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Study on the q -analogue of a certain family of linear operators

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Abstract: In this paper, we introduce the q -analogue of a certain family of linear operators in geometric function theory. Our main purpose is to define some subclasses of analytic functions by means of the q -analogue of linear operators and investigate various inclusion relationships with integral preserving properties.

Key words: Analytic functions, q -difference operator, q -Hurwitz–Lerch zeta function, q -Srivastava–Attiya operator, q -multiplier transformation

1. Introduction

Let \mathbf{A} denote the class of analytic functions $f(z)$ in the open unit disk $\mathbf{O} = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Subordination of two functions f and g is denoted by $f \prec g$ and defined as $f(z) = g(w(z))$, where $w(z)$ is the Schwartz function in \mathbf{O} (see [18]). Let S , S^* , and C denote the subclasses of \mathbf{A} of univalent functions, starlike functions, and convex functions, respectively. We recall here some basic definitions and details of q -calculus that are used in this paper.

The q -difference operator, which was introduced by Jackson [12], is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}; \quad q \neq 1, z \neq 0 \quad (1.2)$$

for $q \in (0, 1)$. It is clear that $\lim_{q \rightarrow 1^-} D_q f(z) = f'(z)$, where $f'(z)$ is the ordinary derivative of the function. For more properties of D_q see [5–7, 16, 27].

It can easily be seen that for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ and $z \in \mathbf{O}$

$$D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q z^{n-1}, \quad (1.3)$$

where

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \dots + q^{n-1}. \quad (1.4)$$

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We have the following rules of D_q :

$$D_q(af(z) \pm bg(z)) = aD_qf(z) \pm bD_qg(z), \tag{1.5}$$

$$D_q(f(z)g(z)) = f(qz)D_q(g(z)) + g(z)D_q(f(qz)), \tag{1.6}$$

$$D_q\left(\frac{f(z)}{g(z)}\right) = \frac{D_q(f(z))g(z) - f(z)D_q(g(z))}{g(qz)g(z)}, \quad g(qz)g(z) \neq 0, \tag{1.7}$$

$$D_q(\log f(z)) = \frac{D_q(f(z))}{f(z)}. \tag{1.8}$$

In the context of geometric function theory, usage of q-difference equations was first applied by Jackson [13], Carmichael [4], Mason [17], Adams [1], and Trijitzinsky [26]. Some properties related with q-function theory were first introduced by Ismail et al. [11]. Moreover, several authors studied many applications of q-calculus associated with generalized subclasses of analytic functions; see [2, 19, 21, 28]. The study of linear operators plays an important and vital role in the field of geometric function theory. In recent days, many well-known researchers are interested in introducing and studying those linear operators in terms of q-analogues. In [15], the authors introduced the q-analogue of the Ruscheweyh derivative operator and studied some of the properties of this differential operator. Noor et al. [22] introduced the q-Bernardi integral operator. Furthermore, by using the concept of q-calculus, Arif et al. [3] defined the q-Noor integral operator and investigated a number of important properties.

Motivated by the work mentioned above, we introduce the q-Srivastava–Attiya operator and the q-multiplier transformation on \mathbf{A} as follows.

First, we give the q-analogue of the Hurwitz–Lerch zeta function by the following series:

$$\varphi_q(s, b; z) = \sum_{n=0}^{\infty} \frac{z^n}{[n+b]_q^s}, \tag{1.9}$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $s \in \mathbb{C}$ when $|z| < 1$, and $Re(s) > 1$ when $|z| = 1$.

From (1.9) we can easily get

$$\begin{aligned} \psi_q(s, b; z) &= [1+b]_q^s \left\{ \varphi_q(s, b; z) - [b]_q^s \right\} \\ &= z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_q}{[n+b]_q} \right)^s z^n. \end{aligned} \tag{1.10}$$

Now by (1.10) and (1.1), we define the q-Srivastava–Attiya operator $J_{q,b}^s : \mathbf{A} \rightarrow \mathbf{A}$ by

$$\begin{aligned} J_{q,b}^s f(z) &= \psi_q(s, b; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{[1+b]_q}{[n+b]_q} \right)^s a_n z^n, \end{aligned} \tag{1.11}$$

where $*$ denotes convolution (or the Hadamard product).

We note that:

(i) If $q \rightarrow 1^-$, then the function $\varphi_q(s, b; z)$ reduces to the Hurwitz–Lerch zeta function and the operator $J_{q,b}^s$ coincides with the Srivastava–Attiya operator; we refer to [24, 25].

(ii) $J_{q,0}^1 f(z) = \int_0^z \frac{f(t)}{t} d_q t$. (q-Alexander operator).

(iii) $J_{q,b}^1 f(z) = \frac{[1+b]_q}{z^b} \int_0^z t^{b-1} f(t) d_q t$. (q-Bernardi operator [22]).

(iv) $J_{q,1}^1 f(z) = \frac{[2]_q}{z} \int_0^z f(t) d_q t$. (q-Libera operator [22]).

For any real number s , the q-analogue of multiplier transformation $I_{q,s}^b : \mathbf{A} \rightarrow \mathbf{A}$ is defined as follows:

$$I_{q,s}^b f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[n+b]_q}{[1+b]_q} \right)^s a_n z^n, \tag{1.12}$$

where $f \in \mathbf{A}$ and $b > -1$. It is noted that for nonnegative integers s and $b = 0$, the operator $I_{q,s}^b$ becomes the well-known Salagean q-differential operator [8], which was studied by several authors [9, 10].

From (1.11) and (1.12), we easily have the following identities:

$$zD_q \left(J_{q,b}^{s+1} f(z) \right) = \left(1 + \frac{[b]_q}{q^b} \right) J_{q,b}^s f(z) - \frac{[b]_q}{q^b} J_{q,b}^{s+1} f(z), \tag{1.13}$$

$$zD_q \left(I_{q,s}^b f(z) \right) = \left(1 + \frac{[b]_q}{q^b} \right) I_{q,s+1}^b f(z) - \frac{[b]_q}{q^b} I_{q,s}^b f(z). \tag{1.14}$$

Using the q-derivative, we now define some new classes of analytic functions below.

Let Φ be the class of analytic functions $\varphi(z)$, which are univalent convex functions in \mathbf{O} with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ in \mathbf{O} .

Definition 1.1 A function $f \in \mathbf{A}$ is said to be in the class $ST_q(\varphi)$ if it satisfies the following condition:

$$\frac{zD_q f(z)}{f(z)} \prec \varphi(z),$$

where D_q is the q-difference operator.

Analogously, a function $f \in \mathbf{A}$ is said to be in the class $CV_q(\varphi)$ if and only if

$$zD_q f(z) \in ST_q(\varphi). \tag{1.15}$$

Using the operators defined above, we define the following:

Definition 1.2 Let $f \in \mathbf{A}$, s be real, and $b > -1$. Then

$$f \in ST_{q,b}^s(\varphi) \text{ if and only if } J_{q,b}^s f(z) \in ST_q(\varphi),$$

and

$$f \in CV_{q,b}^s(\varphi) \text{ if and only if } J_{q,b}^s f(z) \in CV_q(\varphi). \tag{1.16}$$

It is clear that

$$f \in CV_{q,b}^s(\varphi) \text{ if and only if } z(D_q f) \in ST_{q,b}^s(\varphi). \tag{1.17}$$

Special cases:

(i) If $s = 0$ and $\varphi(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$), then $ST_{q,b}^s(\varphi)$ reduces to the class $S_q^*(A, B)$ introduced by Noor et al. [22]. Moreover, if $q \rightarrow 1^-$, then $S_q^*(A, B)$ coincides with the class $S^*[A, B]$ (see [14]).

(ii) If $s = 0$ and $\varphi(z) = \frac{1}{1-qz}$, then $ST_{q,b}^s(\varphi)$ reduces to the class ST_q studied by Noor [20].

(iii) If $s = 0$ and $\varphi(z) = \frac{1+z}{1-qz}$, then $ST_{q,b}^s(\varphi)$ reduces to the class S_q^* introduced by Noor et al. [21].

Definition 1.3 Let $f \in \mathbf{A}$, s be real, and $b > -1$. Then

$$f \in \widetilde{ST}_{q,s}^b(\varphi) \text{ if and only if } I_{q,s}^b f(z) \in ST_q(\varphi),$$

and

$$f \in \widetilde{CV}_{q,s}^b(\varphi) \text{ if and only if } I_{q,s}^b f(z) \in CV_q(\varphi).$$

It is obvious that

$$f \in \widetilde{CV}_{q,s}^b(\varphi) \text{ if and only if } z(D_q f) \in \widetilde{ST}_{q,s}^b(\varphi). \tag{1.18}$$

Remark 1.4 It is easy to check that for $0 < q < 1$, the function $f_1(z) = \frac{z}{1-qz}$, $z \in \mathbf{O}$, belongs to $ST_q(\varphi)$ and the function $f_2(z) = \frac{1+z}{1-qz}$, $z \in \mathbf{O}$, belongs to $CV_q(\varphi)$. Therefore, our newly defined classes are not empty; consequently, we proceed to discuss our main results.

2. Main results

We need the following lemma to prove our results:

Lemma 2.1 [23] Let β and γ be complex numbers with $\beta \neq 0$ and let $h(z)$ be analytic in \mathbf{O} with $h(0) = 1$ and $Re\{\beta h(z) + \gamma\} > 0$. If $p(z) = 1 + p_1z + p_2z^2 + \dots$ is analytic in \mathbf{O} , then

$$p(z) + \frac{zD_q p(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that $p(z) \prec h(z)$.

2.1. Inclusion results

Theorem 2.2 Let $\varphi(z)$ be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{O}$. Then, for $s > 0$, $b \in \mathbb{N}$,

$$ST_{q,b}^s(\varphi) \subset ST_{q,b}^{s+1}(\varphi).$$

Proof Let $f \in ST_{q,b}^s(\varphi)$ and we set

$$\frac{zD_q \left(J_{q,b}^{s+1} f(z) \right)}{J_{q,b}^{s+1} f(z)} = p(z), \tag{2.1}$$

where $p(z)$ is analytic in \mathbf{O} with $p(0) = 1$.

Using identity (1.13) and (2.1), we get

$$\frac{zD_q \left(J_{q,b}^{s+1} f(z) \right)}{J_{q,b}^{s+1} f(z)} = \left(1 + \frac{[b]_q}{q^b} \right) \frac{J_{q,b}^s f(z)}{J_{q,b}^{s+1} f(z)} - \frac{[b]_q}{q^b},$$

or equivalently

$$\left(1 + \frac{[b]_q}{q^b} \right) \frac{J_{q,b}^s f(z)}{J_{q,b}^{s+1} f(z)} = p(z) + \gamma_q, \quad \left(\text{for } \gamma_q = \frac{[b]_q}{q^b} \right),$$

and q-logarithmic differentiation yields

$$\frac{zD_q \left(J_{q,b}^s f(z) \right)}{J_{q,b}^s f(z)} = p(z) + \frac{zD_q p(z)}{p(z) + \gamma_q}. \tag{2.2}$$

Since $f \in ST_{q,b}^s(\varphi)$, from (2.2) we have

$$p(z) + \frac{zD_q p(z)}{p(z) + \gamma_q} \prec \varphi(z).$$

By applying Lemma 2.1, we conclude that $p(z) \prec \varphi(z)$. Consequently, $\frac{zD_q \left(J_{q,b}^{s+1} f(z) \right)}{J_{q,b}^{s+1} f(z)} \prec \varphi(z)$ implies $f \in ST_{q,b}^{s+1}(\varphi)$. □

Theorem 2.3 *Let $s, b,$ and φ be given the same as in Theorem 2.2. Then*

$$CV_{q,b}^s(\varphi) \subset CV_{q,b}^{s+1}(\varphi).$$

Proof Let $f \in CV_{q,b}^s(\varphi)$. Then, by Alexander’s type relation (1.17), we can write $z(D_q f) \in ST_{q,b}^s(\varphi)$. From Theorem 2.2, we know that $ST_{q,b}^s(\varphi) \subset ST_{q,b}^{s+1}(\varphi)$, so we have $z(D_q f) \in ST_{q,b}^{s+1}(\varphi)$. We get the required result by again using Alexander’s type relation (1.17). □

Corollary 2.4 *Let s and b be given the same as in Theorem 2.2. Then, for $\varphi(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$),*

$$ST_{q,b}^s \left(\frac{1+Az}{1+Bz} \right) \subset ST_{q,b}^{s+1} \left(\frac{1+Az}{1+Bz} \right).$$

Moreover, for $A = 0$ and $B = -q$ and for $A = 1$ and $B = -q$,

$$ST_{q,b}^s \left(\frac{1}{1-qz} \right) \subset ST_{q,b}^{s+1} \left(\frac{1}{1-qz} \right) \text{ and } ST_{q,b}^s \left(\frac{1+z}{1-qz} \right) \subset ST_{q,b}^{s+1} \left(\frac{1+z}{1-qz} \right),$$

respectively.

One can easily prove the following results by using similar arguments as used before.

Theorem 2.5 Let $\varphi(z)$ be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{O}$. Then, for $s > 0$, $b \in \mathbb{N}$,

$$\widetilde{ST}_{q,s+1}^b(\varphi) \subset \widetilde{ST}_{q,s}^b(\varphi)$$

and

$$\widetilde{CV}_{q,s+1}^b(\varphi) \subset \widetilde{CV}_{q,s}^b(\varphi).$$

2.2. Integral preserving property

Theorem 2.6 Let $f \in ST_{q,b}^s(\varphi)$. Then $J_{q,b}^1 f \in ST_{q,b}^s(\varphi)$, where

$$J_{q,b}^1 f(z) = \frac{[1+b]_q}{z^b} \int_0^z t^{b-1} f(t) d_q t. \tag{2.3}$$

Proof Let $f \in ST_{q,b}^s(\varphi)$. If we set $F(z) = J_{q,b}^1 f(z)$,

$$\frac{zD_q \left(J_{q,b}^s F(z) \right)}{J_{q,b}^s F(z)} = Q(z), \tag{2.4}$$

where $Q(z)$ is analytic in \mathbf{O} with $Q(0) = 1$.

From (2.3) we can write

$$\frac{D_q \left(z^b F(z) \right)}{[1+b]_q} = z^{b-1} f(z).$$

Using the product rule of the q-difference operator, we get

$$zD_q F(z) = \left(1 + \frac{[b]_q}{q^b} \right) f(z) - [b]_q F(z). \tag{2.5}$$

From (2.2), (2.5), and (1.11), we have

$$Q(z) = \left(1 + \frac{[b]_q}{q^b} \right) \frac{zD_q \left(J_{q,b}^s f(z) \right)}{J_{q,b}^s F(z)} - [b]_q.$$

On q-logarithmic differentiation, we get

$$\frac{zD_q \left(J_{q,b}^s f(z) \right)}{J_{q,b}^s f(z)} = Q(z) + \frac{zD_q Q(z)}{Q(z) + [b]_q}. \tag{2.6}$$

Since $f \in ST_{q,b}^s(\varphi)$, (2.6) implies

$$Q(z) + \frac{zD_q Q(z)}{Q(z) + [b]_q} \prec \varphi(z).$$

Now, by applying Lemma 2.1, we conclude $Q(z) \prec \varphi(z)$. Consequently, $\frac{zD_q \left(J_{q,b}^s F(z) \right)}{J_{q,b}^s F(z)} \prec \varphi(z)$. Hence,

$F \in ST_{q,b}^s(\varphi)$. □

On similar arguments as used in Theorem 2.3, one can easily prove the following result.

Theorem 2.7 *Let $f \in CV_{q,b}^s(\varphi)$. Then $J_{q,b}^1 f \in CV_{q,b}^s(\varphi)$, where $J_{q,b}^1 f$ is defined by (2.3).*

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