[Turkish Journal of Mathematics](https://dctubitak.researchcommons.org/math)

Volume 44 | Number 1 Article 22

1-1-2020

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 319 – 333
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doi:10.3906/mat-1904-56

A generalization of the Minkowski distance and new definitions of the central conics

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Received: 08.04.2019 • **Accepted/Published Online:** 06.01.2019 • **Final Version:** 20.01.2020

Abstract: In this paper, we give a generalization of the well-known Minkowski distance family in the n-dimensional Cartesian coordinate space. Then we consider three special cases of this family, which are also generalizations of the taxicab, Euclidean, and maximum metrics, respectively, and we determine some circle properties of them in the real plane. While we determine some properties of circles of these generalized distances, we discover a new definition of ellipses, and then we also determine a similar definition of hyperbolas, which will be new members among different metrical definitions of central conics in the Euclidean plane.

Key words: Minkowski distance, l_p-norm, l_p-metric, taxicab distance, Manhattan distance, Euclidean distance, maximum distance, Chebyshev distance, ellipse, hyperbola, central conics, asymptote, eccentrix, eccentricity

1. Introduction

Beyond mathematics, distances, especially the well-known Minkowski distance (also known as l_p-metric) with its special cases of taxicab (also known as l_1 or Manhattan), Euclidean (also known as l_2), and maximum (also known as l_∞ or Chebyshev) distances, are very important keys for many application areas such as data mining, machine learning, pattern recognition, and spatial analysis.

Here, we give a generalization of the Minkowski distance in the n-dimensional Cartesian coordinate space, which gives a new metric family for $p \geq 1$, and we consider this generalization for special cases $p = 1$, $p = 2$, and $p \to \infty$, which we call the generalized taxicab, Euclidean, and maximum metrics, respectively. Then we determine the circles of the special cases in the real plane, and we see that circles of the generalized taxicab and maximum metrics are parallelograms and circles of the generalized Euclidean metric are ellipses, as expected conclusions by the derivation method of the generalized Minkowski distance. While we determine some properties of the generalized Euclidean distance, we discover and determine a new metrical definition for the central conics, ellipses, and hyperbolas, which can be referenced by "two-eccentrices" definition, similar to their well-known "two-foci", "focus-directrix", and "focus-circular directrix" definitions (see [8, 16] for definitions and their analogues in normed (or Minkowski) planes; also see [10] for the same subject on the taxicab plane).

Throughout this paper, symmetry about a line is used in the Euclidean sense and angle measurement is in Euclidean radian. The terms square, rectangle, rhombus, parallelogram, ellipse, hyperbola, and conic are used in the Euclidean sense, and the center of them stands for their center of symmetry.

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2010 *AMS Mathematics Subject Classification:* 51K05, 51K99, 51N20.

2. A generalization of the Minkowski distance

It is known that the symmetric gauge-function of $\mathbb{R}^n$,

$$l_p(Ax) = \left(\sum_{i=1}^{n} (\lambda_i |v_{i1}x_1 + ... + v_{in}x_n|)^p \right)^{1/p} \tag{2.1}$$

for $p \geq 1$, derived by the well-known l_p-norm $l_p(x) = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ for $p \geq 1$ and the linear transformation

$$A : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1 v_{11} & ... & \lambda_1 v_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_n v_{n1} & ... & \lambda_n v_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{2.2}$$

where $v_i = (v_{i1}, ..., v_{in})$ for $i \in \{1, ..., n\}$, are linearly independent n unit vectors, determining a norm that is a generalization of the l_p-norm (see [1, 7, 17]). Then, on the other hand, one can get a new induced metric family from the generalized norm, which is also a generalization of the l_p-metric derived from the l_p-norm. In the following definition, we define this new distance family with its special cases, in the n-dimensional Cartesian coordinate space, using linearly independent n unit vectors $v_1, ..., v_n$ and n positive real numbers $\lambda_1, ..., \lambda_n$, supposing λ_i weights are initially determined and fixed:

Definition 2.1 *Let $X = (x_1, ..., x_n)$ and $Y = (y_1, ..., y_n)$ be two points in $\mathbb{R}^n$. For linearly independent n unit vectors $v_1, ..., v_n$ where $v_i = (v_{i1}, ..., v_{in})$, and positive real numbers $p, \lambda_1, ..., \lambda_n$ where $p \geq 1$, the function $d_{p(v_1,...,v_n)} : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by*

$$d_{p(v_1,...,v_n)}(X, Y) = \left(\sum_{i=1}^{n} (\lambda_i |v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|)^p \right)^{1/p} \tag{2.3}$$

*is called the $(v_1, ..., v_n)$**-Minkowski** (or $l_{p(v_1,...,v_n)}$) **distance function** in $\mathbb{R}^n$, and real number $d_{p(v_1,...,v_n)}(X, Y)$ is called the $(v_1, ..., v_n)$**-Minkowski distance** between points X and Y. In addition, if $p = 1$, $p = 2$, and $p \to \infty$, then $d_{p(v_1,...,v_n)}(X, Y)$ is called the $(v_1, ..., v_n)$**-taxicab distance**, $(v_1, ..., v_n)$**-Euclidean distance,** and $(v_1, ..., v_n)$**-maximum distance** between points X and Y, respectively, and we denote them by $d_{T(v_1,...,v_n)}(X, Y)$, $d_{E(v_1,...,v_n)}(X, Y)$, and $d_{M(v_1,...,v_n)}(X, Y)$, respectively.*

Here, since $\sigma \leq d_{p(v_1,...,v_n)}(X, Y) \leq \sigma n^{1/p}$ where $\sigma = \max\limits_{i \in \{1,...,n\}} \{\lambda_i |v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|\}$, we have $\lim\limits_{p \to \infty} d_{p(v_1,...,v_n)}(X, Y) = \max\limits_{i \in \{1,...,n\}} \{\lambda_i |v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|\}$, and so

$$d_{M(v_1,...,v_n)}(X, Y) = \max_{i \in \{1,...,n\}} \{\lambda_i |v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|\}. \tag{2.4}$$

Remark 2.2 *In the n-dimensional Cartesian coordinate space, let $\Psi_P^{v_i}$ denote the hyperplane through point P and perpendicular to the unit vector v_i for $i \in \{1, ..., n\}$. Since the Euclidean distance between point Y and hyperplane $\Psi_X^{v_i}$ (or the point X and hyperplane $\Psi_Y^{v_i}$) is*

$$d_E(Y, \Psi_X^{v_i}) = |v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|, \tag{2.5}$$

the $(v_1, ..., v_n)$-Minkowski distance between points X and Y is

$$d_{p(v_1,...,v_n)}(X, Y) = \left(\sum_{i=1}^{n} (\lambda_i d_E(Y, \Psi_X^{v_i}))^p \right)^{1/p}, \tag{2.6}$$

which is the geometric interpretation of $(v_1, ..., v_n)$-Minkowski distance. In other words, the $(v_1, ..., v_n)$-Minkowski distance between points X and Y is determined by the sum of weighted Euclidean distances from one of the points to hyperplanes through the other point, each of which is perpendicular to one of the vectors $v_1, ..., v_n$. Clearly, for $\lambda_i = 1$ and unit vectors $v_1, ..., v_n$ where $v_{ii} = 1$ and $v_{ij} = 0$ for $i \neq j$, $i, j \in \{1, ..., n\}$, we have $d_{p(v_1,...,v_n)}(X, Y) = d_p(X, Y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$, which is the well-known ***Minkowski*** *(or l_p)* ***distance*** *between the points X and Y, that gives the well-known* ***taxicab***, ***Euclidean***, *and* ***maximum*** *distances denoted by $d_T(X, Y)$, $d_E(X, Y)$, and $d_M(X, Y)$, for $p = 1$, $p = 2$, and $p \to \infty$, respectively (see [6, pp. 94, 301]).*

The following corollaries are clear from the properties of l_p-geometry:

Corollary 2.3 *If W, X and Y, Z are pairs of distinct points such that the lines determined by them are the same or parallel, then $d_{p(v_1,...,v_n)}(W, X)/d_{p(v_1,...,v_n)}(Y, Z) = d_E(W, X)/d_E(Y, Z)$.*

Corollary 2.4 *Circles and spheres of $(v_1, ..., v_n)$-Minkowski distance are symmetric about their center.*

Corollary 2.5 *Translation by any vector preserves $(v_1, ..., v_n)$-Minkowski distance.*

Remark 2.6 *Instead of unit vectors, one can define the $(v_1, ..., v_n)$-Minkowski distance for any linearly independent n vectors $v_1, ..., v_n$ as follows:*

$$d'_{p(v_1,...,v_n)}(X, Y) = \left(\sum_{i=1}^{n} \left(\lambda_i \frac{|v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|}{(v_{i1}^2 + ... + v_{in}^2)^{1/2}}\right)^p\right)^{1/p}, \tag{2.7}$$

or one can define it by unit vectors $v_1, ..., v_n$ and positive real numbers $\mu_1, ..., \mu_n$ as follows:

$$d''_{p(v_1,...,v_n)}(X, Y) = \left(\sum_{i=1}^{n} \mu_i \left(|v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|\right)^p\right)^{1/p}, \tag{2.8}$$

which also determine metric families for $p \geq 1$, generalizing the Minkowski distance. However, then we have

$$d'_{p(v_1,...,v_n)}(X, Y) = d'_{p(k_1 v_1,...,k_n v_n)}(X, Y) \text{ for any } k_i \in \mathbb{R} - \{0\}, \tag{2.9}$$

and

$$d''_{M(v_1,...,v_n)}(X, Y) = \lim_{p\to\infty} d_{p(v_1,...,v_n)}(X, Y) = \max_{i\in\{1,...,n\}} \{|v_{i1}(x_1 - y_1) + ... + v_{in}(x_n - y_n)|\}, \tag{2.10}$$

which is independent of μ_i.

In the next sections, we investigate circles of (v_1, v_2)-Minkowski distance in the real plane for $p = 1$, $p = 2$, and $p \to \infty$, and we call them (v_1, v_2)-taxicab, (v_1, v_2)-Euclidean, and (v_1, v_2)-maximum circles, respectively, having the case of $p = 2$ at the end, in which we use circles of the other two cases. While we investigate the circles, we use the coordinate axes x and y as usual instead of x_1 and x_2, and throughout the paper, we denote by l_1 and l_2 the lines through center C of a (v_1, v_2)-Minkowski circle and perpendicular to unit vectors v_1 and v_2, respectively; that is, $l_i = \Psi_C^{v_i}$.

3. Circles of the generalized taxicab metric in $\mathbb{R}^2$

By Definition 2.1 and Remark 2.2, the (v_1, v_2)-taxicab distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is

$$\begin{aligned} d_{T(v_1,v_2)}(P_1, P_2) &= \lambda_1 \left| v_{11}(x_1 - x_2) + v_{12}(y_1 - y_2) \right| + \lambda_2 \left| v_{21}(x_1 - x_2) + v_{22}(y_1 - y_2) \right| \\ &= \lambda_1 \, d_E(P_2, l_1) + \lambda_2 \, d_E(P_2, l_2), \end{aligned}$$

that is, the sum of weighted Euclidean distances from point P_2 to lines l_1 and l_2, which are passing through P_1 and perpendicular to the vectors v_1 and v_2, respectively. For vectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$, $d_{T(v_1,v_2)}$ in $\mathbb{R}^2$ is the same as the (slightly) generalized taxicab metric (also known as the weighted taxicab metric) defined in [19] (see also [3–5]). In addition, for unit vectors v_1 and v_2 such that $v_1 \perp v_2$ and $v_{12}/v_{11} = m$ where $v_{11} \neq 0$, $d_{T(v_1,v_2)}$ in $\mathbb{R}^2$ is the same as the m-generalized taxicab metric $d_{T_g(m)}$ defined in [2].

Since the generalized taxicab geometry is derived by an affinity from the corresponding l_p-geometry, that is, the taxicab geometry (see [11]), one can easily prove the following theorem, which determines the circles of the generalized taxicab metric $d_{T(v_1,v_2)}$ in $\mathbb{R}^2$. See Figure 1 for examples of the unit (v_1, v_2)-taxicab circles.

Theorem 3.1 *Every (v_1, v_2)-taxicab circle is a parallelogram with the same center, each of whose diagonals is perpendicular to v_1 or v_2. In addition, if $\lambda_1 = \lambda_2$ then it is a rectangle, if $v_1 \perp v_2$ then it is a rhombus, and if $\lambda_1 = \lambda_2$ and $v_1 \perp v_2$ then it is a square.*

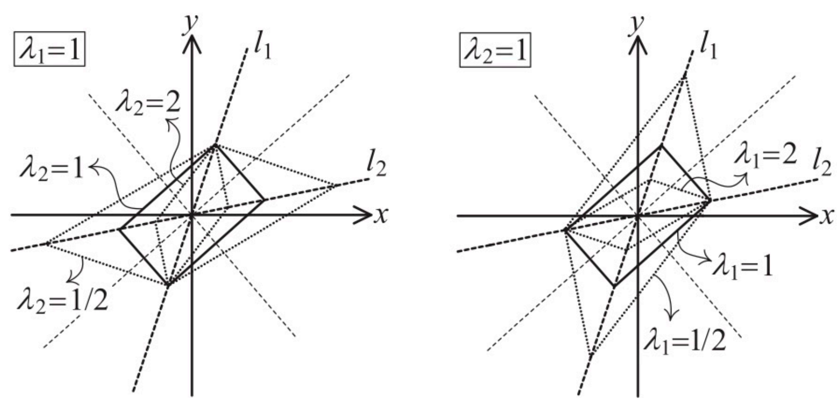

Figure 1. The unit (v_1, v_2)-taxicab circles for $v_1 = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, $v_2 = \left(\frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$.

Now let us consider the case of $\lambda_1 = \lambda_2 = 1$: We know that a (v_1, v_2)-taxicab circle with center C and radius r, that is, the set of all points P satisfying the equation

$$d_E(P, l_1) + d_E(P, l_2) = r,$$

is a rectangle with the same center, whose diagonals are on the lines l_1 and l_2 and sides are parallel to angle bisectors of the lines l_1 and l_2. Besides, if $v_1 \perp v_2$ then the (v_1, v_2)-taxicab circle is a square with the same properties. On the other hand, for a point Q_i on both line l_i and the (v_1, v_2)-taxicab circle (see Figure 2), it is clear that

$$d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.$$

The following theorem shows that every rectangle is a (v_1, v_2)-taxicab circle with the same center for a proper generalized taxicab metric $d_{T(v_1,v_2)}$ with $\lambda_1 = \lambda_2 = 1$:

Theorem 3.2 *Every rectangle with sides of lengths* $2a$ *and* $2b$ *is a* (v_1, v_2)*-taxicab circle with the same center and the radius* $\frac{2ab}{\sqrt{a^2+b^2}}$*, for* $\lambda_1 = \lambda_2 = 1$ *and linearly independent unit vectors* v_1 *and* v_2*, each of which is perpendicular to a diagonal of the rectangle.*

Proof Without loss of generality, let us consider a rectangle with center C and sides of lengths $2a$ and $2b$, as in Figure 3. Denote the diagonal lines of the rectangle by d_1 and d_2. Clearly, C is the intersection point of d_1 and d_2. Draw two lines d_2' and d_2'', each of them passing through a vertex on d_1 and parallel to d_2. Since sides of the rectangle are angle bisectors of pair of lines d_1, d_2' and d_1, d_2'', we have

$$d_E(P, d_1) + d_E(P, d_2) = d_E(d_2, d_2') = d_E(d_2, d_2''). \tag{3.1}$$

On the other hand, for the area of the rectangle we have

$$4ab = 2\sqrt{a^2 + b^2}\, d_E(d_2, d_2'), \text{ and so } d_E(d_2, d_2') = \frac{2ab}{\sqrt{a^2 + b^2}}. \tag{3.2}$$

Then, for every point P on the rectangle, we have

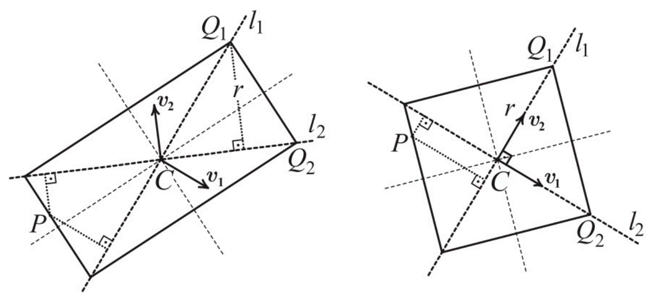

Figure 2. (v_1, v_2)-taxicab circles with center C and radius r, for $\lambda_1 = \lambda_2 = 1$.

$$d_E(P, d_1) + d_E(P, d_2) = \frac{2ab}{\sqrt{a^2 + b^2}}. \tag{3.3}$$

Thus, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to a diagonal, the rectangle is a (v_1, v_2)-taxicab circle with center C and radius $2ab/\sqrt{a^2 + b^2}$. □

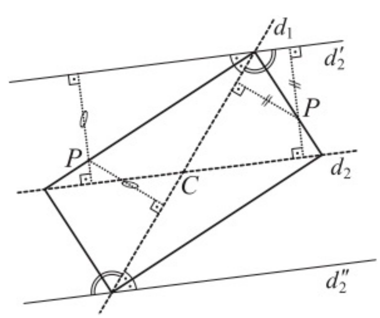

Figure 3. A rectangle with center C and sides of lengths $2a$ and $2b$.

4. Circles of the generalized maximum metric in $\mathbb{R}^2$

By Definition 2.1 and Remark 2.2, the (v_1, v_2)-maximum distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is

$$\begin{aligned} d_{M(v_1,v_2)}(P_1, P_2) &= \max\{\lambda_1|v_{11}(x_1-x_2)+v_{12}(y_1-y_2)|\,, \lambda_2|v_{21}(x_1-x_2)+v_{22}(y_1-y_2)|\} \\ &= \max\{\lambda_1\, d_E(P_2, l_1), \lambda_2\, d_E(P_2, l_2)\}, \end{aligned}$$

that is, the maximum of weighted Euclidean distances from point P_2 to lines l_1 and l_2, which are passing through P_1 and perpendicular to the vectors v_1 and v_2, respectively.

Since the generalized maximum geometry is derived by an affinity from the corresponding l_p-geometry, that is, the maximum geometry (see [15]), one can easily prove the following theorem, which determines the circles of the generalized maximum metric $d_{M(v_1,v_2)}$ in $\mathbb{R}^2$. See Figure 4 for examples of the unit (v_1, v_2)-maximum circles.

Theorem 4.1 *Every (v_1, v_2)-maximum circle is a parallelogram with the same center, each of whose sides is perpendicular to v_1 or v_2. In addition, if $\lambda_1 = \lambda_2$ then it is a rhombus, if $v_1 \perp v_2$ then it is a rectangle, and if $\lambda_1 = \lambda_2$ and $v_1 \perp v_2$ then it is a square.*

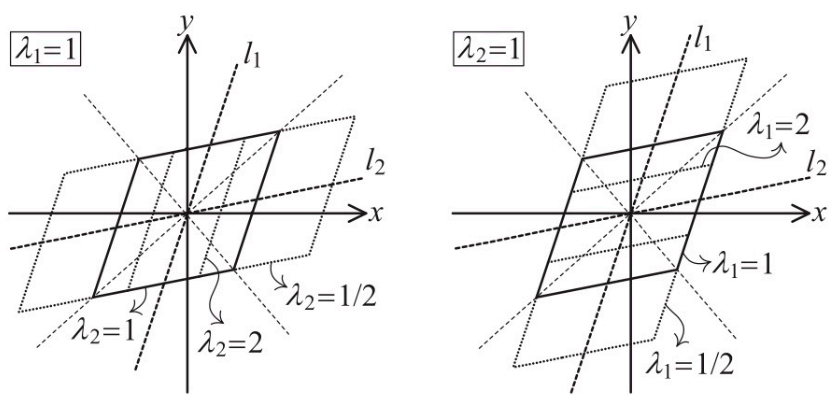

Figure 4. The unit (v_1, v_2)-maximum circles for $v_1 = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, $v_2 = \left(\frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$.

Now let us consider the case of $\lambda_1 = \lambda_2 = 1$: We know that a (v_1, v_2)-maximum circle with center C and radius r, that is, the set of all points P satisfying the equation

$$\max\{d_E(P, l_1), d_E(P, l_2)\} = r,$$

is a rhombus with the same center, whose sides are parallel to the lines l_1 and l_2, and whose diagonals are on angle bisectors of the lines l_1 and l_2. Besides, if $v_1 \perp v_2$ then the (v_1, v_2)-maximum circle is a square with the same properties. On the other hand, for a point Q_i on both line l_i and the (v_1, v_2)-maximum circle (see Figure 5), it is clear that

$$d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.$$

The following theorem shows that every rhombus is a (v_1, v_2)-maximum circle with the same center for a proper generalized maximum metric $d_{M(v_1,v_2)}$ with $\lambda_1 = \lambda_2 = 1$:

Theorem 4.2 *Every rhombus with diagonals of lengths* $2e$ *and* $2f$ *is a* (v_1, v_2)*-maximum circle with the same center and the radius* $\frac{ef}{\sqrt{e^2+f^2}}$*, for* $\lambda_1 = \lambda_2 = 1$ *and linearly independent unit vectors* v_1 *and* v_2*, each of which is perpendicular to a side of the rhombus.*

Proof Without loss of generality, let us consider a rhombus with center C and diagonals of lengths $2e$ and $2f$, as in Figure 6. Denote by d_1 and d_2 two distinct lines each through C and parallel to a side of the rhombus. Since diagonals are angle bisectors of consecutive sides, we have

$$\max\{d_E(P, d_1), d_E(P, d_2)\} = d_E(V, d_1) = d_E(V, d_2) \tag{4.1}$$

for any vertex V of the rhombus. On the other hand, for the area of the rhombus we have

$$2ef = 2\sqrt{e^2 + f^2}\, d_E(V, d_1), \text{ and so } d_E(V, d_1) = \frac{ef}{\sqrt{e^2 + f^2}}. \tag{4.2}$$

Then, for every point P on the rhombus, we have

$$\max\{d_E(P, d_1), d_E(P, d_2)\} = \frac{ef}{\sqrt{e^2 + f^2}}. \tag{4.3}$$

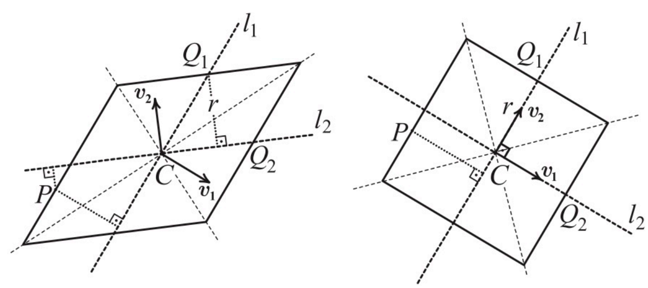

Figure 5. (v_1, v_2)-maximum circles with center C and radius r, for $\lambda_1 = \lambda_2 = 1$.

Thus, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to a side, the rhombus is a (v_1, v_2)-maximum circle having center C and radius $ef/\sqrt{e^2 + f^2}$. □

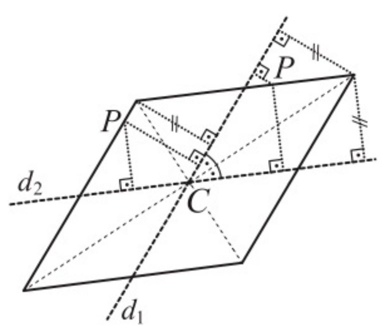

Figure 6. A rhombus with center C and diagonals of lengths $2e$ and $2f$.

5. Circles of the generalized Euclidean metric in $\mathbb{R}^2$

By Definition 2.1 and Remark 2.2, the (v_1, v_2)-Euclidean distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ is

$$\begin{aligned} d_{E(v_1,v_2)}(P_1, P_2) &= \left[(\lambda_1|v_{11}(x_1-x_2)+v_{12}(y_1-y_2)|)^2+(\lambda_2|v_{21}(x_1-x_2)+v_{22}(y_1-y_2)|)^2\right]^{1/2} \\ &= \left[(\lambda_1\, d_E(P_2, l_1))^2 + (\lambda_2\, d_E(P_2, l_2))^2\right]^{1/2}, \end{aligned}$$

that is, the square root of the sum of the square of weighted Euclidean distances from the points P_2 to the lines l_1 and l_2, which are passing through P_1 and perpendicular to the vectors v_1 and v_2, respectively. Notice that by the Pythagorean theorem, for $\lambda_1 = \lambda_2 = 1$ and perpendicular unit vectors v_1 and v_2, we have

$$d_{E(v_1,v_2)}(P_1, P_2) = d_E(P_1, P_2). \tag{5.1}$$

Since the generalized Euclidean geometry is derived by an affinity from the corresponding l_p-geometry, that is, the Euclidean geometry, one can easily prove the following theorem, which determines the circles of the generalized Euclidean metric $d_{E(v_1,v_2)}$ in the real plane. See Figure 7 for examples of the unit (v_1, v_2)-Euclidean circles.

Theorem 5.1 *Every (v_1, v_2)-Euclidean circle is an ellipse with the same center. In addition, if $\lambda_1 = \lambda_2$ then its axes are angle bisectors of the lines l_1 and l_2, if $v_1 \perp v_2$ then its axes are the lines l_1 and l_2, and if $\lambda_1 = \lambda_2$ and $v_1 \perp v_2$ then it is a Euclidean circle with the same center.*

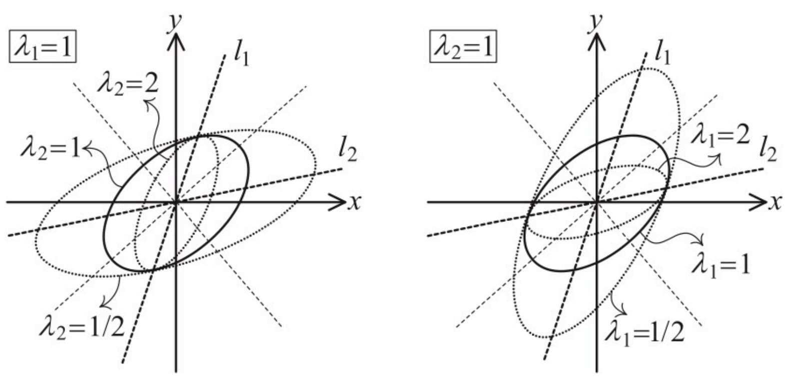

Figure 7. The unit (v_1, v_2)-Euclidean circles for $v_1 = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, $v_2 = \left(\frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$.

Now let us consider the case of $\lambda_1 = \lambda_2 = 1$: We know that a (v_1, v_2)-Euclidean circle with center C and radius r, that is, the set of all points P satisfying the equation

$$\left[(d_E(P, l_1))^2 + (d_E(P, l_2))^2\right]^{1/2} = r,$$

is an ellipse with the same center, whose axes are angle bisectors of the lines l_1 and l_2, such that the major axis is the angle bisector of the nonobtuse angle between l_1 and l_2. In addition, if $v_1 \perp v_2$ then (v_1, v_2)-Euclidean circle is a Euclidean circle having the same center and the radius. In addition, for a point Q_i on both line l_i and the (v_1, v_2)-Euclidean circle (see Figure 8), it is clear that

$$d_E(Q_1, l_2) = d_E(Q_2, l_1) = r.$$

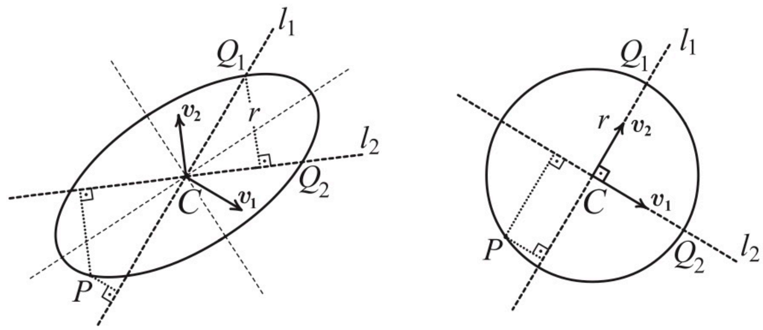

Figure 8. (v_1, v_2)-Euclidean circles with center C and radius r, for $\lambda_1 = \lambda_2 = 1$.

The following theorem determines some relations between parameters of a (v_1, v_2)-Euclidean circle and the ellipse related to it:

Theorem 5.2 *If a* (v_1, v_2)*-Euclidean circle with radius* r *for* $\lambda_1 = \lambda_2 = 1$ *is an ellipse with the same center, having semimajor axis* a *and semiminor axis* b*, then*

$$r = \frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}, \quad a = \frac{r}{\sqrt{1-\cos\theta}}, \quad \textit{and} \quad b = \frac{r}{\sqrt{1+\cos\theta}},$$

where θ *is the nonobtuse angle between* v_1 *and* v_2*, and* $\cos\theta = |v_{11}v_{21} + v_{12}v_{22}|$.

Proof Let a (v_1, v_2)-Euclidean circle with radius r for $\lambda_1 = \lambda_2 = 1$ be an ellipse with the same center, having semimajor axis a and semiminor axis b, and let θ be the nonobtuse angle between v_1 and v_2. Then the nonobtuse angle between lines l_1 and l_2 is equal to θ, and the axes of the ellipse are angle bisectors of the lines l_1 and l_2, such that the major axis of it is the angle bisector of the nonobtuse angle between l_1 and l_2. Using similar right triangles whose hypotenuses are a and b (see Figure 9), one gets $\sin\frac{\theta}{2} = \frac{r}{a\sqrt{2}}$, $\cos\frac{\theta}{2} = \frac{r}{b\sqrt{2}}$, $\tan\frac{\theta}{2} = \frac{r}{\sqrt{2a^2-r^2}} = \frac{\sqrt{2b^2-r^2}}{r}$, and so $\tan\frac{\theta}{2} = \frac{b}{a}$, $\sin\theta = \frac{r^2}{ab}$, and $\cos\theta = 1 - \frac{r^2}{a^2} = \frac{r^2}{b^2} - 1$. Then we have

$$r = \frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}, \; a = \frac{r}{\sqrt{1-\cos\theta}}, \quad \text{and} \quad b = \frac{r}{\sqrt{1+\cos\theta}}.$$

Besides, one can derive that

$$\sin\theta = \frac{2ab}{\sqrt{a^2+b^2}}, \; \cos\theta = \frac{a^2-b^2}{a^2+b^2}, \text{ and } \tan\theta = \frac{2ab}{a^2-b^2}.$$

In addition, by $|\langle v_1, v_2\rangle| = \|v_1\| \, \|v_2\| \cos\theta$ it follows immediately that $\cos\theta = |v_{11}v_{21} + v_{12}v_{22}|$ and $\sin\theta = |v_{11}v_{22} - v_{12}v_{21}|$. □

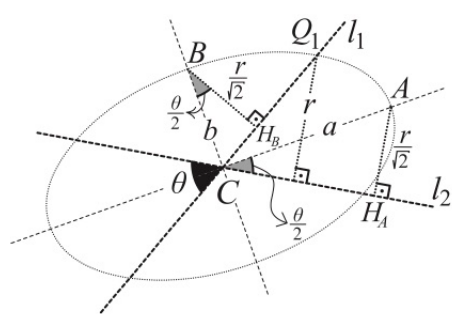

Figure 9. A (v_1, v_2)-Euclidean circle and an ellipse that are the same.

Notice that r^2 is the harmonic mean of a^2 and b^2, so we have $b \leq r \leq a$, where the equality holds only for the case $a = b = r$. Another fact is that the chords derived by lines l_1 and l_2 have the same length, and if $d_E(C, Q_i) = R$ then $\sin\theta = \frac{r}{R}$, and we get $R = \sqrt{a^2 + b^2}/\sqrt{2}$ and $Rr = ab$. However, since chords derived by lines l_1 and l_2 are conjugate diameters by the following theorem, the last two equalities can also be derived by the first and the second theorems of Appollonius, which are:

(1) The sum of the squares of any two conjugate semidiameters is equal to $a^2 + b^2$.

(2) The area of the parallelogram determined by two coterminous conjugate semidiameters is equal to ab (see [12, pp. 1800–1803]).

Theorem 5.3 *The chords derived by lines l_1 and l_2 are conjugate diameters of the ellipse.*

Proof We know that diameters parallel to any pair of supplemental chords (which are formed by joining the extremities of any diameter to a point lying on the ellipse) are conjugate (see [12, p. 1805]). Since l_1 and l_2 are parallel to a pair of supplemental chords formed by joining the extremities of the minor axis to one of the extremities of the major axis, the chords derived by lines l_1 and l_2 are conjugate diameters of the ellipse. □

Remark 5.4 *Since the chords derived by lines l_1 and l_2 are conjugate, l_1 is parallel to the tangent lines through the extremities of the chord determined by l_2, and vice versa. It is clear that the tangent lines through the extremities of these conjugate diameters determine a rhombus with sides of length $2R$, circumscribing the ellipse. Since l_1 and l_2 are symmetric about the axes of the ellipse, the diagonals of the rhombus are on the axes of the ellipse, and they have lengths $2\sqrt{2}a$ and $2\sqrt{2}b$. Similarly, the chords determined by the axes of the ellipse are also conjugate, and the tangent lines through the extremities of these conjugate diameters determine a rectangle with sides of lengths $2a$ and $2b$, circumscribing the ellipse. Since $\tan\frac{\theta}{2} = \frac{b}{a}$, the diagonals of this rectangle are on the lines l_1 and l_2, and they have length $2\sqrt{2}R$ (see Figure 10). Notice that there is an ellipse similar to the prior, through eight vertices of these rectangle and the rhombus, whose semimajor and semiminor axes are equal to $\sqrt{2}a$ and $\sqrt{2}b$, respectively (see Figure 10).*

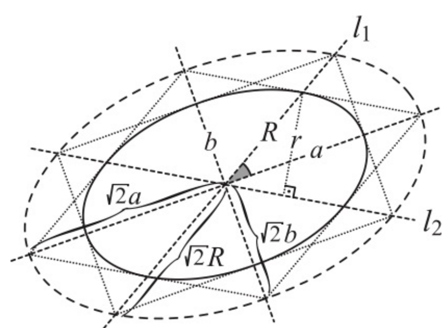

Figure 10. The rectangle and the rhombus determined by an ellipse.

Remark 5.5 *Observe that the rhombus derived by the tangent lines of a (v_1, v_2)-Euclidean circle, through the extremities of the conjugate diameters determined by the lines l_1 and l_2, is the (v_1, v_2)-maximum circle, and the rectangle whose vertices are the midpoints of sides of this rhombus is the (v_1, v_2)-taxicab circle, having the same center and radius of the (v_1, v_2)-Euclidean circle, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2. Figure 11 illustrates (v_1, v_2)-taxicab, (v_1, v_2)-Euclidean, and (v_1, v_2)-maximum circles with the same center and radius for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to one of the lines l_1 and l_2.*

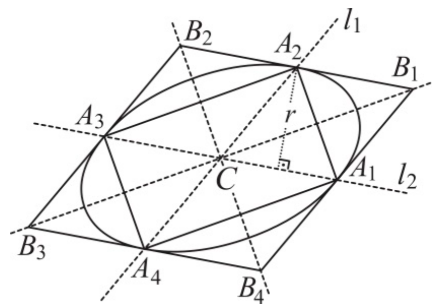

Figure 11. (v_1, v_2)-Minkowski circles with the same center and radius.

The following theorem shows that every ellipse is a (v_1, v_2)-Euclidean circle with the same center for a proper generalized Euclidean metric $d_{E(v_1,v_2)}$ with $\lambda_1 = \lambda_2 = 1$:

Theorem 5.6 *Every ellipse with a semimajor axis a and semiminor axis b is a (v_1, v_2)-Euclidean circle with the same center and the radius $\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to a diagonal line of the rectangle circumscribing the ellipse, whose sides are parallel to the axes of the ellipse.*

Proof Since Euclidean distances are preserved under rigid motions, without loss of generality, let us consider the ellipse with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Take the lines $l_1 : bx - ay = 0$ and $l_2 : bx + ay = 0$ passing through the origin, since they are diagonal lines of the rectangle circumscribing the ellipse, whose sides are parallel to the axes of the ellipse. Thus, for every point $P = (x_0, y_0)$ on the ellipse, we have

$$[(d_E(P, l_1))^2 + (d_E(P, l_2))^2]^{1/2} = \left[\frac{2(bx_0^2 + ay_0^2)}{a^2 + b^2}\right]^{1/2} = \frac{\sqrt{2}ab}{\sqrt{a^2 + b^2}}, \tag{5.2}$$

which is a constant. Thus, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to one of the lines l_1 and l_2, the ellipse is a (v_1, v_2)-Euclidean circle with center O and radius $\sqrt{2}ab/\sqrt{a^2 + b^2}$. □

Now, by Theorem 5.1 and Theorem 5.2, for any positive real number r and two distinct lines l_1 and l_2 intersecting at a point C, every point P satisfying the equation

$$(d_E(P, l_1))^2 + (d_E(P, l_2))^2 = r^2 \tag{5.3}$$

is on an ellipse with center C, semimajor axis $a = \frac{r}{\sqrt{1-\cos\theta}}$, and semiminor axes $b = \frac{r}{\sqrt{1+\cos\theta}}$, where θ is the nonobtuse angle between lines l_1 and l_2, and lines l_1 and l_2 are diagonal lines of the rectangle circumscribing the ellipse, whose sides are parallel to the axes of the ellipse. Conversely, by Theorem 5.6, any point P on this ellipse satisfies the following equation:

$$(d_E(P, l_1))^2 + (d_E(P, l_2))^2 = \frac{2\frac{r^2}{1-\cos\theta}\frac{r^2}{1+\cos\theta}}{\frac{r^2}{1-\cos\theta} + \frac{r^2}{1+\cos\theta}} = r^2. \tag{5.4}$$

Obviously, for every ellipse, there is a unique pair of lines l_1 and l_2, and there is a unique constant r^2, which is the square of the distance from an intersection point of the ellipse and one of the lines l_1 and l_2 to the other one of them. Notice that we discover a new definition of the ellipse:

Definition 5.7 *In the Euclidean plane, an* ellipse *is the set of all points of the sum of squares whose distances to two intersecting fixed lines are constant. We call each such fixed line an* eccentrix *of the ellipse, and we call the chord determined by an eccentrix the* eccentric diameter *of the ellipse, and half of an eccentric diameter the* eccentric radius *of the ellipse.*

Clearly, eccentrices of an ellipse determine the eccentricity and thus the shape of the ellipse, and vice versa, since the eccentricity is

$$e = \sqrt{1 - \tfrac{b^2}{a^2}} = \sqrt{1 - \tan^2 \tfrac{\theta}{2}}, \tag{5.5}$$

where θ is the nonobtuse angle between the eccentrices. Notice that ellipses with the same eccentrices, or more generally ellipses having the same angle between their eccentrices, are similar, since they have the same eccentricity (see Figure 12).

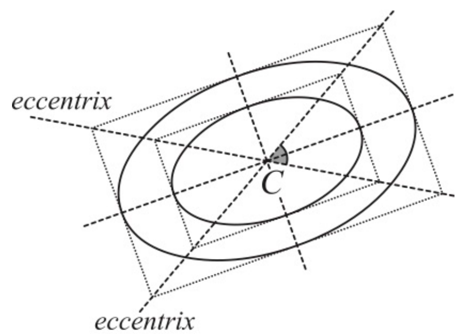

Figure 12. Ellipses with the same eccentrices.

Notice that, in Sections 3 and 4, "two-eccentrices" definitions and similar properties can also be given for the rectangle and rhombus. We leave them to the reader, and we give the following fundamental conclusions related to this new "two-eccentrices" definition of the ellipse:

Corollary 5.8 *Given a constant $c \in \mathbb{R}^+$ and two fixed lines l_1 and l_2 intersecting at a point C, having the nonobtuse angle θ between them, the ellipse with constant c and eccentrices l_1 and l_2 is the ellipse with center C, having semimajor axis $a = \frac{\sqrt{c}}{\sqrt{1-\cos\theta}}$ and semiminor axis $b = \frac{\sqrt{c}}{\sqrt{1+\cos\theta}}$, such that the major axis of the ellipse is the angle bisector of θ. In addition, this ellipse is a (v_1, v_2)-Euclidean circle with respect to the (v_1, v_2)-Euclidean metric, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to one of the eccentrices of the ellipse, having center C and radius $\sqrt{c}$, which is the Euclidean distance from the intersection point of an eccentrix and the ellipse to the other eccentrix.*

Corollary 5.9 *Given an ellipse with center C, semimajor axis a, and semiminor axis b, the eccentrices of this ellipse are diagonal lines of the rectangle circumscribing the ellipse whose sides are parallel to the axes of the ellipse, and the constant of this ellipse is $\frac{2a^2b^2}{a^2+b^2}$, which is the square of the Euclidean distance from the intersection point of an eccentrix and the ellipse to the other eccentrix. In addition, this ellipse is a (v_1, v_2)-Euclidean circle with center C and radius $\frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$, with respect to the (v_1, v_2)-Euclidean metric, for $\lambda_1 = \lambda_2 = 1$ and linearly independent unit vectors v_1 and v_2, each of which is perpendicular to one of the eccentrices of the ellipse.*

Corollary 5.10 *A (v_1, v_2)-Euclidean circle with center C and radius r is an ellipse whose constant is r^2 and eccentrices are the lines through C and perpendicular to v_1 and v_2. In addition, the semimajor and semiminor axes of this ellipse are $a = \frac{r}{\sqrt{1-\cos\theta}}$ and $b = \frac{r}{\sqrt{1+\cos\theta}}$ where $\cos\theta = |v_{11}v_{21} + v_{12}v_{22}|$.*

Remark 5.11 *Clearly, an ellipse can be determined uniquely by its axes, semimajor axis a and minor axis b, or simply a rectangle with sides of lengths $2a$ and $2b$, whose sides are parallel to the axes or the ellipse. Here, we see that it can also be determined uniquely by its eccentrices with the angle between them and eccentric radius R, or simply a rhombus with sides of length $2R$, whose sides are parallel to the eccentrices of the ellipse, having the relation $R = \sqrt{a^2 + b^2}/\sqrt{2}$ (see Figure 13). While diagonals of the rectangle (whose length is equal to $2\sqrt{2}R$) give eccentrices of the ellipse, diagonals of the rhombus (whose lengths are equal to $2\sqrt{2}a$ and $2\sqrt{2}b$) give axes of the ellipse (see Figure 10). Obviously, when such a rectangle is given, one can construct the related rhombus, and vice versa. Notice that four intersection points of diagonals of the rectangle and sides of the rhombus,and four intersection points of diagonals of the rhombus and sides of the rectangle, are on the ellipse (see also [9, 13] for construction ofan ellipse from a pair of conjugate diameters).*

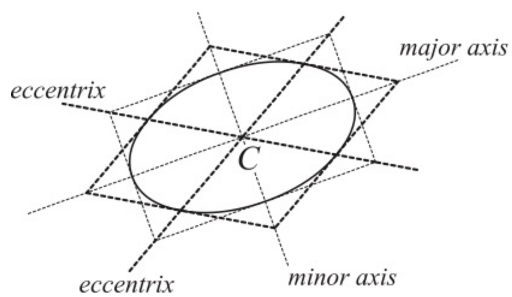

Figure 13. Axes and eccentrices of an ellipse.

Here, we have two important questions that remain to be answered. The first one is: What is the set of all points of the difference of squares whose distances to two intersecting fixed lines are constant, and does this set determine a definition for the hyperbola? It is not difficult to see that for any two intersecting fixed lines this set determines a pair of conjugate hyperbolas having perpendicular asymptotes, which are the angle bisectors of the fixed lines. Let us consider the same lines $l_i : v_{i1}x + v_{i2}y = 0$ for linear independent unit vectors v_i where $i = 1, 2$, and the point $X = (x, y)$ satisfying the condition

$$\left|(d_E(X, l_1))^2 - (d_E(X, l_2))^2\right| = k \tag{5.6}$$

for $k \in \mathbb{R}^+$, which gives the equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0, \tag{5.7}$$

where $A = (v_{11}^2 - v_{21}^2)$, $B = (v_{12}^2 - v_{22}^2)$, $C = 2(v_{11}v_{12} - v_{21}v_{22})$, $D = E = 0$, and $F = \mp k$. Since $\Delta \neq 0$ and $\delta < 0$, this equation determines two hyperbolas (see [20, pp. 232–233]). Moreover, by the theorem given in [18], if $v_{12} \neq v_{22}$ then slopes m_1 and m_2 of the asymptotes of these hyperbolas are the distinct real roots of the same quadratic equation:

$$(v_{12}^2 - v_{22}^2)m^2 + 2(v_{11}v_{12} - v_{21}v_{22})m + (v_{11}^2 - v_{21}^2) = 0, \tag{5.8}$$

and if $v_{12} = v_{22}$ then the hyperbolas have the same asymptotes and one of them is vertical and the other one is horizontal. By determining the roots of equation (5.8), one can also see that the asymptotes are the angle bisectors or the lines l_1 and l_2. Thus, we get a pair of conjugate hyperbolas having perpendicular asymptotes, which are the angle bisectors of the fixed lines l_1 and l_2, and not a definition of the hyperbola.

Then the next question is as follows: Can a similar definition be given for the hyperbola, related to the same two intersecting fixed lines, or in other words, a "two-eccentrices" definition of the other central conic? The answer is "yes" as we expected it to be, since all known definitions of central conics exist for both ellipse and hyperbola. Fortunately, determining this new definition will be easier. In [14], it was shown that every hyperbola has the following properties:

(1) For any point P on hyperbola with the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the product of distances from P to the asymptotes (having equations $bx - ay = 0$ and $bx + ay = 0$) is constant $\frac{a^2b^2}{a^2+b^2} = \frac{b^2}{e^2}$, where e is the eccentricity.

(2) The set of points whose product of distances to the lines $bx - ay = 0$ and $bx + ay = 0$ is constant $\frac{a^2b^2}{a^2+b^2}$ consists of the union of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Even though they do not state or indicate that these properties determine a new definition for hyperbolas, with a little arrangement, these properties obviously give the following definition that we seek:

Definition 5.12 *In the Euclidean plane, a* hyperbola *is the set of all such points in only one of the two pairs of opposite regions determined by two intersecting fixed lines for the product whose distances to these two intersecting fixed lines is constant.*

By the properties above, it is clear that the two intersecting fixed lines are the asymptotes of the hyperbola. However, we can see them also as *eccentrices* of the hyperbola, since they determine the eccentricity and thus the shape of the hyperbola, as we do for the ellipse. Then we have the "two-eccentrices" definition of the hyperbola, which completes the "two-eccentrices" definition of central conics in the Euclidean plane.

Remark 5.13 *The eccentrices of the ellipse with the equation* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and the asymptotes (or the eccentrices) of conjugate hyperbolas with the equations* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *and* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ *are the same (see Figure 14).*

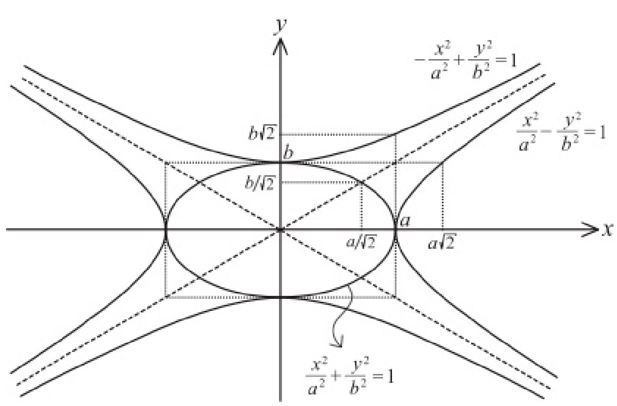

Figure 14. Central conics determined by the same semiaxes a and b or the same eccentrices $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, and constants $\frac{2a^2b^2}{a^2+b^2}$ and $\frac{a^2b^2}{a^2+b^2}$ for ellipse and hyperbola, respectively.

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