $S$  contains an  $FG$ -submodule  $Y$  such that  $S = A_1 \oplus Y$ . In particular,  $A$  contains a simple  $FG$ -submodule of infinite  $F$ -dimension. The  $FG$ -composition length of  $A/Y$  is less than the  $FG$ -composition length of  $A$ , a situation that was considered above. The result follows.  $\square$

### Proof of Corollary A

Suppose that  $A$  contains a nonzero proper  $FG$ -submodule, but  $A$  is not  $FG$ -quasifinite. Then  $A$  contains a nonzero proper  $FG$ -submodule  $B$  such that  $A/B$  is a simple  $FG$ -module. Being proper,  $B$  has finite  $F$ -dimension, but then Corollary 2.6 shows that  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ . Since  $B \neq 0$  it follows that  $C$  is a proper  $FG$ -submodule and we obtain a contradiction, because  $C$  has infinite  $F$ -dimension. This contradiction proves the result.  $\square$

### Proof of Corollary B

Suppose that  $A$  is a monolithic  $FG$ -module and let  $B$  be its  $FG$ -monolith. Then  $B$  is a nonzero simple  $FG$ -submodule of finite codimension over  $F$ . Since  $A$  has infinite dimension,  $\mathbf{dim}_F(B)$  is infinite. Then Corollary 3.5 shows that  $A$  contains an  $FG$ -submodule  $C$  such that  $A = B \oplus C$ . In particular,  $C$  has finite  $F$ -dimension and we obtain a contradiction since  $C$  is nonzero. This contradiction proves the result.  $\square$

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