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The intrinsic metric and geodesics on the Sierpinski gasket $SG(3)$

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Abstract: We give an explicit expression for the intrinsic metric on the Sierpinski gasket $SG(3)$ (the mod-3 Sierpinski gasket) via code representation of its points. We also investigate the geodesics of $SG(3)$ and determine the number of geodesics between two points.

Key words: Sierpinski gasket, mod-3 Pascal–Sierpinski gasket, intrinsic metric, geodesic

1. Introduction

The Sierpinski gasket is one of the classical examples in fractal geometry. This set can be considered as the attractor of an iterated function system (IFS) consisting of three similitudes with scaling ratios $1/2$ (see [2] for the notion of IFS). The family of Sierpinski gaskets $\{SG(n) \mid 1 < n \in \mathbb{N}\}$ can be considered as an important generalization of the classical Sierpinski gasket. Mathematicians work on the elements of this family (and on the so-called irregular Sierpinski gaskets generated by these fractals) especially in the fields of Brownian motion, random walk, graph theory, and stochastic process etc. (see [1, 3, 5]).

First we give a small brief for this family of Sierpinski gaskets. Start with a equilateral triangle S_0. Divide S_0 into four smaller equilateral triangles using the midpoints of the edges of S_0. Removing the middle triangle we get S_1 (see Figure 1b) and repeat this procedure on each remaining equilateral triangle to obtain S_2. Continuing this procedure, we obtain a nested sequence of sets $S_0 \supset S_1 \supset \cdots \supset S_i \supset \cdots$. The (classical) Sierpinski gasket is the intersection of these sets.

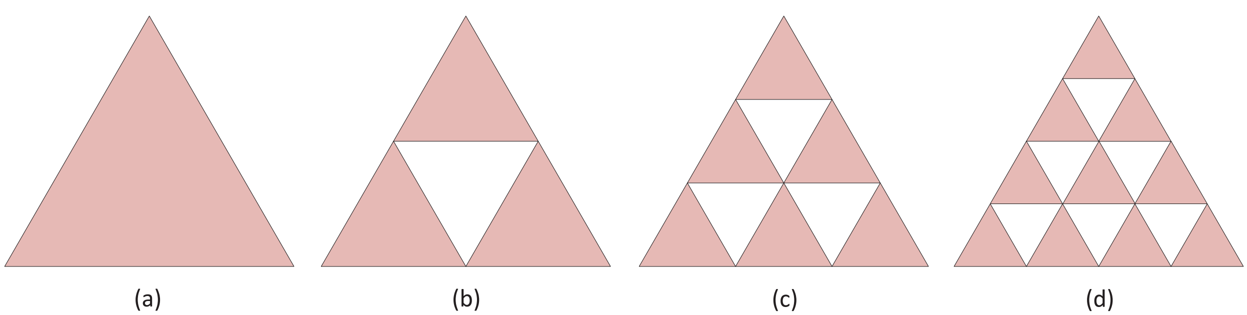

Figure 1. (a) The equilateral triangle S_0, the first stage of the construction of (b) $SG(2)$, (c) $SG(3)$, and (d) $SG(4)$.

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Apply this procedure by dividing the edges of the equilateral triangle into n equal-length parts to obtain $n(n+1)/2$ equilateral triangles as mentioned in Figures 1c and 1d for $n=3$ and $n=4$, respectively. At the end of the same procedure, the intersection of the related nested sets is called the Sierpinski gasket $SG(n)$. For $n=2$, the classical Sierpinski gasket is obtained and represented by $SG(2)$. $SG(2)$, $SG(3)$, and $SG(4)$ are shown in Figure 2.

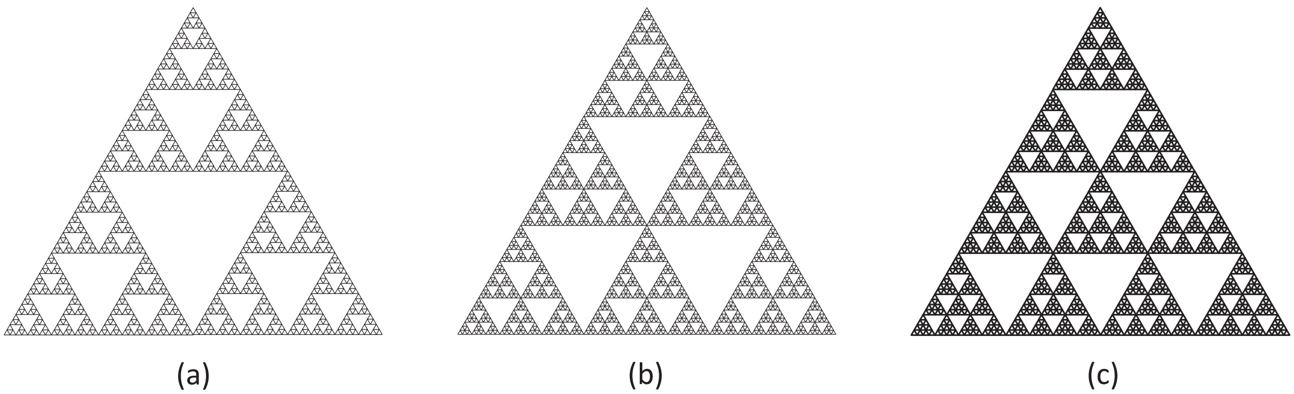

Figure 2. The Sierpinski gaskets (a) $SG(2)$, (b) $SG(3)$, and (c) $SG(4)$.

In [12], for a prime number p, the authors define the mod-p Sierpinski gasket or the Pascal–Sierpinski gasket (with the inspiration of the Pascal triangle and the divisibility of the numbers in this triangle by p) which coincides with $SG(p)$ introduced above. For example, the mod-2 Sierpinski gasket is $SG(2)$ and the mod-3 Sierpinski gasket is $SG(3)$.

In this work, we investigate the intrinsic metric and geodesics on the Sierpinski gasket $SG(3)$. The intrinsic metric on a set A which is obtained by taking into account the paths on the structure can be defined as

$$d(x,y)=\inf\{\delta \mid \delta \text{ is the length of a rectifiable curve in A joining } x \text{ and } y\}$$

for $x,y\in A$ (for details see [4]).

In several works, the intrinsic metric on the self-similar sets such as classical Sierpinski gasket, Vicsek fractal, and Sierpinski carpet was constructed and defined by using different techniques (see [7–11]). Strichartz defines the intrinsic metric via barycentric coordinates (for details see [17]). In [14], Romik gives an expression of the intrinsic metric on the discrete Sierpinski gasket. In [6], Cristea gives a formula of the intrinsic metric on the Sierpinski carpet by using carpet coordinates and show the equivalence of the Euclidean metric and the intrinsic metric on the Sierpinski carpet. In [13, 15, 16], the authors use code representations of the points of the classical Sierpinski gasket and Vicsek fractal to express the intrinsic metric and classified the geodesics on the related set. They prove that there exist at most 5 geodesics between two points in the Sierpinski gasket $SG(2)$. In [9], the authors investigate the geodesics on the m-dimensional (classical) Sierpinski gasket (for $m>2$) and prove that there exist at most 8 geodesics between two points.

In Section 2, we give a formula for the intrinsic metric of the Sierpinski gasket $SG(3)$ (using code representations of the points) and use this formula to prove some geometric results. In Section 3, we investigate the geodesics of $SG(3)$ with respect to the intrinsic metric, and contrary to the classical Sierpinski gasket $SG(2)$ case, we prove that the number of geodesics between two points in $SG(3)$ can be infinitely different numbers.

2. Code representations and the intrinsic metric

2.1. Code representations of the points of $SG(3)$

$SG(3)$ is the union of its six similitude copies (with similarity ratios $1/3$) as seen in Figure 3. It can be obtained also as the attractor of the iterated function system $\{\mathbb{R}^2; f_1, f_2, f_3, f_4, f_5, f_6\}$ where

$$\begin{array}{lll} f_1(x,y) = \left(\frac{1}{3}x, \frac{1}{3}y\right) & f_2(x,y) = \left(\frac{1}{3}x + \frac{2}{3}, \frac{1}{3}y\right) & f_3(x,y) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y + \frac{\sqrt{3}}{3}\right) \\ f_4(x,y) = \left(\frac{1}{3}x + \frac{1}{2}, \frac{1}{3}y + \frac{\sqrt{3}}{6}\right) & f_5(x,y) = \left(\frac{1}{3}x + \frac{1}{6}, \frac{1}{3}y + \frac{\sqrt{3}}{6}\right) & f_6(x,y) = \left(\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}y\right) \end{array}.$$

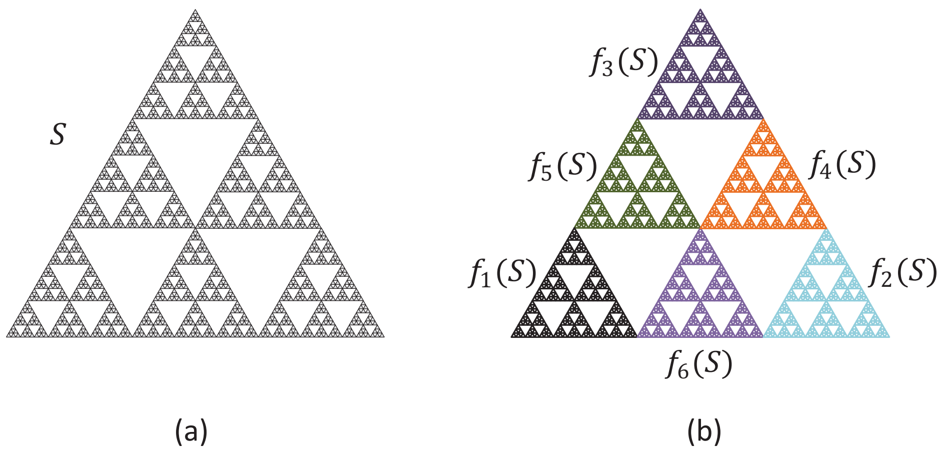

Figure 3. (a) $SG(3)$ and (b) the attractor $S = SG(3)$ as the union of its six similitude copies.

From now on we will use the notation S instead of $SG(3)$ for abbreviation.

Let S_1, S_2, S_3, S_4, S_5, and S_6 be the just-touching parts of S as indicated in Figure 4. Note that $S_i = f_i(S)$ for all $i = 1, 2, \ldots, 6$ and $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \cup S_6$. For a word $\sigma = \sigma_1\sigma_2 \cdots \sigma_k \in \{1, 2, \ldots, 6\}^k$ with length k, let $S_\sigma := f_\sigma(S)$ where $f_\sigma = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}$. We call S_σ as the subgasket of level k (with the code σ, see Figure 4 for examples). We set $\sigma = \emptyset$ if $k = 0$, and $S_\sigma = S$ (which is the unique subgasket of level 0). We give some examples in Figure 5 for codes of some subgaskets of level 1 and level 2.

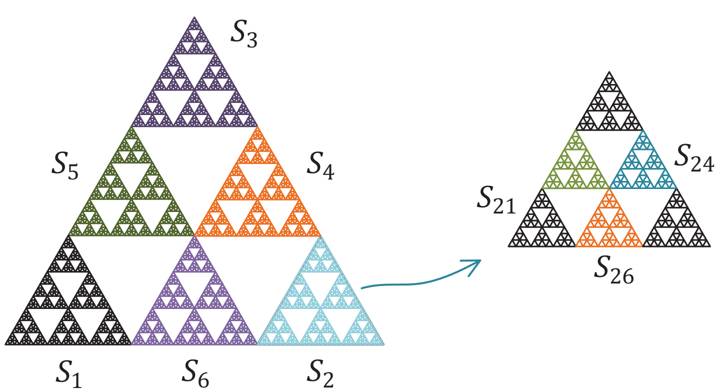

Figure 4. Some subgaskets of level 1 and level 2.

Let $p_1p_2 \ldots p_{k-1}p_kp_{k+1} \ldots$ be a representation of a point $p \in S$. It is obvious that

$$S_{p_1} \supset S_{p_1p_2} \supset S_{p_1p_2p_3} \supset \cdots \supset S_{p_1p_2 \ldots p_k} \supset \cdots$$

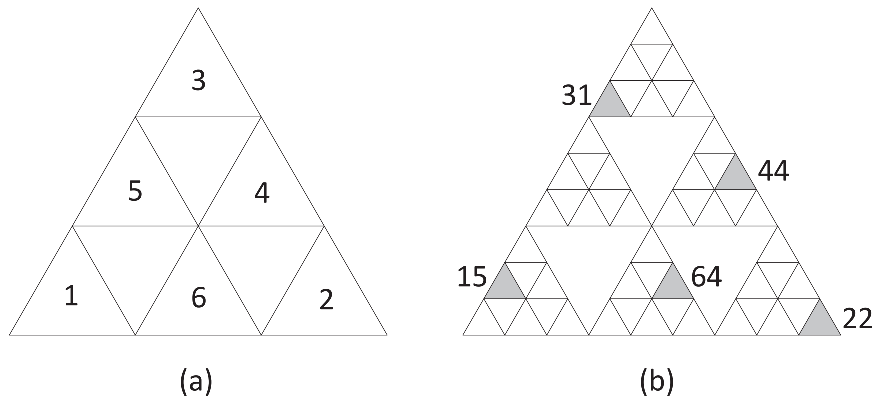

Figure 5. Codes of the subgaskets of level 1 (a) and codes of some subgaskets of level 2 (b).

for $p_i \in \{1,2,\ldots,6\}$ (for all $i>0$), and by the Cantor Intersection Theorem, the infinite intersection

$$\bigcap_{k=1}^{\infty} S_{p_1 p_2 \ldots p_k}$$

is a singleton, say $\{p\}$ where $p \in S$. We call the sequence $p_1 p_2 \ldots p_k \ldots$ as a code representation of the point p. Note that, if $p \in S$ is the intersection point of any two subgaskets of S_σ for some σ word with length $k>0$ (such a "vertex" point is called a junction point of S) then p has two or three different representations.

Let p be the unique point of the singleton $S_{\sigma x} \cap S_{\sigma y}$ where $x \neq y$. If $x, y \in \{4,5,6\}$ then p has three different representations and otherwise p has two different representations. For example, if p is the unique point of the singleton $S_{\sigma 4} \cap S_{\sigma 5}$ then $\sigma 4\overline{1} = \sigma 4111\ldots$, $\sigma 5\overline{2} = \sigma 5222\ldots$ and $\sigma 6\overline{3} = \sigma 6333\ldots$ are three different representations of the point p (where $\overline{x}$ stands for infinite repetition throughout the paper). For another example, if p is the unique point of the singleton $S_{\sigma 1} \cap S_{\sigma 5}$ then $\sigma 1\overline{3} = \sigma 1333\ldots$ and $\sigma 5\overline{1} = \sigma 5111\ldots$ are two different representations of the point p. We use (for example) $3_{\sigma 1}$ or $1_{\sigma 5}$ for this point. More generally, let 1_σ, 2_σ, and 3_σ be the vertices of the convex hull of the subgasket S_σ as indicated in Figure 6a. Note that $\sigma\overline{1}$, $\sigma\overline{2}$, and $\sigma\overline{3}$ are the code representations of these vertex points, respectively.

2.2. The intrinsic metric on $SG(3)$

Let $x \in \{1,2,3,4,5,6\}$. If the number $y \in \{1,2,3,4,5,6\}$ satisfies $|x-y| = 3$ then we call y as the conjugate of x and denote it by $\widetilde{x}$. More clearly, $\widetilde{1} = 4, \widetilde{2} = 5$, $\widetilde{3} = 6, \widetilde{4} = 1$, $\widetilde{5} = 2$, and $\widetilde{6} = 3$.

For a given number $x \in \{1,2,3\}$, let x' and x'' denote the numbers that satisfy $x' < x''$, $x + x' + x'' = 6$ and $x', x'' \in \{1,2,3\}$.

Let $p_1 p_2 \ldots p_{k-1} p_k p_{k+1} \ldots$ be a code representation of a point $p \in S = SG(3)$ and let $\sigma = p_1 p_2 \ldots p_k$. The length of a shortest path between $p \in S_\sigma$ and the vertex t_σ of S_σ (for all $t \in \{1,2,3\}$) is the sum of the lengths of the shortest paths between all pairs of vertices $t_{p_\sigma p_{k+1} \cdots p_{i-1}} = t_{p_1 p_2 \cdots p_{i-1}}$ and $t_{p_\sigma p_{k+1} \cdots p_i} = t_{p_1 p_2 \cdots p_i}$ for all $i \geq k+1$ (see Figure 6b). Note that $t_{p_1 p_2 \cdots p_{i-1}}$ and $t_{p_1 p_2 \cdots p_i}$ can be considered as two vertices of two subgaskets of (same) level i since $t_{p_1 p_2 \cdots p_{i-1}} = t_{p_1 p_2 \cdots p_{i-1} t}$. Therefore, the length of a shortest path a geodesic between these vertices can be 0, $1/3^i$ or $2/3^i$ since the length of the edges of the convex hull of a subgasket of level i is $1/3^i$. If $p_i = t$ then these vertices would be the same point, so the length of a geodesic between

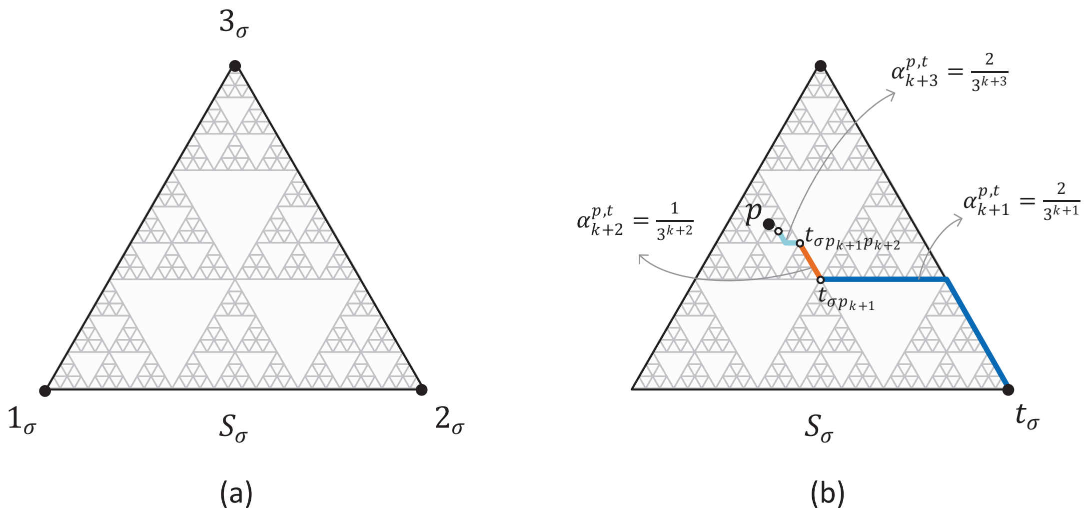

Figure 6. Vertices of a subgasket S_σ (a) and shortest paths between some vertices (for $t = 2$, $p_{k+1} = 5, p_{k+2} = 4$ and $p_{k+3} = 5$) (b).

them would be 0. If $p_i > 3$ and $p_i \neq \widetilde{t}$ then these vertices would be the vertices of the same subgasket of level i, so the length of a geodesic between them would be $1/3^i$. Otherwise, these vertices cannot lie in the same subgasket; thus, the length of a geodesic between them would be $2/3^i$.

As a result, the length of a shortest path (a geodesic) between p and the vertex t_σ of the subgasket S_σ of level k that contains p can be expressed as

$$d^{\sigma}_{p,t} = \sum_{i=k+1}^{\infty} \frac{\alpha_i^{p,t}}{3^i} \tag{2.1}$$

for $t \in \{1, 2, 3\}$ where

$$\alpha_i^{p,t} = \begin{cases} 0 & , \quad p_i = t \\ 1 & , \quad p_i > 3, p_i \neq \widetilde{t} \\ 2 & , \quad \text{otherwise} \end{cases} .$$

Theorem 2.1 *Let $a_1a_2\ldots a_k\ldots$ and $b_1b_2\ldots b_k\ldots$ be two representations of the points $a, b \in SG(3)$ respectively such that $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$ (we assume $a_k < b_k$ for simplicity). The length of a shortest path (or geodesic distance) between a and b can be expressed as*

$$d(a,b) = \begin{cases} \frac{1}{3^k} + \min\{d^{\mu}_{a,b_k} + d^{\nu}_{b,a_k}\,,\, d^{\mu}_{a,c_k} + d^{\nu}_{b,c_k} + \frac{1}{3^k}\} & , \quad a_k, b_k \in \{1,2,3\} \\ \min\{d^{\mu}_{a,\widetilde{a}_k} + d^{\nu}_{b,\widetilde{b}_k}\,,\, d^{\mu}_{a,e_k} + d^{\nu}_{b,e_k} + \frac{1}{3^k}\} & , \quad a_k, b_k \in \{4,5,6\} \\ \frac{1}{3^k} + d^{\nu}_{b,a_k} + \min\{d^{\mu}_{a,a'_k}\,,\, d^{\mu}_{a,a''_k}\} & , \quad a_k = \widetilde{b_k} \\ \min\{d^{\mu}_{a,9-a_k-b_k} + d^{\nu}_{b,a_k}\,,\, d^{\mu}_{a,\widetilde{b_k}} + d^{\nu}_{b,\widetilde{b_k}} + \frac{1}{3^k}\} & , \quad \textit{otherwise} \end{cases} \tag{2.2}$$

where $c_k = 6 - a_k - b_k$, $e_k = 6 - \widetilde{a}_k - \widetilde{b_k}$, $\mu = a_1a_2\ldots a_{k-1}a_k$ and $\nu = a_1a_2\ldots a_{k-1}b_k$.

Proof Let $a_1a_2\dots a_k\dots$ and $b_1b_2\dots b_k\dots$ be two representations respectively of the points $a,b\in S=SG(3)$ such that $a_i=b_i$ for $i=1,2,\dots,k-1$ and $a_k<b_k$, i.e. $a,b\in S_\sigma$ for $\sigma=a_1a_2\dots a_{k-1}$ and $a\in S_\mu, b\in S_\nu$ where $\mu=a_1a_2\dots a_{k-1}a_k=\sigma a_k$, $\nu=a_1a_2\dots a_{k-1}b_k=\sigma b_k$ (see Figure 7).

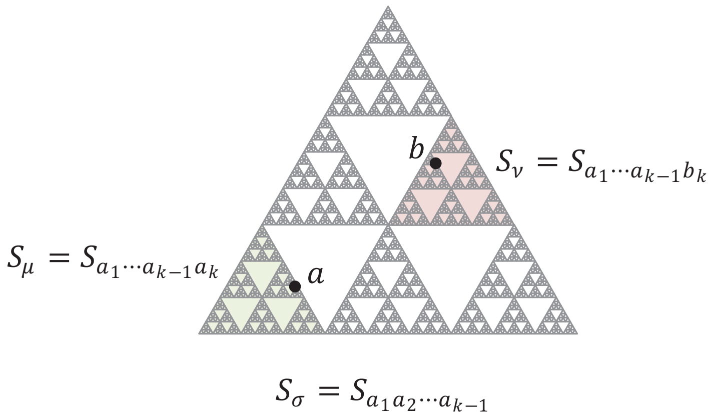

Figure 7. The subgaskets S_μ and S_ν of level k that contain the points a and b respectively.

Case I ($a_k,b_k\in\{1,2,3\}$): Let $a_k=1$ and $b_k=2$. As shown in Figure 8a, the shortest paths between a and b must pass through either the line $3_{\sigma1}3_{\sigma2}=3_\mu3_\nu$ or the line $2_{\sigma1}1_{\sigma2}=2_\mu1_\nu$ (in this special case, we have $\mu=\sigma1$ and $\nu=\sigma2$). First, consider the shortest path passing through the line $3_{\sigma1}3_{\sigma2}$ whose length is $2/3^k$. To calculate the length of this path, we also need to calculate the shortest distance between the points "a and $3_{\sigma1}$" and distance between the points "$3_{\sigma2}$ and b" which are $d^\mu_{a,c_k}=d^\mu_{a,3}$ and $d^\nu_{b,c_k}=d^\nu_{b,3}$ respectively (note that $c_k=3$ for this case). Thus, the length of the related shortest path is $d^\mu_{a,c_k}+d^\nu_{b,c_k}+\frac{2}{3^k}$. Now consider the shortest path passing through the line $2_{\sigma1}1_{\sigma2}$ whose length is $1/3^k$. To calculate the length of this path, we need to calculate the shortest distance between "a and $2_{\sigma1}$" and distance between "$1_{\sigma2}$ and b" which are $d^\mu_{a,b_k}=d^\mu_{a,2}$ and $d^\nu_{b,a_k}=d^\nu_{b,1}$ respectively. Thus, the length of the related shortest path is $d^\mu_{a,b_k}+d^\nu_{b,a_k}+\frac{1}{3^k}$. Minimum of the lengths of these two paths would be the shortest distance between a and b (for the cases $a_k=1,b_k=3$ and $a_k=2,b_k=3$ the formula can be obtained using similar argument).

Case II ($a_k,b_k\in\{4,5,6\}$): Let $a_k=5$ and $b_k=6$ ($\mu=\sigma5$, $\nu=\sigma6$). As shown in Figure 8b, the shortest paths between a and b must pass through either the points $2_{\sigma5}(=3_{\sigma6})$ or the line $1_{\sigma5}1_{\sigma6}$. It is obvious that to calculate the length of the shortest path passing through the point $3_{\sigma6}$ we need to calculate the shortest distance between the points "a and $2_{\sigma5}$" where 2 is the conjugate of $a_k=5$ ($\widetilde{5}=2$), and the shortest distance between the points "b and $3_{\sigma6}$" where 3 is the conjugate of $b_k=6$ ($\widetilde{6}=3$). Thus, the length of the related shortest path is $d^\mu_{a,\widetilde{a}_k}+d^\nu_{b,\widetilde{b_k}}=d^\mu_{a,2}+d^\nu_{b,3}$.

Now consider the shortest path passing through the line $1_{\sigma5}1_{\sigma6}$ whose length is $1/3^k$. To calculate the length of this path, we need to calculate the shortest distance between "a and $1_{\sigma5}$" and distance between "$1_{\sigma6}$ and b" which are $d^\mu_{a,e_k}=d^\mu_{a,1}$ and $d^\nu_{b,e_k}=d^\nu_{b,1}$ respectively. Thus, the length of the related shortest path is $d^\mu_{a,e_k}+d^\nu_{b,e_k}+\frac{1}{3^k}$. Minimum of the lengths of these two paths would be the shortest distance between a and b (for the cases $a_k=4,b_k=5$, and $a_k=4,b_k=6$ the formula can be obtained using similar argument).

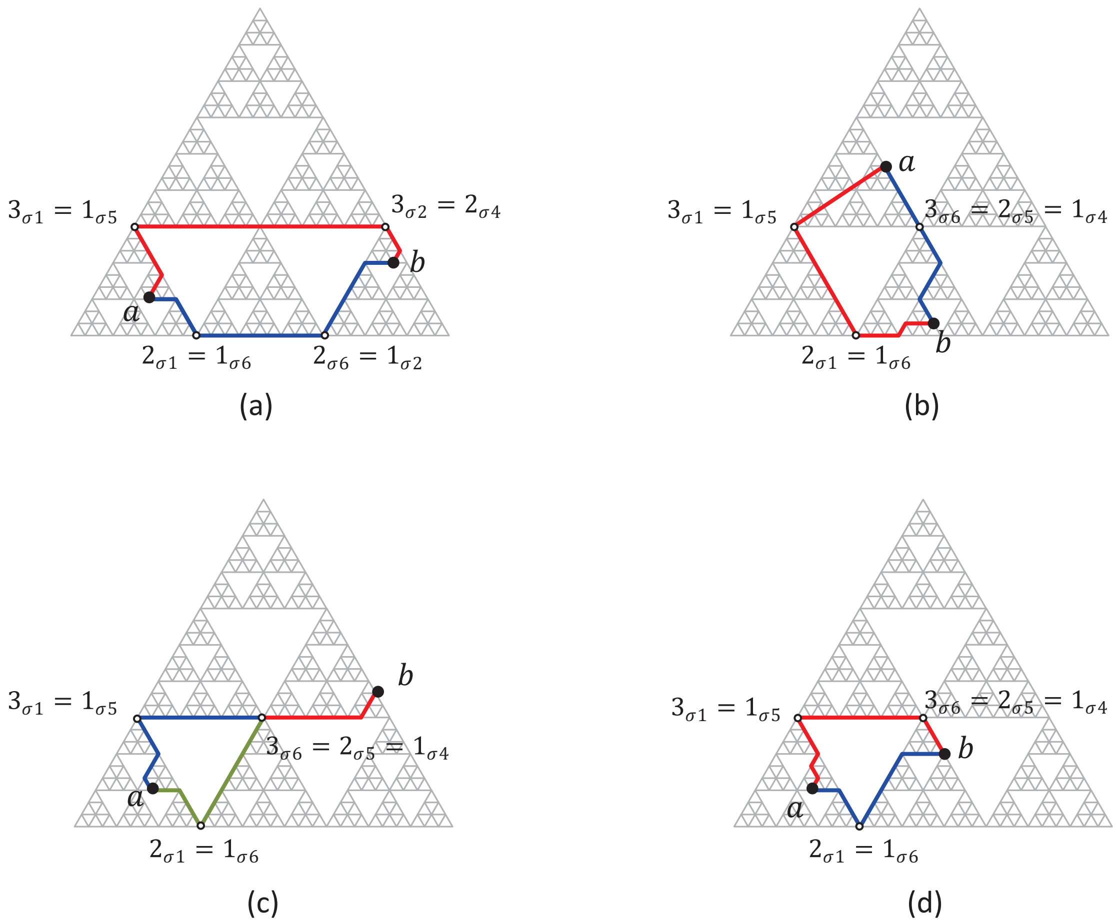

Figure 8. Possible shortest paths (geodesics) between a and b where (a) $a_k = 1,\ b_k = 2$, (b) $a_k = 5,\ b_k = 6$, (c) $a_k = 1,\ b_k = 4$ and (d) $a_k = 1,\ b_k = 6$.

Case III ($a_k = \widetilde{b_k}$): Let $a_k = 1$ and $b_k = 4$ ($\mu = \sigma 1,\ \nu = \sigma 4$). As shown in Figure 8c, the shortest paths between a and b must pass through the points $1_{\sigma 4} (= 3_{\sigma 6} = 2_{\sigma 5})$. The distance between the points b and $1_{\sigma 4}$ is $d^{\nu}_{b,a_k} = d^{\nu}_{b,1}$ (note that $a_k = \widetilde{b_k}$). The shortest paths between the points a and $1_{\sigma 4}$ must pass through either the line $2_{\sigma 1} 1_{\sigma 4}$ or the line $3_{\sigma 1} 1_{\sigma 4}$ whose length is $1/3^k$. Thus, there exits two different paths between a and $1_{\sigma 4}$ with lengths $1/3^k + d^{\mu}_{a,a'_k} = 1/3^k + d^{\mu}_{a,2}$ and $1/3^k + d^{\mu}_{a,a''_k} = 1/3^k + d^{\mu}_{a,3}$ respectively (note that $a'_k = 2$ and $a''_k = 3$ since $a_k = 1$). For the cases $a_k = 2, b_k = 5$, and $a_k = 3, b_k = 6$ the formula can be obtained using similar argument.

Case IV (otherwise): In this case, a and b lie in the adjacent subgaskets of level k. For example, let $a_k = 1$ and $b_k = 6$ ($\mu = \sigma 1,\ \nu = \sigma 6$). As shown in Figure 8d, the shortest path between a and b must pass through either the points $2_{\sigma 1} (= 1_{\sigma 6})$ or the line $3_{\sigma 1} 3_{\sigma 6}$. The length of the shortest path passing through the point $2_{\sigma 1} (= 1_{\sigma 6})$ is $d^{\mu}_{a,9-a_k-b_k} + d^{\nu}_{b,a_k} = d^{\mu}_{a,2} + d^{\nu}_{b,1}$ as the sum of the shortest distance between the points "a and $2_{\sigma 1} (= 1_{\sigma 6})$" and the shortest distance between the points "b and $1_{\sigma 6} (= 2_{\sigma 1})$" (notice that $2 = 9 - a_k - b_k$). To calculate the length of the shortest path passing through the line $3_{\sigma 1} 3_{\sigma 6}$ we need to sum the geodesic distance between "a and $3_{\sigma 1}$", the geodesic distance between "b and $3_{\sigma 6}$" and $1/3^k$ which give us $d^{\mu}_{a,\widetilde{b_k}} + d^{\nu}_{b,\widetilde{b_k}} + \frac{1}{3^k} = d^{\mu}_{a,3} + d^{\nu}_{b,3} + \frac{1}{3^k}$. For the cases "$a_k = 1, b_k = 5$", "$a_k = 2, b_k = 6$", "$a_k = 2, b_k = 4$", "$a_k = 3, b_k = 4$", "$a_k = 3, b_k = 5$" the formula can be obtained using similar argument.

□

In Example 2.2 below, we compute the geodesic distance between two points which will be used in the proof of Theorem 3.4 in the next section.

Example 2.2 *Consider the points $a, b \in S$ such that one of the code representations of a is $1\mu_1\mu_2\overline{1} = 155\cdots566\cdots6\overline{1}$ where $\mu_1 = 55\cdots5$ and $\mu_2 = 66\cdots6$ are two words with length $n \in \mathbb{N}$ and $m \in \mathbb{N}$ respectively, and one of the code representations of b is $1222\cdots = 1\overline{2}$. Note that the point b is the vertex point 2_1 which is the right vertex of the subgasket S_1 (of level 1). Using Equation (2.1) we obtain*

$$\begin{aligned} d^1_{a,2} &= \sum_{i=2}^{\infty} \frac{\alpha_i^{a,2}}{3^i} \\ &= \frac{2}{3^2} + \cdots + \frac{2}{3^{n+1}} + \frac{1}{3^{n+2}} + \cdots + \frac{1}{3^{n+m+1}} + \frac{2}{3^{n+m+2}} + \frac{2}{3^{n+m+3}} + \cdots \\ &= \frac{2}{3^2}\frac{1}{1-1/3} - \frac{1}{3^{n+2}}\left(1 + \frac{1}{3} + \cdots + \frac{1}{3^{m-1}}\right) \\ &= \frac{1}{3} - \frac{1}{2\cdot 3^{n+1}} + \frac{1}{2\cdot 3^{n+m+1}}. \end{aligned}$$

Since b is a vertex point (of S_1), we get immediately $d(a,b) = d^1_{a,2}$ without using the formula (2.2). In case $n = 2$ and $m = 3$, we obtain $\frac{230}{729}$ as the geodesic distance between a and b whose code representations are $155666\overline{1}$ and $1\overline{2}$ respectively.

Remark 2.3 *One can verify easily that the formula of the intrinsic metric (2.2) does not depend on the code representations of the points.*

Proposition 2.4 *The distance function d defined in Theorem 2.1 is a strictly intrinsic metric on S.*

Proof From the construction, the claim is obvious from the fact that d is defined as the minimum of the lengths of the geodesics. □

Lemma 2.5 *Let $a \in S_\sigma$ where S_σ is the subgasket of level k. The sum of the geodesic distances between a and the vertices of the subgasket S_σ is $2/3^k$:*

$$d(a, 1_\sigma) + d(a, 2_\sigma) + d(a, 3_\sigma) = \frac{2}{3^k}.$$

Proof Remember that the length of the shortest path between a and the vertex t_σ of the subgasket S_σ of level k contains a can be computed by Equation (2.1). Using the formula we get

$$d(a, 1_\sigma) + d(a, 2_\sigma) + d(a, 3_\sigma) \quad = \quad d^{\sigma}_{a,1} + d^{\sigma}_{a,2} + d^{\sigma}_{a,3} = \sum_{i=k+1}^{\infty} \frac{\alpha_i^{a,1} + \alpha_i^{a,2} + \alpha_i^{a,3}}{3^i}.$$

Since

$$\alpha_i^{a,1} = \begin{cases} 0 & , & a_i = 1 \\ 1 & , & a_i > 3, a_i \neq 4 \\ 2 & , & \text{otherwise} \end{cases} , \quad \alpha_i^{a,2} = \begin{cases} 0 & , & a_i = 2 \\ 1 & , & a_i > 3, a_i \neq 5 \\ 2 & , & \text{otherwise} \end{cases} , \quad \alpha_i^{a,3} = \begin{cases} 0 & , & a_i = 3 \\ 1 & , & a_i > 3, a_i \neq 6 \\ 2 & , & \text{otherwise} \end{cases} ,$$

we obtain $\alpha_i^{a,1} + \alpha_i^{a,2} + \alpha_i^{a,3} = 4$ for $i \geq k+1$ which implies

$$d(a, 1_\sigma) + d(a, 2_\sigma) + d(a, 3_\sigma) = \sum_{i=k+1}^{\infty} \frac{4}{3^i} = \frac{2}{3^k}.$$

□

3. Geodesics of the Sierpinski gasket $SG(3)$

In the classical Sierpinski gasket ($SG(2)$) case, it is shown that the number of geodesics between two points can be 1, 2, 3, 4, or at most 5 (see [16]). In [9], the authors generalized the previous result to the higher dimensional case and proved that the number of geodesics between two points can be 1, 2, 3, 4, 5, 6, or at most 8 on the (n-dimensional) Sierpinski gasket (on the Sierpinski Tetrahedron for example). However, on the Sierpinski gasket $SG(3)$, we prove that the number of geodesics between two points can be infinitely many (see Figure 9 for examples). Lemma 3.1 and Theorem 3.4 give an idea about how many geodesics can be between two different points. (We use the notation $\mathbb{N}^*$ for the set $\mathbb{N} \cup \{0\}$ throughout the paper.)

Lemma 3.1 *Let $a \in S_\sigma$ where S_σ is a subgasket of level k and let $t \in \{1, 2, 3\}$. Then the number of geodesics between a and $t_\sigma \neq a$ is either one of the numbers 2^n, $3 \cdot 2^n$ for some $n \in \mathbb{N}^*$ or ∞. Moreover, for all $n \in \mathbb{N}^*$ there exists $a \in S_\sigma$ for some σ such that the number of geodesics between a and t_σ is exactly 2^n or $3 \cdot 2^n$.*

Proof Using the symmetry properties, without loss of generality, we may assume that $t = 1$. Let $a \in S_\sigma$ where S_σ is a subgasket of level k and let $a_1 a_2 \ldots a_k \ldots$ be one of the code representations of a. Consider all geodesics starting at the vertex 1_σ and ending at the point a (see Figure 10 for examples).

Assume that a is a vertex point of the subgasket $S_\mu \subset S_\sigma$ where μ is a word with length $m = k + l$. In this case the number of geodesics would be finite and a geodesic starting at the vertex 1_σ and ending at the point a must pass through the vertex points $1_\sigma, 1_{\sigma a_{k+1}}, \ldots, 1_{\sigma a_{k+1} \ldots a_{m-1}}$ and $a = t_{\sigma a_{k+1} \ldots a_m}$ (note that, a is one of the points $1_{\sigma a_{k+1} \ldots a_m}, 2_{\sigma a_{k+1} \ldots a_m}$ or $3_{\sigma a_{k+1} \ldots a_m}$ since it is a vertex point of S_μ).

The number of geodesics between 1_σ and $1_{\sigma a_{k+1}}$ can be 0, 1, or at most 2 (indeed, if $a_{k+1} = 1$ then it would be 0, if $a_{k+1} = 2$ then it would be 2 and otherwise it would be 1). Similarly, the number of geodesics between $1_{\sigma a_{k+1}}$ and $1_{\sigma a_{k+1} a_{k+2}}$ can be 0, 1, or at most 2. More generally, the number of geodesics between $1_{\sigma a_{k+1} \ldots a_{i-1}}$ and $1_{\sigma a_{k+1} \ldots a_i}$ (which are also the vertex points of two different subgaskets of (the same) level $k+i$) can be 0, 1, or at most 2 for $k < i < m$ (see Figure 11a). Thus, the number of geodesics between the vertex points 1_σ and $1_{\sigma a_{k+1} \ldots a_{m-1}}$ can be obtained by multiplying the numbers of these (partial) geodesics as 2^n for some $n \in \mathbb{N}^*$ or 0 (note that the total geodesic number is 0 if $a_i = 1$ for all $k < i < m$ which gives $1_\sigma = 1_{\sigma a_{k+1} \ldots a_{m-1}}$). As the final step, we need to find the number of geodesics between $1_{\sigma a_{k+1} \ldots a_{m-1}}$ and $a = t_{\sigma a_{k+1} \ldots a_m}$ which can be 0 (if $a_m = 1, t = 1$), 1, 2 or 3. In fact, $a = 3_{\sigma a_{k+1} \ldots a_{m-1} 4}$ or $a = 2_{\sigma a_{k+1} \ldots a_{m-1} 4}$ if

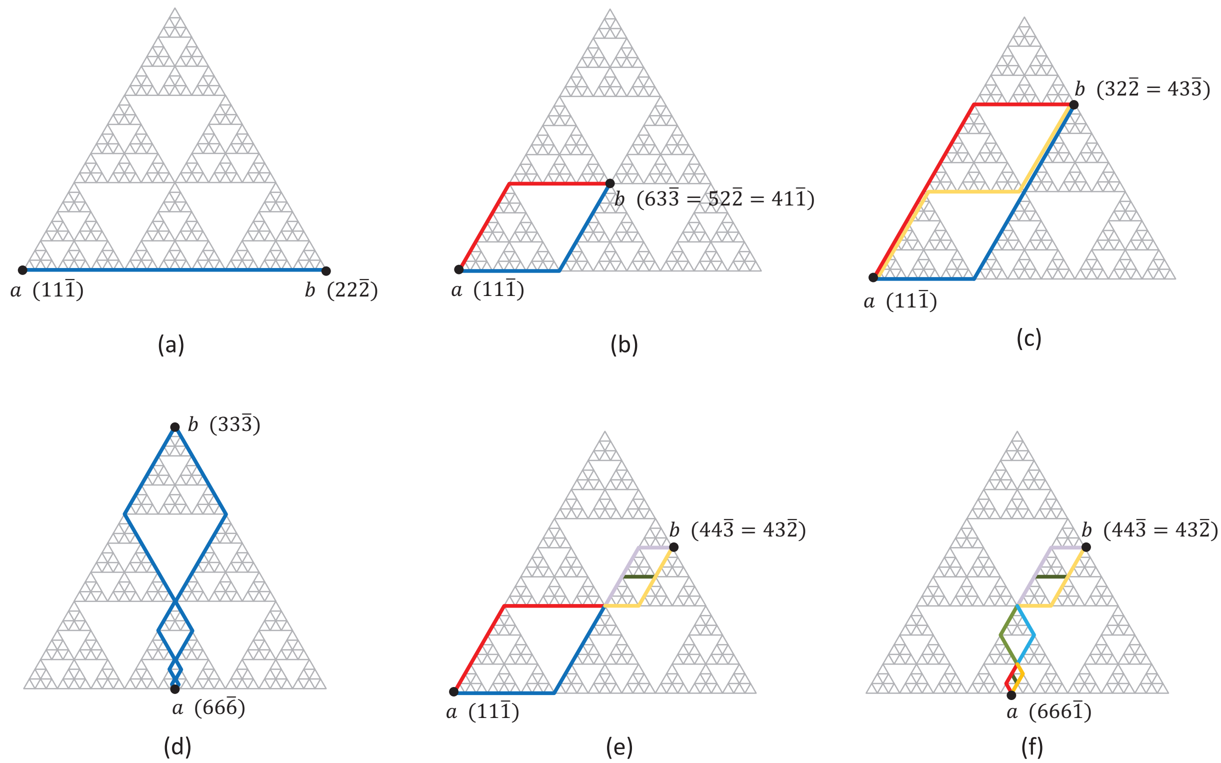

Figure 9. Examples of geodesics between the points a and b; (a) 1 geodesic, (b) 2 geodesics, (c) 3 geodesics, (d) infinite geodesics, (e) 6 geodesics, and (f) 12 geodesics (the word in parentheses is one of the code representation of the related point).

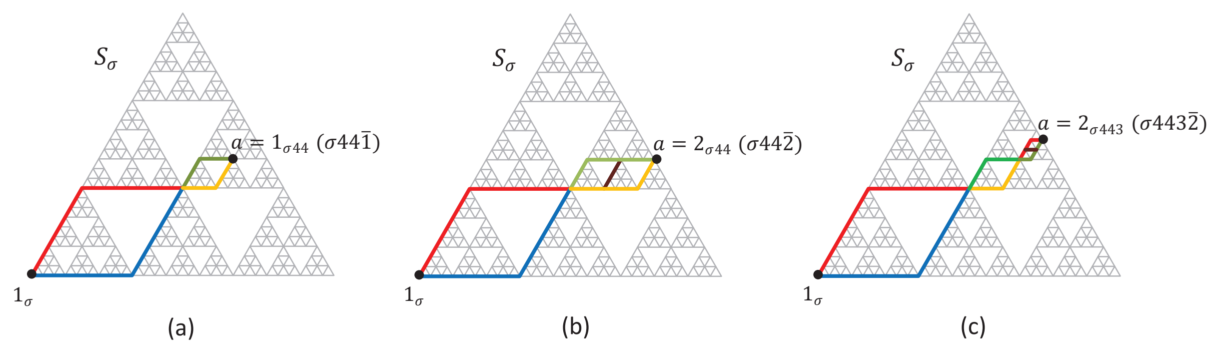

Figure 10. Examples of geodesics between the points 1_σ and a; (a) $2^2 = 4$ geodesics, (b) $3 \cdot 2^1 = 6$ geodesics and (c) $3 \cdot 2^2 = 12$ geodesics (the word in parentheses is one of the code representations of the related point).

and only if the number of geodesics between $1_{\sigma a_{k+1} \ldots a_{m-1}}$ and a is 3 (see Figure 11b). Therefore, the number of geodesics between the vertex point 1_σ and the point a can be 2^n or $3 \cdot 2^n$ for some $n \in \mathbb{N}^*$.

Assume that a is not a vertex point of any subgasket in S_σ. Using the similar argument (if necessary, infinite times) we obtain the number of geodesics either ∞ or 2^n for some $n \in \mathbb{N}^*$ (since a is not a vertex point, there does not exist 3 (partial) geodesics in the final step as in the previous case). For example, if $t = 3$ and the code representation of a is of the form $\sigma 666 \cdots = \sigma\overline{6}$, then the number of geodesics is ∞ (note that $6 = \widetilde{3}$) (Furthermore, if the number of geodesics between a and t_σ is ∞ then the code representation of a must be of the form $\sigma\nu\,\widetilde{t}\,\widetilde{t}\,\widetilde{t}\cdots$ for some ν).

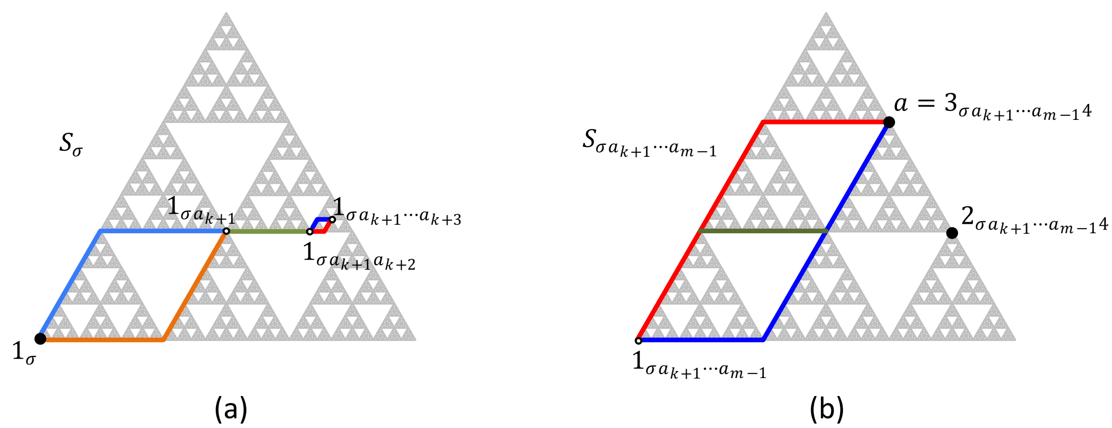

Figure 11. Examples of geodesics between the vertex points $1_{\sigma a_{k+1}\cdots a_{i-1}}$ and $1_{\sigma a_{k+1}\cdots a_i}$ (a), three geodesics between the vertex points $1_{\sigma a_{k+1}\cdots a_{m-1}}$ and $a = 3_{\sigma a_{k+1}\cdots a_{m-1}4}$ (b).

To see the second part of the statement, due to the symmetry of the self-similar set we can assume $t = 1$ for simplicity. The points $1_{\sigma 4\cdots 4}$ and $2_{\sigma 4\cdots 4}$ which are the vertex points of two subgaskets (of level $k+n$ and level $k+n+1$ respectively) would be the desired points. □

Lemma 3.2 *Let $a \in S_\sigma$ where S_σ is a subgasket of level k. If the number of geodesics between a and t_σ is $3 \cdot 2^n$ for some $n \in \mathbb{N}^*$ or ∞, then the number of geodesics between a and t^*_σ is 2^m for some $m \in \mathbb{N}^*$ for each $t^* \neq t$.*

Proof For simplicity assume that $t = 1$ and $t^* = 2$. As mentioned in the proof of Lemma 3.1, if the number of geodesics between a and 1_σ is finite and 3 (or a multiple of 3) then there exist (just) two different code representations of a as $\sigma\nu\overline{2}$ or $\sigma\mu\overline{3}$ for some ν and μ (indeed, these equal-length words ν and μ differ only in the last letters). If the number of geodesics between a and 2_σ is 3 (or a multiple of 3) then a code representation of a must be of the form $\sigma\upsilon\overline{1}$ or $\sigma\tau\overline{3}$ for some υ and τ. Thus, we obtain three different code representations of a which is a contradiction. Note that if the point b has three different code representations then the number of geodesics between b and t_σ (for all $t \in \{1,2,3\}$) must be an even number which is obvious from the proof of Lemma 3.1. If the number of geodesics between a and 1_σ is ∞ then a is not a vertex point of a subgasket. Thus, the number of geodesics between a and 1_σ cannot be a multiple of 3 as indicated in the proof of Lemma 3.1. □

Remark 3.3 *The numbers of geodesics between the point a and the (three) vertex points can be an even number at the same time. Lemma 3.2 says that more than one of these numbers cannot be a multiple of 3. For example, consider the point $3_{\sigma 6}$ for some σ. The number of geodesics between $3_{\sigma 6}$ and t_σ is 2 for all $t \in \{1,2,3\}$.*

Theorem 3.4 *Let $a, b \in S$. The number of geodesics between a and b is one of the numbers $3^u\,2^m$, $3\,2^m + 3\,2^n$, $3^u\,2^m + 2^n$ or ∞ for some $u \in \{0,1,2\}$ and $m, n \in \mathbb{N}^*$. Moreover, given one of the numbers $3^u\,2^m$, $2^m + 2^n$, $3 \cdot 2^m + 2^n$ (for all $u \in \{0,1,2\}, m, n \in \mathbb{N}^*$), there exist $a, b \in S$ such that the number of geodesics between them is exactly the given number.*

Proof Since a and b are different points then there exists a subgasket S_σ such that $a, b \in S_\sigma$ of level k and $a \in S_{\sigma i}$, $b \in S_{\sigma j}$ for some $i, j \in \{1,2,3,4,5,6\}$ such that $i \neq j$. Assume that $a \in S_{\sigma 1}$ and $b \in S_{\sigma 2}$. Any geodesics between a and b must pass through "Gate-1" or "Gate-2" segments whose lengths are $2/3^{k+1}$ and $1/3^{k+1}$ respectively as mentioned in Figure 12a.

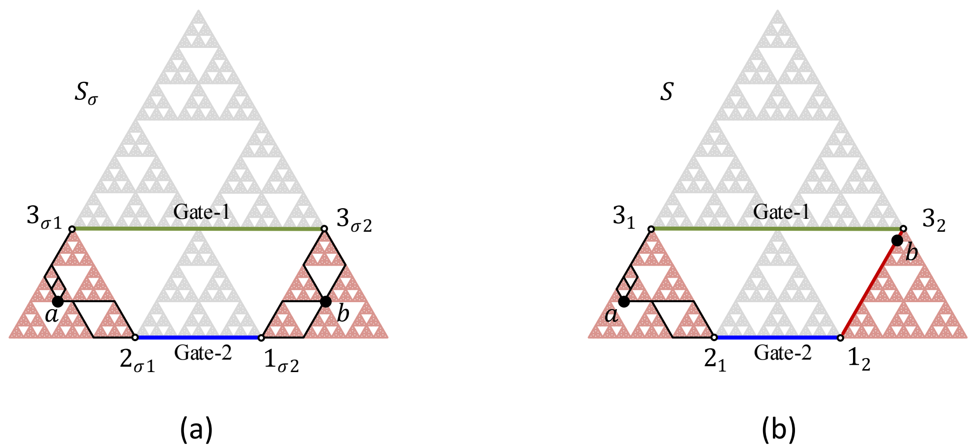

Figure 12. Gate-1 and Gate-2 between two nonadjacent subgaskets of level $k+1$ (a), the point b on the line segment between 1_2 and 3_2 (b).

If the geodesics use Gate-1 then, by Lemma 3.1, the number of geodesics between "a and $3_{\sigma 1}$" and "b and $3_{\sigma 2}$" are $3^r 2^{n_1}$ and $3^s 2^{n_2}$ respectively for some $r, s \in \{0,1\}, n_1, n_2 \in \mathbb{N}^*$ and by multiplying these numbers we obtain the total geodesic number between a and b as $3^u 2^m$ for some $u \in \{0,1,2\}, m \in \mathbb{N}^*$. If the geodesics pass through Gate-2 then the total number of geodesics would be the same form. Obviously, some of the geodesics pass through Gate-1 while others may pass through Gate-2; thus, we need the sum of these numbers to find total number. Note that, by Lemma 3.2, the number of geodesics between a (or b) and the vertex points "$3_{\sigma 1}$ and $2_{\sigma 1}$" (or "$3_{\sigma 2}$ and $1_{\sigma 2}$") cannot be both a multiple of 3. Thus, in this case the total geodesic number can be $2^m + 2^n$, $3 \cdot 2^m + 2^n$, $3^2 \cdot 2^m + 2^n$, $3 \cdot 2^m + 3 \cdot 2^n$ for some $m, n \in \mathbb{N}^*$. Therefore, the number of geodesics between a and b must be of the form $3^u 2^m$ or $3 \cdot 2^m + 3 \cdot 2^n$ or $3^u 2^m + 2^n$ for some $u \in \{0,1,2\}$, $m, n \in \mathbb{N}^*$.

If a and b are in different subgaskets, the same argument can be applied (notice that the length of a gate can be 0, $1/3^{k+1}$ or $2/3^{k+1}$ depending on the positions of a and b).

The second part of the theorem can be proved using Lemma 3.1. Firstly, let $m \in \mathbb{N}^*$. Consider the points a and b whose code representations are $133\sigma\overline{1} = 13366\cdots6\overline{1}$ and $2336\overline{1}$ respectively where $\sigma = 66\cdots6$ is the word with length $m+1$. The number of geodesics between a and $3_{\sigma 1}$ is $3 \cdot 2^m$ and the number of geodesics between b and $3_{\sigma 2}$ is 3. Thus, the number of geodesics between a and b is $3^2 2^m$ (we choose a and b in the subgaskets S_{133} and S_{233} to ensure that geodesics only pass through Gate-1). Similarly, the number of geodesics between a and b whose code representations are $133\sigma\overline{1}$ and $233\overline{1}$ is $3 \cdot 2^m$ since the number of geodesics between b and $3_{\sigma 2}$ is 1, and the number of geodesics between a and b whose code representations are $133\sigma\overline{3}$ and $233\overline{1}$ is 2^m since the number of geodesics between a and $3_{\sigma 1}$ is 2^m and the number of geodesics between b and $3_{\sigma 2}$ is 1.

Now, we show that there exist a pair of points such that the number of geodesics between them is $2^n + 3 \cdot 2^m$ for $m, n \in \mathbb{N}$ (for $m = n = 0$ it is obvious for the points whose code representations are $12\overline{1}$ and $42\overline{1}$). Consider the point $a \in S_1$ whose code representation is $1\sigma\mu\overline{1} = 155\cdots566\cdots6\overline{1}$ where $\sigma = 55\cdots5$ and $\mu = 66\cdots6$ are the words with length n and $m+1$ respectively. The number of geodesics between a and the vertex 2_1 is 2^n and the number of geodesics between a and the vertex 3_1 is $3 \cdot 2^m$. Using Equation (2.1), one can easily compute (as done in Example 2.2) the lengths of these geodesics as

$$d^1_{a,2} = \frac{1}{3} - \frac{1}{2 \cdot 3^{n+1}} + \frac{1}{2 \cdot 3^{n+m+2}}$$

and

$$d^1_{a,3} = \frac{1}{6} + \frac{1}{2 \cdot 3^{n+1}}.$$

It is obvious that $K = d^1_{a,2} - d^1_{a,3} > 0$. Let $b \in S_2$ be the point lies on the line segment between the vertices 1_2 and 3_2 such that $d^2_{b,3} = K/2$. Then, using Theorem 2.1, we obtain

$$d(a,b) = d^1_{a,2} + 1/3 + (1/3 - K/2) = d^1_{a,3} + 2/3 + K/2$$

which implies that there exist different geodesics between a and b using both Gate-1 and Gate-2 (see Figure 12b). Thus, summing the number of geodesics passing through Gate-1 and Gate-2, we obtain the number of geodesics between a and b as $2^n + 3 \cdot 2^m$. Note that there exists just one geodesic between "b and $1_{\sigma 2}$" (or "b and $3_{\sigma 2}$") (It would be a nice result to find one code representation of the point b).

We finally show that there exist a pair of points such that the number of geodesics between them is $2^n + 2^m$ for $m, n \in \mathbb{N}$ (for $m = n = 0$ it is obvious). Consider the point $a \in S_1$ whose code representation is $1\sigma\mu\overline{1} = 155\cdots566\cdots6\overline{3}$ where $\sigma = 55\cdots5$ and $\mu = 66\cdots6$ are the words with length $n-1$ and m respectively. The number of geodesics between a and the vertex 2_1 is 2^n and the number of geodesics between a and the vertex 3_1 is 2^m. Similarly, using the above argument, one can find $b \in S_2$ that lies on the line segment between the vertices 1_2 and 3_2 such that there exist different geodesics between a and b using both Gate-1 and Gate-2. Therefore, the number of geodesics between a and b is $2^n + 2^m$. $\square$

Does there exist a pair of points $a, b \in S$ such that the number of geodesics between them is exactly $3^2\,2^m + 2^n$ or $3\,2^m + 3\,2^n$ for given $m, n \in \mathbb{N}^*$? For some small and special values of m and n, the claim is true. However, the general situation remains a question worth solving.

Example 3.5 *Let $m = 4, n = 3$. It is known that there exist pairs of points such that the number of geodesics between them is one of the numbers $16, 48, 56$, or 144 by Theorem 3.4. Taking $m = 12, n = 10$ we get $2^n + 3 \cdot 2^m = 13312$ geodesics. Notice that we may not always find a pair of points such that the number of geodesics between them is a given number. For example, by the first part of Theorem 3.4, one can easily show that there does not exist a pair of points such that the number of geodesics between them is (the odd number) $3 + 3^2 \cdot 2^m$ for all $m \in \mathbb{N}$. On the other hand, there are infinite geodesics between the points a and b whose code representations (for example) are $\overline{3}$ and $\overline{6}$.*

References

[1] Barlow MT, Hambly BM. Transition density estimates for brownian motion on scale irregular Sierpinski gaskets. Annales de l'Institut Henri Poincaré Probabilités et Statistiques 1997; 33 (5): 531-557.

[2] Barnsley M. Fractals Everywhere. San Diego, CA, USA: Academic Press, 1988.

[3] Hilfer R, Blumen A. Renormalisation on Sierpinski-type fractals. Journal of Physics A: Mathematical and General 1984; 17 (10): 537-545. doi: 10.1088/0305-4470/17/10/004

[4] Burago D, Burago Y, Ivanov S. A Course in Metric Geometry. USA: AMS, 2001.

[5] Chang SC, Chen LC. Number of connected spanning subgraphs on the Sierpinski gasket. Discrete Mathematics and Theoretical Computer Science 2019; 11 (1): 55-78.

[6] Cristea LL. A geometric property of the Sierpinski carpet. Quaestiones Mathematicae 2005; 28: 251-262. doi: 10.2989/16073600509486126

[7] Cristea LL, Steinsky B. Distances in Sierpinski graphs and on the Sierpinski gasket. Aequationes Mathematicae 2013; 8 (3): 201-219. doi: 10.1007/s00010-013-0197-7

[8] Grabner P, Tichy RF. Equidistribution and Brownian motion on the Sierpinski gasket. Monatshefte für Mathematik 1998; 125 (2): 147-164.

[9] Gu J, Ye Q, Xi L. Geodesics of higher-dimensional Sierpinski gasket. Fractals 2019; 27 (4): 1950049. doi: 10.1142/S0218348X1950049X

[10] Güneri M, Saltan M. Intrinsic metric formulas on some self-similar sets via the code representation. Fractal and Fractional 2019; 3 (1): 1-13. doi: 10.3390/fractalfract3010013

[11] Hinz AM, Schief A. The average distance on the Sierpinski gasket. Probability Theory and Related Fields 1990; 87 (1): 129-138.

[12] Holter NS, Lakhtakia A, Varadan VK, Varadan VV, Messier R. On a new class of planar fractals: the Pascal-Sierpinski gaskets. Journal of Physics A: Mathematical and General 1986; 19 (9): 1753-1759. doi: 10.1088/0305-4470/19/9/047

[13] Özdemir Y, Saltan M, Demir B. The intrinsic metric on the box fractal. Bulletin of the Iranian Mathematical Society 2019; doi: 10.1007/s41980-018-00197-w

[14] Romik D. Shortest paths in the Tower of Hanoi graph and finite automata, SIAM Journal on Discrete Mathematics 2006; 20 (3): 610-622. doi: 10.1137/050628660

[15] Saltan M, Özdemir Y, Demir B. An explicit formula of the intrinsic metric on the Sierpinski gasket via code representation. Turkish Journal of Mathematics 2018; 42 (2): 716-725. doi: 10.3906/mat-1702-55

[16] Saltan M, Özdemir Y, Demir B. Geodesics of the Sierpinski gasket. Fractals 2018; 26 (3): 1850024. doi: 10.1142/S0218348X1850024X

[17] Strichartz RS. Isoperimetric estimates on Sierpinski gasket type fractals. Transactions of the American Mathematical Society 1999; 351 (5): 1705-1752.