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Coefficient estimates for the class of quasi q -convex functions

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Abstract: In this paper we introduce and investigate the class of $P_q(\lambda, \beta, A, B)$, which is called quasi q -starlike and quasi q -convex with respect to the values of the parameter λ . We give coefficient bounds estimates and the results for the main theorem.

Key words: q -Derivative, starlike function, convex function, q -starlike function, q -convex function, quasi q -starlike function, quasi q -convex function, close-to-quasi q -starlike function, close-to-quasi q -convex function

1. Introduction

Let A denote the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0.$$

We say that the function $f(z)$ is subordinate to $g(z)$ and can be represented as $f \prec g$, if there exists a function $w(z)$ such that $w(0) = 0$, $|w(z)| < 1$, and $f(z) = g(w(z))$. If $g(z)$ is univalent the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [3]). $S^*(\alpha)$ and $C(\alpha)$ are starlike and convex functions of order α respectively such that

$$S^*(\alpha) = \left\{ f \in A : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in U \right\}$$

$$C(\alpha) = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, z \in U \right\}$$

for $\alpha = 0$, $S^* = S^*(0)$ and $C = C(0)$ are respectively starlike and convex functions in U (see [3]). Jackson (see [4]) introduced q -derivative operator of the functions $f(z)$ as follows:

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z}, & \text{if } z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases} \quad (1.1)$$

for $q \in (0, 1)$. It is clear that

$$\lim_{q \rightarrow 1^-} D_q f(z) = f'(z).$$

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For $n \in N$, $D_q z^n = [n]_q z^{n-1}$, $[n]_q = \frac{1-q^n}{1-q}$, $[0]_q = 0$. Recently Uçar Özkan (see [8]) studied some properties of q -starlike and q -close to convex functions. Ramachandran [6] studied q -starlike and q -convex functions with respect to symmetric points. Polatoğlu [5] investigated q -starlike functions and obtained growth and distortion theorems for this class.

Quasi-starlike and quasi-convex functions were studied by Altıntaş (see [1]). Altıntaş and Mustafa studied q -starlike and q -convex functions (see [2]).

We have the following properties:

Remark 1.1 We have symbols for the following classes:

- i. $CV(\beta)$ is the class of convex functions of order β .
- ii. $ST(\beta)$ is the class of starlike functions of order β .
- iii. $CCV(\alpha)$ is the class of close-to-convex of order α .
- iv. $CST(\alpha)$ is the class of close-to-starlike of order α .
- v. $QCV(\alpha)$ is the class of quasi-convex of order α .
- vi. $QST(\alpha)$ is the class of quasi-starlike of order α .
- vii. $CC_qV(\alpha)$ is the class of close-to- q convex of order α .
- viii. $CS_qT(\alpha)$ is the class of close-to- q starlike order α .
- ix. $QC_qV(\alpha)$ is the class of quasi- q convex of order α .
- x. $QS_qT(\alpha)$ is the class of quasi- q starlike of order α .

Remark 1.2 We have definitions of the following classes.

- i. $CV = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \right\}$.
- ii. $ST = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \right\}$.
- iii. $CV(\beta) = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \beta, \quad 0 \leq \beta < 1 \right\}$.
- iv. $ST(\beta) = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta, \quad 0 \leq \beta < 1 \right\}$.
- v. $CCV(\beta) = \left\{ f \in A : \operatorname{Re} \frac{f'(z)}{g'(z)} > \beta, \quad 0 \leq \beta < 1, \quad g \in CV \right\}$.
- vi. $CST(\beta) = \left\{ f \in A : \operatorname{Re} \frac{f(z)}{g(z)} > \beta, \quad 0 \leq \beta < 1, \quad g \in ST \right\}$.
- vii. $CCV(\beta, A, B) = \left\{ f \in A : \frac{f'(z)}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1 \right\}$.

- viii. $CST(\beta, A, B) = \left\{ f \in A : \frac{f(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1 \right\}$.
- ix. $QCV(\beta, A, B) = \left\{ f \in A : \frac{(zf'(z))'}{g'(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1 \right\}$.
- x. $QST(\beta, A, B) = \left\{ f \in A : \frac{zf'(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1 \right\}$.
- xi. $QC_qV(\beta, A, B) = \left\{ f \in A : \frac{D_q(zD_q(z))}{D_qg(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in CV(\beta), \quad -1 \leq B < A \leq 1, \quad q \in (0, 1) \right\}$.
- xii. $QS_qT(\beta, A, B) = \left\{ f \in A : \frac{zD_qf(z)}{g(z)} \prec \frac{1+Az}{1+Bz}, \quad g \in ST(\beta), \quad -1 \leq B < A \leq 1, \quad q \in (0, 1) \right\}$.

Remark 1.3 i. $f \in CV(\beta) \Rightarrow zf' \in ST(\beta)$, $(0 \leq \beta < 1)$.

ii. $f \in CCV(\alpha) \Rightarrow zf' \in CST(\alpha)$, $(0 \leq \alpha < 1)$.

iii. $f \in QCV(\alpha) \Rightarrow zf' \in QST(\alpha)$.

iv. $f \in QC_qV(\alpha) \Rightarrow zf' \in QS_qT(\alpha)$.

Lemma 1.4 *If $h(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in U and $h(z) \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, then we have*

$$Reh(z) > \frac{1-A}{1-B}. \tag{1.2}$$

(see [7]).

Lemma 1.5 *If $f(z) \in C_qV(\beta)$ or $Re \left[1 + \frac{zD_q^2f(z)}{D_qf(z)} \right] > \beta$, $(0 \leq \beta < 1)$, then*

$$\sum_{n=2}^{\infty} ([n]_q - \beta)[n]_q a_n < 1 - \beta.$$

For the proof of this lemma, we use the following equation:

$$D_q^2f(z) = D_q(D_qf(z)). \tag{1.3}$$

(see[2].Corollary 2.5)

Definition 1.6 *A function $f(z) \in A$ in the form of (1.1) is said to be in the class $P_q(\lambda, \beta, A, B)$ if the following condition is satisfied:*

$$\frac{D_qf(z) + \lambda zD_q^2f(z)}{D_qg(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.4}$$

where $g(z) \in CV(\beta)$, $q \in (0, 1)$, $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $-1 \leq B < A \leq 1$.

If $\lambda = 0$ then we have

$$\frac{D_qf(z)}{D_qg(z)} = \frac{zD_qf(z)}{zD_qg(z)} = \frac{zD_qf(z)}{h(z)} \prec \frac{1+Az}{1+Bz},$$

and $h(z) \in ST(\beta)$, so $f(z) \in QS_qT(\beta, A, B)$.

If $\lambda = 1$ then we have

$$\frac{D_q(zD_qf(z))}{D_qg(z)} \prec \frac{1 + Az}{1 + Bz},$$

and $g(z) \in CV(\beta)$. Hence, $f(z) \in QC_qV(\beta, A, B)$.

2. Main results

Theorem 2.1 *If $f(z) \in P_q(\lambda, \beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} [n]_q [1 + \lambda[n - 1]_q] a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}, \tag{2.1}$$

where $q \in (0, 1)$, $0 \leq \lambda \leq 1$, $0 \leq \beta < 1$, $\alpha = \frac{1-A}{1-B}$, $-1 \leq B < A \leq 1$.

Proof From Lemma 1.3 and Definition 1.5 we have

$$h(z) \prec \frac{1 + Az}{1 + Bz} \Rightarrow Re h(z) > \frac{1 - A}{1 - B} = \alpha$$

and

$$Re \frac{1 - \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} - \lambda \sum_{n=2}^{\infty} [n]_q [n - 1]_q a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} [n]_q b_n z^{n-1}} > \alpha.$$

If we choose z real and $z \rightarrow 1^-$ then we have

$$\sum_{n=2}^{\infty} [n]_q [1 + \lambda[n - 1]_q] a_n \leq 1 - \alpha + \alpha \sum_{n=2}^{\infty} [n]_q b_n. \tag{2.2}$$

From Lemma 1.4 if $g(z) \in C_qV(\beta)$ and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$, ($b_n \geq 0$), then we let

$$\sum_{n=2}^{\infty} ([n]_q - \beta) [n]_q b_n \leq 1 - \beta,$$

or

$$([2]_q - \beta) \sum_{n=2}^{\infty} [n]_q b_n \leq 1 - \beta,$$

and

$$\sum_{n=2}^{\infty} [n]_q b_n \leq \frac{1 - \beta}{[2]_q - \beta}. \tag{2.3}$$

Using (2.3) in (2.2) we find the equality (2.1). The result is sharp for the function

$$f(z) = f_n(z) = z - \frac{1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}}{[n]_q [1 + \lambda][n - 1]_q} z^n.$$

□

From Theorem 2.1 for the different values of q, λ, β, A, B we can obtain the following results and all results are sharp.

Corollary 2.2 *If $f \in QC_qV(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} [n]_q (1 + [n-1]_q) a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}. \quad (2.4)$$

We let $\lambda = 1$ in Theorem 2.1.

Corollary 2.3 *If $f \in QCV(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{2 - \beta}, \quad (2.5)$$

We let $q \rightarrow 1^-$ and $\lambda = 1$ in Theorem 2.1.

Corollary 2.4 *If $f \in QCV(0, A, B)$ then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1 - \alpha + \frac{\alpha}{2} = \frac{2 - \alpha}{2}. \quad (2.6)$$

We let $q \rightarrow 1^-$, $\lambda = 1$ and $\beta = 0$ in Theorem 2.1.

Corollary 2.5 *If $f \in QCV(0, 1, -1)$ then we have*

$$\sum_{n=2}^{\infty} n^2 a_n \leq 1 \quad (2.7)$$

We let $q \rightarrow 1^-$, $\lambda = 1$, $B = -1$, and $A = 1$ in Theorem 2.1.

Corollary 2.6 *If $f \in QS_qT(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} [n]_q a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}. \quad (2.8)$$

We let $\lambda = 0$ in Theorem 2.1.

Corollary 2.7 *If $f \in QST(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} n a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}, \quad (2.9)$$

We let $q \rightarrow 1^-$ and $\lambda = 0$ in Theorem 2.1.

Corollary 2.8 *If $f \in QST(0, A, B)$ then we have*

$$\sum_{n=2}^{\infty} na_n \leq (1 - \alpha + \frac{\alpha}{2}) = \frac{2 - \alpha}{2}. \tag{2.10}$$

We let $q \rightarrow 1^-$, $\lambda = 0$, and $\beta = 0$ in Theorem 2.1.

Corollary 2.9 *If $f \in QST(0, 1, -1)$ then we have*

$$\sum_{n=2}^{\infty} na_n \leq 1. \tag{2.11}$$

We let $q \rightarrow 1^-$, $\lambda = 0$, $\beta = 0$, $A = 1$, and $B = -1$ in Theorem 2.1.

Corollary 2.10 *If $f \in CC_qV(\beta, A, B)$ then we have*

$$\sum_{n=2}^{\infty} [n]_q a_n \leq 1 - \alpha + \alpha \frac{1 - \beta}{[2]_q - \beta}. \tag{2.12}$$

We let $\lambda = 0$ in Theorem 2.1.

Corollary 2.11 *If $f \in CC_qV(\beta, A, B) \Leftrightarrow f \in QS_qT(\beta, A, B)$ we let $\lambda = 0$ in Theorem 2.1 and then we have*

$$\frac{D_q f(z)}{D_q g(z)} = \frac{z D_q f(z)}{z D_q g(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.13}$$

and $g(z) \in CV(\beta) \Rightarrow z D_q g(z) = h(z)$, or

$$\frac{z D_q f(z)}{h(z)} \prec \frac{1 + Az}{1 + Bz}, \tag{2.14}$$

and $h(z) \in ST(\beta)$ so $f(z) \in QS_qT(\beta, A, B)$. Hence, we have $CC_qV(\beta, A, B) = QS_qT(\beta, A, B)$.

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