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On \mathbf{H} -curvature of (α, β) -metrics

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Abstract: The non-Riemannian quantity \mathbf{H} was introduced by Akbar-Zadeh to characterization of Finsler metrics of constant flag curvature. In this paper, we study two important subclasses of Finsler metrics in the class of so-called (α, β) -metrics, which are defined by $F = \alpha\varphi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a closed 1-form on a manifold. We prove that every polynomial metric of degree $k \geq 3$ and exponential metric has almost vanishing \mathbf{H} -curvature if and only if $\mathbf{H} = 0$. In this case, F reduces to a Berwald metric. Then we prove that every Einstein polynomial metric of degree $k \geq 3$ and exponential metric satisfies $\mathbf{H} = 0$. In this case, F is a Berwald metric.

Key words: Polynomial metrics, exponential metric, almost vanishing \mathbf{H} -curvature.

1. Introduction

Let (M, F) be a Finsler manifold. Then a global vector field \mathbf{G} is induced by the Finsler metric F on slit tangent bundle TM_0 , which in a standard coordinate (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are scalar functions on TM_0 [8].

In [1], Akbar-Zadeh considered a non-Riemannian quantity \mathbf{H} obtained from the mean Berwald curvature \mathbf{E} by the covariant horizontal differentiation along geodesics. He proved that for a Weyl metric, the flag curvature \mathbf{K} is a scalar function on the manifold $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{H} = \mathbf{0}$ [7]. The quantity $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$ is defined as the covariant derivative of mean Berwald curvature along geodesics. In local coordinates,

$$H_{ij} = \frac{1}{2} \left[y^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial x^m} - 2G^m \frac{\partial^4 G^k}{\partial y^i \partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^i} \frac{\partial^3 G^k}{\partial y^j \partial y^k \partial y^m} - \frac{\partial G^m}{\partial y^j} \frac{\partial^4 G^k}{\partial y^i \partial y^k \partial y^m} \right].$$

A Finsler metric F on an n -dimensional manifold M is called of almost vanishing H -curvature if

$$\mathbf{H} = \frac{n+1}{2} F^{-1} \theta \mathbf{h}, \quad (1.1)$$

where $\theta := \theta_i(x) y^i$ is a 1-form on M and $\mathbf{h} = h_{ij} dx^i \otimes dx^j$ is the angular tensor [7].

Najafi et al. [6] proved that every R-quadratic metric satisfies $\mathbf{H} = 0$. Then, Najafi et al. [7] generalized the Akbar-Zadeh theorem and proved that a Finsler metric F has almost isotropic flag curvature $\mathbf{K} = 3\theta/F + \sigma$ if and only if it has almost vanishing H -curvature, where $\theta = \theta_i(x) y^i$ is a 1-form and $\sigma = \sigma(x)$ is a scalar

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function on manifold. Mo [4] found a new equation between H -curvature and Riemannian curvature on a Finsler manifold. Tayebi and Najafi [13] showed that every m -th root metric with almost vanishing H -curvature satisfies $\mathbf{H} = 0$. Moreover, Xia [23] proved that a Randers metric has almost isotropic S -curvature if and only if it is of almost vanishing H -curvature. Recently, Zohrehvand and Rezaii [26] have obtained necessary and sufficient conditions for a square metric to be of almost vanishing H -curvature.

Randers metric and square metric belong to the class of (α, β) -metrics. Therefore, in order to find explicit examples of Finsler metrics of almost vanishing H -curvature, we consider (α, β) -metrics. An (α, β) -metric is a Finsler metric of the form $F := \alpha\varphi(s)$, $s = \beta/\alpha$, where $\varphi = \varphi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M (see [9, 14, 16, 18–20, 22]). A polynomial (α, β) -metric of degree k is given by $\varphi := (1 + s)^k$, $s = \beta/\alpha$, where $k \in \mathbb{N}$. This class of metrics contains Randers metrics ($k = 1$) and square metrics ($k = 2$) as special cases. In this paper, we consider polynomial (α, β) -metrics with almost vanishing H -curvature and prove the following.

Theorem 1.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α, β) -metric of degree m ($m \geq 3$) on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M . Then F has almost vanishing H -curvature if and only if $\mathbf{H} = 0$. In this case, F is a Berwald metric.*

Example 1.2 *Theorem 1.1 does not hold for the polynomial (α, β) -metrics of degree 1, i.e. the Randers-type Finsler metrics. For example, the standard Funk metric on the Euclidean unit ball is defined by*

$$F(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 - |x|^2} + \frac{\langle x, y \rangle}{1 - |x|^2}, \quad y \in T_x B^n(1) \simeq \mathbb{R}^n,$$

where \langle, \rangle and $|\cdot|$ denote the Euclidean inner product and norm on \mathbb{R}^n , respectively. F is a Randers metric and it is easy to see that β is a closed 1-form. By a simple calculation, it follows that $\mathbf{H} = 0$ while F is not Berwald metric.

For an (α, β) -metric $F := \alpha\varphi(s)$, $s = \beta/\alpha$, let us define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma_{ik}^j dx^k$ denote the Levi-Civita connection forms of α . Let

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ r_j &:= b^i r_{ij}, & r &:= b^i b^j r_{ij}, & s_j &:= b^i s_{ij}, & r_0 &:= r_j y^j, & s_0 &:= s_j y^j, \\ r_{i0} &:= r_{ij} y^j, & r_{00} &:= r_{ij} y^i y^j, & s_{i0} &:= s_{ij} y^j, & s^i_j &:= a^{im} s_{mj}, & r^i_j &:= a^{im} r_{mj}, \\ q_{ij} &:= r_{im} s^m_j, & t_{ij} &:= s_{im} s^m_j, & q_i &:= b^i q_{ij}, & t_i &:= b^i t_{ij}. \end{aligned}$$

Now, we can give another example.

Example 1.3 *Theorem 1.1 does not hold for the polynomial (α, β) -metrics of degree 2, generally. For example, let $F = (\alpha + \beta)^2/\alpha$ be the square metric defined by following*

$$\alpha := \frac{\sqrt{|y|^2(1 - |x|^2) + \langle x, y \rangle^2}}{(1 - |x|^2)^2}, \quad \beta := \frac{\langle x, y \rangle}{(1 - |x|^2)^2}. \tag{1.2}$$

F is an (α, β) -metric on the unit ball $\mathbb{B}^n(1) \subset \mathbb{R}^n$. We have

$$b_{i|j} = 2\tau\{(1 + 2b^2)a_{ij} - 3b_i b_j\},$$

where $\tau = (1 - |x|^2)/2$. Thus, β is closed with respect to α . F has constant flag curvature then it satisfies $\mathbf{H} = 0$. Since β is not parallel with respect to α , then F is not a Berwald metric.

Example 1.4 For polynomial (α, β) -metric $F = (\alpha + \beta)^3/\alpha^2$, we have

$$\Theta = \frac{3(4s - 1)}{2(8s^2 - 6B + s - 1)}, \quad Q = \frac{3}{1 - 2s}, \quad \Psi = \frac{3}{-8s^2 + 6B - s + 1}.$$

Suppose that F has almost vanishing H -curvature (1.1). Since β is a closed 1-form, then

$$\begin{aligned} 2H_{jk} = & \left[\frac{h_3}{A^6\alpha^4}r_{00}^2 + \frac{h_4}{A^4\alpha^3}r_{00|0} + \frac{h_5}{A^5\alpha^3}r_{00}r_0 + \frac{h_6}{A^3\alpha^2}r_{0|0} + \frac{h_{13}}{A^4\alpha^2}rr_{00} + \frac{h_{14}}{A^4\alpha^2}r_0^2 \right] b_j b_k \\ & + \left[\frac{h_{17}}{A^6\alpha^4}r_{00}^2 + \frac{h_{18}}{A^4\alpha^3}r_{00|0} + \frac{h_{19}}{A^5\alpha^3}r_{00}r_0 + \frac{h_{20}}{A^3\alpha^2}r_{0|0} + \frac{h_{27}}{A^4\alpha^2}rr_{00} + \frac{h_{28}}{A^4\alpha^2}r_0^2 \right] l_j l_k \\ & + \left[\frac{h_{31}}{A^5\alpha^4}r_{00}^2 + \frac{h_{32}}{A^3\alpha^3}r_{00|0} + \frac{h_{33}}{A^4\alpha^3}r_{00}r_0 + \frac{h_{34}}{A^2\alpha^2}r_{0|0} + \frac{h_{41}}{A^3\alpha^2}rr_{00} + \frac{h_{42}}{A^3\alpha^2}r_0^2 \right] a_{jk} \\ & + \frac{h_{43}}{A^2\alpha}r_{jk|0} + \left[\frac{h_{45}}{A^4\alpha^2}r_{00} + \frac{h_{46}}{A^3\alpha}r_0 \right] r_{jk} + \frac{h_{48}}{A^4\alpha^2}r_{0j}r_{0k} + \left[\frac{h_{51}}{A^6\alpha^4}r_{00}^2 + \frac{h_{52}}{A^4\alpha^3}r_{00|0} \right. \\ & + \frac{h_{53}}{A^5\alpha^3}r_{00}r_0 + \frac{h_{54}}{A^3\alpha^2}r_{0|0} + \frac{h_{61}}{A^4\alpha^2}rr_{00} + \left. \frac{h_{62}}{A^4\alpha^2}r_0^2 \right] [l_k b_j + l_j b_k] \\ & + \left[\frac{h_{71}}{A^4\alpha^2}r_{00} + \frac{h_{72}}{A^3\alpha}r_0 \right] [l_k r_j + l_j r_k] + \left[\frac{h_{74}}{A^5\alpha^3}r_{00} + \frac{h_{75}}{A^4\alpha^2}r_0 + \frac{h_{76}}{A^3\alpha}r \right] (l_k r_{0j} + l_j r_{0k}) \\ & + \left[\frac{h_{89}}{A^4\alpha^2}r_{00} + \frac{h_{90}}{A^3\alpha}r_0 \right] [b_k r_j + b_j r_k] + \left[\frac{h_{92}}{A^5\alpha^3}r_{00} + \frac{h_{93}}{A^4\alpha^2}r_0 + \frac{h_{94}}{A^3\alpha}r \right] [b_k r_{0j} + b_j r_{0k}] \\ & + \frac{h_{104}}{A^3\alpha} [r_k r_{j0} + r_j r_{k0}] + \frac{h_{106}}{A^3\alpha^2} [l_k r_{0j|0} + l_j r_{0k|0}] + \frac{h_{107}}{A^3\alpha^2} [r_{0j|0} b_k + r_{0k|0} b_j] \\ & + \frac{h_{108}}{A^2\alpha} [r_{k|0} b_j + r_{j|0} b_k] + \frac{h_{109}}{A^2\alpha} [l_k r_{j|0} + l_j r_{k|0}], \end{aligned}$$

where $A := 1 + 6B + 6Bs - 9s^2 - 8s^3$, $B := \|\beta\|_\alpha = \sqrt{b^i b_i}$ and $h_i (i = 1, 2, \dots, 109)$ are the polynomials of variations s and B . By using Lemma 3.1, it follows that β satisfies $r_{ij} = 0$ and then it is parallel with respect to α . In this case, F reduces to a Berwald metric.

A Finsler metric $F = F(x, y)$ on an n -dimensional manifold M is called an Einstein metric if its Ricci curvature satisfies $\mathbf{Ric} = (n - 1)\lambda F^2$, where $\lambda = \lambda(x)$ is a scalar function on M . In [2], it is proved that every Einstein polynomial (α, β) -metric is Ricci-flat. In this paper, we prove the following.

Theorem 1.5 Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α, β) -metric of degree m ($m \geq 3$) on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M . Suppose that F is an Einstein metric. Then $\mathbf{H} = 0$. In this case, F is a Berwald metric.

Example 1.6 The Funk metric is an Einstein metric with a closed 1-form. It satisfies $\mathbf{H} = 0$ while it is not a Berwald metric. Then Theorem 1.5 does not hold for the polynomial (α, β) -metrics of degree 1.

Example 1.7 The square metric in Example 1.3 is a Ricci-flat Finsler metric. Moreover, F is an Einstein metric. However, F is not a Berwald metric. Thus, Theorem 1.5 does not hold for the polynomial (α, β) -metrics of degree 2, generally.

Example 1.8 Let $\varphi(s) = (1 + s)^3$ be an Einstein metric. By the Theorem 1.1 in [2], F is Ricci-flat. Suppose that β is a closed 1-form. Then $R_m^m = {}^\alpha R_m^m + T_m^m = 0$, where ${}^\alpha R_m^m$ denotes the Riemannian curvature of α and

$$T_m^m = \left[(n-1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right] \frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha} \left[r_0 \left[(n-1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right] r_{00} + \left[(n-1) \frac{c_7}{A} + \frac{c_8}{A^2} \right] r_{00|0} \right] + \frac{c_{11}}{A^2} (rr_{00} - r_0^2) + \frac{c_{14}}{A} (r_{00}r_m^m - r_{0m}r_0^m + r_{00|m}b^m - r_{0m|0}b^m),$$

$A := 1 + 6B + 6Bs - 9s^2 - 8s^3$ and c_i ($i = 1, \dots, 14$) are polynomials of variations s and B (see [2] for the corrected version of [25]). It follows that $r_{ij} = 0$. Since β is a closed 1-form, then it is parallel with respect to α . In this case, F reduces to a Berwald metric.

The exponential metric is another important (α, β) -metric which is given by $\varphi(s) = e^s$, $s = \beta/\alpha$, (see [10, 15, 24]). Here, we consider exponential (α, β) -metrics with almost vanishing \mathbf{H} -curvature and prove the following.

Theorem 1.9 Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M . Then F has almost vanishing \mathbf{H} -curvature if and only if $\mathbf{H} = 0$. In this case, F is a Berwald metric.

Example 1.10 Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric. At a point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ and in the direction $y = (y^1, \dots, y^n) \in T_x \mathbb{R}^n$, consider the following Riemannian metric α and 1-form β as follows

$$\alpha(x, y) = \sqrt{(y^1)^2 + e^{2x^1} [(y^2)^2 + \dots + (y^n)^2]}, \quad \beta(x, y) := y^1. \tag{1.3}$$

Then $s_{ij} = 0$. In this case, F has constant S -curvature [5]. Thus, F satisfies $\mathbf{H} = 0$ (see [3] and [5]).

Finally, we consider the Einstein exponential metric and prove the following.

Theorem 1.11 Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on a manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a closed 1-form on M . Suppose that F is an Einstein metric. Then $\mathbf{H} = 0$. In this case, F is a Berwald metric.

Example 1.12 Let $F = \alpha e^{\beta/\alpha}$ be an exponential metric, where α and β are defined by (??). Suppose that F is an Einstein metric. Thus, $R_m^m = {}^\alpha R_m^m + T_m^m$, where

$$T_m^m := \left\{ (n-1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right\} \alpha^{-2} r_{00}^2 + \alpha^{-1} \left\{ \left[(n-1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right] r_{00} r_0 + \left[(n-1) \frac{c_7}{A} + \frac{c_8}{A^2} \right] r_{00|0} \right\} + \frac{c_{11}}{A^2} (rr_{00} - r_0^2) + \frac{c_{14}}{A} (r_{00}r_m^m - r_{0m}r_0^m + r_{00|m}b^m - r_{0m|0}b^m), \tag{1.4}$$

$A = 1 + B - s - s^2$ and $c_i, (i = 1, \dots, 14)$, are polynomials of variations s and B (see [2]). Thus, we get $g_2 r_{00}^2 \equiv 0, \text{ mod}(A)$, where $I_2 \equiv g_2, \text{ mod}(A)$, and g_2 is a polynomial of s and B . By Lemma 3.1, it follows that β is a Killing 1-form. Then β is parallel with respect to α . In this case, F reduces to a Berwald metric.

2. Preliminary

Let (M, F) be a Finsler manifold. A global vector field \mathbf{G} is induced by F is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$, where $G^i = G^i(x, y)$ are given by

$$G^i = \frac{1}{4} g^{il} \left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right].$$

\mathbf{G} is called the spray of (M, F) . The projection of an integral curve of the spray \mathbf{G} is called a geodesic in M [12, 22].

For $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow T_x M$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i} |_x$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

\mathbf{B} is called Berwald curvature . F is called a Berwald metric if $\mathbf{B} = 0$.

For $y \in T_x M_0$, define $\mathbf{E}_y : T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{E}_y(u, v) := E_{ij}(y) u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m_{ijm}.$$

\mathbf{E} is called mean Berwald curvature. F is called a weakly Berwald metric if $\mathbf{E} = 0$. By definition, every Berwald metric is a weakly Berwald metric.

For $y \in T_x M_0$, define the linear transformations $\mathbf{R}_y : T_x M \rightarrow T_x M$ with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$, where $\mathbf{R}_y(u) := R^i_k(y) u^k \frac{\partial}{\partial x^i}$ and

$$R^i_k(y) = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in T M_0}$ is called the Riemann curvature (see [11, 17, 21]).

The Ricci curvature $\mathbf{Ric}(x, y)$ is the trace of the Riemann curvature defined by

$$\mathbf{Ric}(x, y) := R^m_m(x, y).$$

A Finsler metric F on an n -dimensional manifold M is called an Einstein metric if the Ricci curvature satisfies

$$\mathbf{Ric} = (n - 1) \sigma F^2, \tag{2.2}$$

where $\sigma = \sigma(x)$ is a scalar function on M .

3. Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. First, we remark that the spray coefficients G^i of an (α, β) -metric $F = \alpha \varphi(s)$, $s = \beta/\alpha$, and the spray coefficients of the Riemannian metric α are related by

following

$$G^i = G_\alpha^i + Q\alpha s^i_0 + (r_{00} - 2Q\alpha s_0)(\Psi b^i + \Theta l^i),$$

where $l^i := \alpha^{-1}y_i$ and

$$Q := \frac{\varphi'}{\varphi - s\varphi'} \quad \Theta := \frac{\varphi\varphi' - s(\varphi\varphi'' + \varphi'\varphi')}{2\varphi[(\varphi - s\varphi') + (B^2 - s^2)\varphi'']}, \quad \Psi := \frac{\varphi''}{2[(\varphi - s\varphi') + (B^2 - s^2)\varphi'']}.$$

Moreover, $B := \|\beta\|_\alpha = \sqrt{b^i b_i}$, where $b^i := a^{ij}b_j$.

Lemma 3.1 *Suppose r_{00} of an (α, β) -metric $F = \alpha\varphi(s)$, $s = \beta/\alpha$, on a manifold M satisfies*

$$Ir_{00}^2 \equiv 0, \quad \text{mod}(as^2 + bs + c), \quad \text{and} \quad I \not\equiv 0, \quad \text{mod}(as^2 + bs + c),$$

where I is a polynomial of B , and s, a, b , and c are polynomials of B and $b \neq 0$. Suppose that r_1 and r_2 are the roots of the equation $as^2 + bs + c = 0$ such that $r_1^2 \neq r_2^2$. Then $r_{ij} = 0$.

Proof The following hold

$$Ir_{00}^2 \equiv 0, \quad \text{mod}(s - r_1) \quad \text{and} \quad Ir_{00}^2 \equiv 0, \quad \text{mod}(s - r_2). \tag{3.1}$$

Let us put

$$I \equiv f_1 \quad \text{mod}(s - r_1) \quad \text{and} \quad I \equiv f_2 \quad \text{mod}(s - r_2),$$

where f_1 and f_2 are polynomials of B . Then we have

$$f_1 r_{00}^2 \equiv 0, \quad \text{mod}(s - r_1) \quad \text{and} \quad f_2 r_{00}^2 \equiv 0, \quad \text{mod}(s - r_2)$$

which imply that

$$r_{00}^2 \equiv 0, \quad \text{mod}(s - r_1) \quad \text{and} \quad r_{00}^2 \equiv 0, \quad \text{mod}(s - r_2).$$

It follows that

$$r_{00} \equiv 0 \quad \text{mod}(s - r_1) \quad \text{and} \quad r_{00} \equiv 0 \quad \text{mod}(s - r_2)$$

Suppose that $r_{00} \neq 0$. Then by the Lemma 4.1 in [25], we get

$$r_{00} = \sigma_1 \alpha^2 (s^2 - r_1^2), \quad \text{and} \quad r_{00} = \sigma_2 \alpha^2 (s^2 - r_2^2), \tag{3.2}$$

where $\sigma_1 = \sigma_1(x)$ and $\sigma_2 = \sigma_2(x)$ are scalar functions on M . By (3.2), we have

$$(\sigma_1 - \sigma_2)\beta^2 + (\sigma_1 r_1^2 - \sigma_2 r_2^2)\alpha^2 = 0.$$

Then $\sigma_1 = \sigma_2$ and $r_1^2 = r_2^2$ which contradict with the assumption. Thus, $r_{00} = 0$. Taking vertical derivations of it twice yields $r_{ij} = 0$. □

Lemma 3.2 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α, β) -metric of degree m on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . Suppose that F has almost vanishing \mathbf{H} -curvature. Then the following holds*

$$f_1 r_{00} s_0 \alpha + f_2 r_{00}^2 + f_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}\left((1 - m^2)s^2 + (2 - m)s + m(m - 1)B + 1\right), \tag{3.3}$$

where f_j , ($j = 1, 2, 3$) are polynomials of variations s and B and they are homogeneous of degree one with respect to s

Proof For the polynomial metric $\varphi(s) = (1 + s)^m$, we have

$$Q = \frac{m}{1 + s - sm}, \quad \Theta = \frac{m(1 + 2s - 2sm)}{2(-m^2s^2 + s^2 - sm + 2s + 1 - Bm + m^2B)},$$

$$\Psi = \frac{m(m - 1)}{2(-m^2s^2 + s^2 - sm + 2s + 1 - Bm + m^2B)}.$$

By assumption, $F = \alpha\varphi(s)$ has almost vanishing \mathbf{H} -curvature, i.e. there exists a 1-form θ on M such that

$$H_{jk} = \frac{n + 1}{2}\theta F_{y^j y^k}, \tag{3.4}$$

where

$$F_{y^j y^k} = \frac{(1 + s)^{m-2}}{\alpha} \left[[1 - (m - 2)s - (m - 1)s^2] a_{jk} + (m^2 - m)b_j b_k - (m^2 - m)(b_j l_k + b_k l_j)s \right. \\ \left. + [(m^2 - 1)s^2 + (m - 2)s - 1] l_j l_k \right] \tag{3.5}$$

$l_i := \alpha_{y^i}$ and

$$2H_{jk} = \left[\frac{h_1}{A^6 D^3 \alpha^3} r_{00} s_0 + \frac{h_2}{A^4 D^3 \alpha^2} s_{0|0} + \frac{h_3}{A^6 \alpha^4} r_{00}^2 + \frac{h_4}{A^4 \alpha^3} r_{00|0} + \frac{h_5}{A^5 \alpha^3} r_{00} r_0 + \frac{h_6}{A^3 \alpha^2} r_{0|0} \right. \\ \left. + \frac{h_7}{A^4 D^4 \alpha} t_0 + \frac{h_8}{A^6 D^4 \alpha^2} s_0^2 + \frac{h_9}{A^4 D^2 \alpha^2} q_{00} + \frac{h_{10}}{A^3 D^2 \alpha} q_0 + \frac{h_{11}}{A^5 D^3 \alpha^2} r_0 s_0 + \frac{h_{12}}{A^4 D^2 \alpha} r s_0 \right. \\ \left. + \frac{h_{13}}{A^4 \alpha^2} r r_{00} + \frac{h_{14}}{A^4 \alpha^2} r_0^2 \right] b_j b_k + \left[\frac{h_{15}}{A^6 D^3 \alpha^3} r_{00} s_0 + \frac{h_{16}}{A^4 D^3 \alpha^2} s_{0|0} + \frac{h_{17}}{A^6 \alpha^4} r_{00}^2 + \frac{h_{18}}{A^4 \alpha^3} r_{00|0} \right. \\ \left. + \frac{h_{19}}{A^5 \alpha^3} r_{00} r_0 + \frac{h_{20}}{A^3 \alpha^2} r_{0|0} + \frac{h_{21}}{A^4 D^4 \alpha} t_0 + \frac{h_{22}}{A^6 D^4 \alpha^2} s_0^2 + \frac{h_{23}}{A^4 D^2 \alpha^2} q_{00} + \frac{h_{24}}{A^3 D^2 \alpha} q_0 \right]$$

$$\begin{aligned}
 & + \frac{h_{25}}{A^5 D^3 \alpha^2} r_0 s_0 + \frac{h_{26}}{A^4 D^2 \alpha} r s_0 + \frac{h_{27}}{A^4 \alpha^2} r r_{00} + \frac{h_{28}}{A^4 \alpha^2} r_0^2 \Big] l_j l_k \\
 & + \left[\frac{h_{29}}{A^5 D^2 \alpha^3} r_{00} s_0 + \frac{h_{30}}{A^3 D^2 \alpha^2} s_{0|0} + \frac{h_{31}}{A^5 \alpha^4} r_{00}^2 + \frac{h_{32}}{A^3 \alpha^3} r_{00|0} + \frac{h_{33}}{A^4 \alpha^3} r_{00} r_0 + \frac{h_{34}}{A^2 \alpha^2} r_{0|0} \right. \\
 & + \frac{h_{35}}{A^4 D^2 \alpha^2} r_0 s_0 + \frac{h_{36}}{A^5 D^3 \alpha^2} s_0^2 + \frac{h_{37}}{A^3 D^3 \alpha} t_0 + \frac{h_{38}}{A^3 D \alpha^2} q_{00} + \frac{h_{39}}{A^2 D \alpha^2} q_0 + \frac{h_{40}}{A^3 D \alpha} r s_0 \\
 & + \frac{h_{41}}{A^3 \alpha^2} r r_{00} + \frac{h_{42}}{A^3 \alpha^2} r_0^2 \Big] a_{jk} + \frac{h_{43}}{A^2 \alpha} r_{jk|0} + \left[\frac{h_{44}}{A^4 D \alpha} s_0 + \frac{h_{45}}{A^4 \alpha^2} r_{00} + \frac{h_{46}}{A^3 \alpha} r_0 \right] r_{jk} \\
 & + \frac{h_{47}}{A^4 D^2} s_k s_j + \frac{h_{48}}{A^4 \alpha^2} r_{0j} r_{0k} + \left[\frac{h_{49}}{A^6 D^3 \alpha^3} r_{00} s_0 + \frac{h_{50}}{A^4 D^3 \alpha^2} s_{0|0} + \frac{h_{51}}{A^6 \alpha^4} r_{00}^2 + \frac{h_{52}}{A^4 \alpha^3} r_{00|0} \right. \\
 & + \frac{h_{53}}{A^5 \alpha^3} r_{00} r_0 + \frac{h_{54}}{A^3 \alpha^2} r_{0|0} + \frac{h_{55}}{A^4 D^4 \alpha} t_0 + \frac{h_{56}}{A^6 D^4 \alpha^2} s_0^2 + \frac{h_{57}}{A^4 D^2 \alpha^2} q_{00} + \frac{h_{58}}{A^3 D^2 \alpha} q_0 \\
 & + \frac{h_{59}}{A^5 D^3 \alpha^2} r_0 s_0 + \frac{h_{60}}{A^4 D^2 \alpha} r s_0 + \frac{h_{61}}{A^4 \alpha^2} r r_{00} + \frac{h_{62}}{A^4 \alpha^2} r_0^2 \Big] (l_k b_j + l_j b_k) + \left[\frac{h_{63}}{A^5 D^3 \alpha} s_0 \right. \\
 & + \frac{h_{64}}{A^5 D^2 \alpha^2} r_{00} + \frac{h_{65}}{A^4 D^2 \alpha} r_0 + \frac{h_{66}}{A^3 D} r \Big] (l_k s_j + l_j s_k) + \left[\frac{h_{67}}{A^4 D^4 \alpha^2} s_0 + \frac{h_{68}}{A^4 D^2 \alpha^3} r_{00} \right. \\
 & + \frac{h_{69}}{A^3 D^2 \alpha^2} r_0 \Big] (l_k s_{j0} + l_j s_{k0}) + \left[\frac{h_{70}}{A^4 D^2 \alpha} s_0 + \frac{h_{71}}{A^4 \alpha^2} r_{00} + \frac{h_{72}}{A^3 \alpha} r_0 \right] (l_k r_j + l_j r_k) \\
 & + \left[\frac{h_{73}}{A^5 D^2 \alpha^2} s_0 + \frac{h_{74}}{A^5 \alpha^3} r_{00} + \frac{h_{75}}{A^4 \alpha^2} r_0 + \frac{h_{76}}{A^3 \alpha} r \right] (l_k r_{0j} + l_j r_{0k}) + \frac{h_{77}}{A^2 D} (l_k q_j + l_j q_k) \\
 & + \frac{h_{78}}{A^3 D \alpha} (l_k q_{0j} + l_j q_{0k}) + \frac{h_{79}}{A^3 D^2 \alpha} (l_k q_{j0} + l_j q_{k0}) + \frac{h_{80}}{A^3 D^3} (l_k t_j + l_j t_k) \\
 & + \left[\frac{h_{81}}{A^5 D^2 \alpha^2} r_{00} + \frac{h_{82}}{A^5 D^3 \alpha} s_0 + \frac{h_{83}}{A^4 D^2 \alpha} r_0 + \frac{h_{84}}{A^3 D} r \right] (b_k s_j + b_j s_k) + \left[\frac{h_{85}}{A^4 D^4 \alpha^2} s_0 \right. \\
 & + \frac{h_{86}}{A^4 D^2 \alpha^3} r_{00} + \frac{h_{87}}{A^3 D^2 \alpha^2} r_0 \Big] (b_k s_{j0} + b_j s_{k0}) + \frac{h_{95}}{A^3 D^2 \alpha} (b_k q_{j0} + b_j q_{k0}) \\
 & + \left[\frac{h_{88}}{A^4 D^2 \alpha} s_0 + \frac{h_{89}}{A^4 \alpha^2} r_{00} + \frac{h_{90}}{A^3 \alpha} r_0 \right] (b_k r_j + b_j r_k) + \left[\frac{h_{91}}{A^5 D^2 \alpha^2} s_0 + \frac{h_{92}}{A^5 \alpha^3} r_{00} \right. \\
 & + \frac{h_{93}}{A^4 \alpha^2} r_0 + \frac{h_{94}}{A^3 \alpha} r \Big] (b_k r_{0j} + b_j r_{0k}) + \frac{h_{96}}{A^3 D \alpha} (b_k q_{0j} + b_j q_{0k}) + \frac{h_{97}}{A^3 D^3} (b_k t_j + b_j t_k) \\
 & + \frac{h_{98}}{A^2 D} (b_k q_j + b_j q_k) + \frac{h_{99}}{A^3 D^3 \alpha} (s_k s_{j0} + s_j s_{k0}) + \frac{h_{100}}{A^3 D} (s_k r_j + s_j r_k) \\
 & + \frac{h_{101}}{A^4 D \alpha} (s_k r_{j0} + s_j r_{k0}) + \frac{h_{102}}{A^3 D \alpha^2} (s_{k0} r_{j0} + s_{j0} r_{k0}) + \frac{h_{103}}{A^2 D \alpha} (s_{k0} r_j + s_{j0} r_k) \\
 & + \frac{h_{104}}{A^3 \alpha} (r_k r_{j0} + r_j r_{k0}) + \frac{h_{105}}{A^2 D} (q_{kj} + q_{jk}) + \frac{h_{106}}{A^3 \alpha^2} (l_k r_{0j|0} + l_j r_{0k|0}) \\
 & + \frac{h_{107}}{A^3 \alpha^2} (r_{0j|0} b_k + r_{0k|0} b_j) + \frac{h_{108}}{A^2 \alpha} (r_{k|0} b_j + r_{j|0} b_k) + \frac{h_{109}}{A^2 \alpha} (l_k r_{j|0} + l_j r_{k|0}) \\
 & + \frac{h_{110}}{A^3 D^2 \alpha} (l_j s_{k|0} + l_k s_{j|0}) + \frac{h_{111}}{A^3 D^2 \alpha} (s_{j|0} b_k + s_{k|0} b_j), \tag{3.6}
 \end{aligned}$$

where

$$A := 1 + m(m - 1)B - (m - 2)s - (m^2 - 1)s^2, \quad D := (m - 1)s - 1,$$

and h_i ($i = 1, 2, \dots, 111$) are the polynomials of variations s and B . Substituting (3.6) in (3.4) and multiplying the result with $A^6 D^4 \alpha^4$ implies that

$$H_{jk} A^6 D^4 \alpha^4 - \frac{n+1}{2} \theta F_{y^j y^k} A^6 D^4 \alpha^4 = 0. \tag{3.7}$$

The following holds

$$\theta F_{y^j y^k} A^6 D^4 \alpha^4 \equiv 0, \quad \text{mod}(A).$$

Then (3.7) is equivalent to the following

$$\begin{aligned} & \left[h_{49} r_{00} s_0 \alpha + h_{51} D^4 r_{00}^2 + h_{56} s_0^2 \alpha^2 \right] (l_j b_k + l_k b_j) + \left[h_1 r_{00} s_0 \alpha + h_3 D^4 r_{00}^2 + h_8 s_0^2 \alpha^2 \right] b_j b_k \\ & + \left[h_{15} r_{00} s_0 \alpha + h_{17} D^4 r_{00}^2 + h_{22} s_0^2 \alpha^2 \right] l_j l_k \equiv 0, \quad \text{mod}(A). \end{aligned} \tag{3.8}$$

Multiplying (3.8) with $b^j b^k$ yields

$$I_1 r_{00} s_0 \alpha + I_2 r_{00}^2 + I_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}(A), \tag{3.9}$$

where I_i ($i = 1, 2, 3$), are polynomials of s and B . Let us put

$$I_1 \equiv f_1 \quad \text{and} \quad I_2 \equiv f_2 \quad \text{and} \quad I_3 \equiv f_3, \quad \text{mod}(A).$$

Then by (3.9), we get (3.3). □

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1: Let β be a closed 1-form. Then (3.6) reduces to the following:

$$\begin{aligned} 2H_{jk} = & \left[\frac{h_3}{A^6 \alpha^4} r_{00}^2 + \frac{h_4}{A^4 \alpha^3} r_{00|0} + \frac{h_5}{A^5 \alpha^3} r_{00} r_0 + \frac{h_6}{A^3 \alpha^2} r_{0|0} + \frac{h_{13}}{A^4 \alpha^2} r r_{00} + \frac{h_{14}}{A^4 \alpha^2} r_0^2 \right] b_j b_k \\ & + \left[\frac{h_{17}}{A^6 \alpha^4} r_{00}^2 + \frac{h_{18}}{A^4 \alpha^3} r_{00|0} + \frac{h_{19}}{A^5 \alpha^3} r_{00} r_0 + \frac{h_{20}}{A^3 \alpha^2} r_{0|0} + \frac{h_{27}}{A^4 \alpha^2} r r_{00} + \frac{h_{28}}{A^4 \alpha^2} r_0^2 \right] l_j l_k \\ & + \left[\frac{h_{31}}{A^5 \alpha^4} r_{00}^2 + \frac{h_{32}}{A^3 \alpha^3} r_{00|0} + \frac{h_{33}}{A^4 \alpha^3} r_{00} r_0 + \frac{h_{34}}{A^2 \alpha^2} r_{0|0} + \frac{h_{41}}{A^3 \alpha^2} r r_{00} + \frac{h_{42}}{A^3 \alpha^2} r_0^2 \right] a_{jk} \\ & + \frac{h_{43}}{A^2 \alpha} r_{jk|0} + \left[\frac{h_{45}}{A^4 \alpha^2} r_{00} + \frac{h_{46}}{A^3 \alpha} r_0 \right] r_{jk} + \frac{h_{48}}{A^4 \alpha^2} r_{0j} r_{0k} + \left[\frac{h_{51}}{A^6 \alpha^4} r_{00}^2 + \frac{h_{52}}{A^4 \alpha^3} r_{00|0} \right. \\ & + \frac{h_{53}}{A^5 \alpha^3} r_{00} r_0 + \frac{h_{54}}{A^3 \alpha^2} r_{0|0} + \frac{h_{61}}{A^4 \alpha^2} r r_{00} + \frac{h_{62}}{A^4 \alpha^2} r_0^2 \left. \right] [l_k b_j + l_j b_k] \\ & + \left[\frac{h_{71}}{A^4 \alpha^2} r_{00} + \frac{h_{72}}{A^3 \alpha} r_0 \right] [l_k r_j + l_j r_k] + \left[\frac{h_{74}}{A^5 \alpha^3} r_{00} + \frac{h_{75}}{A^4 \alpha^2} r_0 + \frac{h_{76}}{A^3 \alpha} r \right] (l_k r_{0j} + l_j r_{0k}) \\ & + \left[\frac{h_{89}}{A^4 \alpha^2} r_{00} + \frac{h_{90}}{A^3 \alpha} r_0 \right] [b_k r_j + b_j r_k] + \left[\frac{h_{92}}{A^5 \alpha^3} r_{00} + \frac{h_{93}}{A^4 \alpha^2} r_0 + \frac{h_{94}}{A^3 \alpha} r \right] [b_k r_{0j} + b_j r_{0k}] \\ & + \frac{h_{104}}{A^3 \alpha} [r_k r_{j0} + r_j r_{k0}] + \frac{h_{106}}{A^3 \alpha^2} [l_k r_{0j|0} + l_j r_{0k|0}] + \frac{h_{107}}{A^3 \alpha^2} [r_{0j|0} b_k + r_{0k|0} b_j] \\ & + \frac{h_{108}}{A^2 \alpha} [r_{k|0} b_j + r_{j|0} b_k] + \frac{h_{109}}{A^2 \alpha} [l_k r_{j|0} + l_j r_{k|0}], \end{aligned} \tag{3.10}$$

By substituting (3.10) in (3.4) and multiplying the result with $A^6\alpha^4$, we get

$$H_{jk}A^6\alpha^4 - \frac{n+1}{2}\theta F_{y^j y^k} A^6\alpha^4 = 0. \tag{3.11}$$

Since

$$\frac{n+1}{2}\theta F_{y^j y^k} A^6\alpha^4 \equiv 0, \quad \text{mod}(A)$$

then (3.11) is equal to the following

$$\left[h_{51}(l_j b_k + l_k b_j) + h_3 b_j b_k + h_{17} l_j l_k \right] r_{00}^2 \equiv 0, \quad \text{mod}(A). \tag{3.12}$$

Multiplying (3.12) with $b^j b^k$ yields

$$I_2 r_{00}^2 \equiv 0, \quad \text{mod}(A),$$

where I_2 is a polynomial of s and B . Then we get

$$f_2 r_{00}^2 \equiv 0 \quad \text{mod}(A),$$

where $I_2 \equiv f_2 \pmod{A}$, and f_2 is a polynomial of s and B and of degree 1 in s . By Lemma 3.1, it follows that β is parallel with respect to α . Plugging this in (3.10) yields $\mathbf{H} = 0$. The converse is trivial. On the other hand, every regular (α, β) -metric is a Berwald metric if and only if β is parallel with respect to α . This completes the proof.

4. Proof of Theorem 1.5

In this section, we are going to prove Theorem 1.5. First, we prove the following.

Lemma 4.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be a polynomial (α, β) -metric of degree m on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . Suppose that F is an Einstein metric. Then the following holds*

$$g_1 r_{00} s_0 \alpha + g_2 r_{00}^2 + g_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}\left((1-m^2)s^2 + (2-m)s + m(m-1)B + 1\right), \tag{4.1}$$

where g_j ($j = 1, 2, 3$), are polynomials of variations B and s .

Proof Let $\varphi(s) = (1+s)^m$ ($m \geq 3$) be an Einstein metric. By the Theorem 1.1 in [2], F is Ricci-flat. Then

$$R_m^m = {}^\alpha R_m^m + T_m^m = 0, \tag{4.2}$$

where ${}^\alpha R_m^m$ denotes the Riemannian curvature of α and

$$\begin{aligned} T_m^m = & \left[(n-1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right] \frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha} \left[\left[(n-1) \frac{c_3}{A^3 D} + \frac{c_4}{A^4 D} \right] r_{00} s_0 + r_0 \left[(n-1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right] r_{00} \right. \\ & \left. + \left[(n-1) \frac{c_7}{A} + \frac{c_8}{A^2} \right] r_{00|0} \right] + \left[\left[(n-1) \frac{c_9}{A^3 D^3} + \frac{c_{10}}{A^4 D^3} \right] s_0^2 + \frac{c_{11}}{A^2} (r r_{00} - r_0^2) \right] \end{aligned}$$

$$\begin{aligned}
 &+ \left[(n-1) \frac{c_{12}}{A^2D} + \frac{c_{13}}{A^3D} \right] r_0 s_0 + \frac{c_{14}}{A} (r_{00} r_m^m - r_{0m} r_0^m + r_{00|m} b^m - r_{0m|0} b^m) \\
 &+ \left[(n-1) \frac{c_{15}}{AD} + \frac{c_{16}}{A^2D} \right] r_{0m} s_0^m + \left[(n-1) \frac{c_{17}}{AD} + \frac{c_{18}}{A^2D} \right] s_{0|0} + \frac{c_{19}}{D^3} s_{0m} s_0^m \Big] \\
 &+ \left[\frac{c_{20}}{A^2D} r s_0 + \left[(n-1) \frac{c_{21}}{AD^2} + \frac{c_{22}}{A^2D^2} \right] s_m s_0^m + \frac{c_{23}}{AD} (3s_m r_0^m - 2s_0 r_m^m + 2r_m s_0^m \right. \\
 &\left. - 2s_{0|m} b^m + s_{m|0} b^m) + \frac{c_{24}}{D} s_{0|m}^m \right] \alpha + \left[\frac{c_{25}}{AD^2} s_m s^m + \frac{c_{26}}{D^2} s_m^i s_i^m \right] \alpha^2, \tag{4.3}
 \end{aligned}$$

$A = 1 + m(m-1)B - (m-2)s - (m^2-1)s^2$, $D := (m-1)s - 1$ and c_i ($i = 1, \dots, 26$), are polynomials of variations s and B (see [2] for the corrected version of [25]). Putting (4.3) in (4.2) and multiplying the result with $A^4 D^3 \alpha^2$ implies that

$${}^\alpha R_m^m A^4 D^3 \alpha^2 + T_m^m A^4 D^3 \alpha^2 = 0.$$

${}^\alpha R_m^m$ is a polynomial with respect to s and B . Since ${}^\alpha R_m^m A^4 D^3 \alpha^2 \equiv 0, \text{ mod}(A)$, then we get $T_m^m A^4 D^3 \alpha^2 \equiv 0, \text{ mod}(A)$. By (4.3), we obtain

$$r_{00} s_0 \alpha c_4 D^2 + r_{00}^2 c_2 D^3 + s_0^2 \alpha^2 c_{10} \equiv 0, \text{ mod}(A).$$

Put

$$c_4 D^2 \equiv g_1 \quad \text{and} \quad c_2 D^3 \equiv g_2 \quad \text{and} \quad c_{10} \equiv g_3 \quad \text{mod}(A).$$

Then we get (4.1). □

Proof of the Theorem 1.5: Let β be a closed 1-form on M . By Lemma 4.1, we get

$$g_2 r_{00}^2 \equiv 0, \text{ mod}(A),$$

where $I_2 \equiv g_2, \text{ mod}(A)$, and g_2 is polynomials of s and B and of degree 1 in s . By Lemma 3.1, it follows that β is Killing. Then β is parallel with respect to α . In this case, F reduces to a Berwald metric.

5. Proof of Theorem 1.9

In this section, we are going to prove Theorem 1.9. First, we prove the following.

Lemma 5.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . Suppose that F has almost vanishing \mathbf{H} -curvature. Then the following holds*

$$h_1 r_{00} s_0 \alpha + h_2 r_{00}^2 + h_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}(-s^2 - s + B + 1), \tag{5.1}$$

where h_j ($j = 1, 2, 3$) are polynomials of variations s and B and of degree one in s .

Proof For the exponential metric $\varphi(s) = e^s$, we have

$$Q = \frac{1}{1-s}, \quad \Theta = \frac{2s-1}{2(s^2+s-B-1)}, \quad \Psi = \frac{1}{2(1+B-s-s^2)}.$$

By assumption, $F = \alpha\varphi(s)$, $s = \beta/\alpha$, has almost vanishing **H**-curvature, i.e. there exists a 1-form θ on M such that

$$H_{jk} = \frac{n+1}{2}\theta F_{y^j y^k}, \tag{5.2}$$

where

$$F_{y^j y^k} = \frac{e^s}{\alpha} \left[(1-s)a_{jk} + b_j b_k - s(b_j l_k + b_k l_j) + (s^2 + s - 1)l_j l_k \right],$$

and

$$\begin{aligned} 2H_{jk} = & \left[\frac{h_1}{\alpha^3 A^6 (s-1)^3} r_{00} s_0 + \frac{h_2}{\alpha^2 A^4 (s-1)^3} s_{0|0} + \frac{h_3}{\alpha^4 A^6} r_{00}^2 + \frac{h_4}{\alpha^3 A^4} r_{00|0} + \frac{h_5}{\alpha^3 A^5} r_{00} r_0 \right. \\ & + \frac{h_6}{\alpha^2 A^3} r_{0|0} + \frac{h_7}{\alpha A^4 (s-1)^4} t_0 + \frac{h_8}{\alpha^2 A^6 (s-1)^4} s_0^2 + \frac{h_9}{\alpha^2 A^4 (s-1)^2} q_{00} + \frac{h_{10}}{\alpha A^3 (s-1)^2} q_0 \\ & + \frac{h_{11}}{\alpha^2 A^5 (s-1)^3} r_0 s_0 + \frac{h_{12}}{\alpha A^4 (s-1)^2} r s_0 + \frac{h_{13}}{\alpha^2 A^4} r r_{00} + \left. \frac{h_{14}}{\alpha^2 A^4} r_0^2 \right] b_j b_k \\ & + \left[\frac{h_{15}}{\alpha^3 A^6 (s-1)^3} r_{00} s_0 + \frac{h_{16}}{\alpha^2 A^4 (s-1)^3} s_{0|0} + \frac{h_{17}}{\alpha^4 A^6} r_{00}^2 + \frac{h_{18}}{\alpha^3 A^4} r_{00|0} + \frac{h_{19}}{\alpha^3 A^5} r_{00} r_0 \right. \\ & + \frac{h_{20}}{\alpha^2 A^3} r_{0|0} + \frac{h_{21}}{\alpha A^4 (s-1)^4} t_0 + \frac{h_{22}}{\alpha^2 A^6 (s-1)^4} s_0^2 + \frac{h_{23}}{\alpha^2 A^4 (s-1)^2} q_{00} + \frac{h_{24}}{\alpha A^3 (s-1)^2} q_0 \\ & + \frac{h_{25}}{\alpha^2 A^5 (s-1)^3} r_0 s_0 + \frac{h_{26}}{\alpha A^4 (s-1)^2} r s_0 + \frac{h_{27}}{\alpha^2 A^4} r r_{00} + \left. \frac{h_{28}}{\alpha^2 A^4} r_0^2 \right] l_j l_k \\ & + a_{jk} \left[\frac{h_{29}}{\alpha^3 A^5 (s-1)^2} r_{00} s_0 + \frac{h_{30}}{\alpha^2 A^3 (s-1)^2} s_{0|0} + \frac{h_{31}}{\alpha^4 A^5} r_{00}^2 + \frac{h_{32}}{\alpha^3 A^3} r_{00|0} + \frac{h_{33}}{\alpha^3 A^4} r_{00} r_0 \right. \\ & + \frac{h_{34}}{\alpha^2 A^2} r_{0|0} + \frac{h_{35}}{\alpha^2 A^4 (s-1)^2} r_0 s_0 + \frac{h_{36}}{\alpha^2 A^5 (s-1)^3} s_0^2 + \frac{h_{37}}{\alpha A^3 (s-1)^3} t_0 + \frac{h_{38}}{\alpha^2 A^3 (s-1)} q_{00} \\ & + \frac{h_{39}}{\alpha^2 A^2 (s-1)} q_0 + \frac{h_{40}}{\alpha A^3 (s-1)} r s_0 + \frac{h_{41}}{\alpha^2 A^3} r r_{00} + \left. \frac{h_{42}}{\alpha^2 A^3} r_0^2 \right] + \frac{h_{43}}{\alpha A^2} r_{jk|0} + r_{jk} \left[\frac{h_{44} s_0}{\alpha A^4 (s-1)} \right. \\ & + \frac{h_{45}}{\alpha^2 A^4} r_{00} + \left. \frac{h_{46}}{\alpha A^3} r_0 \right] + \frac{h_{47}}{A^4 (s-1)^2} s_k s_j + \frac{h_{48}}{\alpha^2 A^4} r_{0j} r_{0k} + \left[\frac{h_{49}}{\alpha^3 A^6 (s-1)^3} r_{00} s_0 \right. \\ & + \frac{h_{50}}{\alpha^2 A^4 (s-1)^3} s_{0|0} + \frac{h_{51}}{\alpha^4 A^6} r_{00}^2 + \frac{h_{52}}{\alpha^3 A^4} r_{00|0} + \frac{h_{53}}{\alpha^3 A^5} r_{00} r_0 + \frac{h_{54}}{\alpha^2 A^3} r_{0|0} + \frac{h_{55}}{\alpha A^4 (s-1)^4} t_0 \\ & + \frac{h_{56}}{\alpha^2 A^6 (s-1)^4} s_0^2 + \frac{h_{57}}{\alpha^2 A^4 (s-1)^2} q_{00} + \frac{h_{58}}{\alpha A^3 (s-1)^2} q_0 + \frac{h_{59} r_0}{\alpha^2 A^5 (s-1)^3} s_0 + \frac{h_{60} r}{\alpha A^4 (s-1)^2} s_0 \\ & + \frac{h_{61}}{\alpha^2 A^4} r r_{00} + \left. \frac{h_{62}}{\alpha^2 A^4} r_0^2 \right] (l_k b_j + l_j b_k) + \left[\frac{h_{63}}{\alpha A^5 (s-1)^3} s_0 + \frac{h_{64}}{\alpha^2 A^5 (s-1)^2} r_{00} + \frac{h_{65}}{\alpha A^4 (s-1)^2} r_0 \right. \\ & + \left. \frac{h_{66}}{A^3 (s-1)} r \right] (l_k s_j + l_j s_k) + (l_k s_{j0} + l_j s_{k0}) \left[\frac{h_{67}}{\alpha^2 A^4 (s-1)^4} s_0 + \frac{h_{68}}{\alpha^3 A^4 (s-1)^2} r_{00} \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{h_{69}}{\alpha^2 A^3 (s-1)^2} r_0 \Big] + (l_k r_j + l_j r_k) \Big[\frac{h_{70}}{\alpha A^4 (s-1)^2} s_0 + \frac{h_{71}}{\alpha^2 A^4} r_{00} + \frac{h_{72}}{\alpha A^3} r_0 \Big] \\
 & + (l_k r_{0j} + l_j r_{0k}) \Big[\frac{h_{73}}{\alpha^2 A^5 (s-1)^2} s_0 + \frac{h_{74}}{\alpha^3 A^5} r_{00} + \frac{h_{75}}{\alpha^2 A^4} r_0 + \frac{h_{76}}{\alpha A^3} r \Big] + \frac{h_{77}}{A^2 (s-1)} (l_k q_j + l_j q_k) \\
 & + \frac{h_{78}}{\alpha A^3 (s-1)} (l_k q_{0j} + l_j q_{0k}) + \frac{h_{79}}{\alpha A^3 (s-1)^2} (l_k q_{j0} + l_j q_{k0}) + \frac{h_{80}}{A^3 (s-1)^3} (l_k t_j + l_j t_k) \\
 & + (b_k s_j + b_j s_k) \Big[\frac{h_{81}}{\alpha^2 A^5 (s-1)^2} r_{00} + \frac{h_{82}}{\alpha A^5 (s-1)^3} s_0 + \frac{h_{83}}{\alpha A^4 (s-1)^2} r_0 + \frac{h_{84}}{A^3 (s-1)} r \Big] \\
 & + (b_k s_{j0} + b_j s_{k0}) \Big[\frac{h_{85}}{\alpha^2 A^4 (s-1)^4} s_0 + \frac{h_{86}}{\alpha^3 A^4 (s-1)^2} r_{00} + \frac{h_{87}}{\alpha^2 A^3 (s-1)^2} r_0 \Big] \\
 & + (b_k r_j + b_j r_k) \Big[\frac{h_{88}}{\alpha A^4 (s-1)^2} s_0 + \frac{h_{89}}{\alpha^2 A^4} r_{00} + \frac{h_{90}}{\alpha A^3} r_0 \Big] + (b_k r_{0j} + b_j r_{0k}) \Big[\frac{h_{91}}{\alpha^2 A^5 (s-1)^2} s_0 \\
 & + \frac{h_{92}}{\alpha^3 A^5} r_{00} + \frac{h_{93}}{\alpha^2 A^4} r_0 + \frac{h_{94}}{\alpha A^3} r \Big] + \frac{h_{95}}{\alpha A^3 (s-1)^2} (b_k q_{j0} + b_j q_{k0}) + \frac{h_{96}}{\alpha A^3 (s-1)} (b_k q_{0j} + b_j q_{0k}) \\
 & + \frac{h_{97}}{A^3 (s-1)^3} (b_k t_j + b_j t_k) + \frac{h_{98}}{A^2 (s-1)} (b_k q_j + b_j q_k) + \frac{h_{99}}{\alpha A^3 (s-1)^3} (s_k s_{j0} + s_j s_{k0}) \\
 & + \frac{h_{100}}{A^3 (s-1)} (s_k r_j + s_j r_k) + \frac{h_{101}}{\alpha A^4 (s-1)} (s_k r_{j0} + s_j r_{k0}) + \frac{h_{102}}{\alpha^2 A^3 (s-1)} (s_{k0} r_{j0} + s_{j0} r_{k0}) \\
 & + \frac{h_{103}}{\alpha A^2 (s-1)} (s_{k0} r_j + s_{j0} r_k) + \frac{h_{104}}{\alpha A^3} (r_k r_{j0} + r_j r_{k0}) + \frac{h_{105}}{A^2 (s-1)} (q_{kj} + q_{jk}) \\
 & + \frac{h_{106}}{\alpha^2 A^3} (l_k r_{0j|0} + l_j r_{0k|0}) + \frac{h_{107}}{\alpha^2 A^3} (r_{0j|0} b_k + r_{0k|0} b_j) + \frac{h_{108}}{\alpha A^2} (r_{k|0} b_j + r_{j|0} b_k) \\
 & + \frac{h_{109}}{\alpha A^2} (l_k r_{j|0} + l_j r_{k|0}) + \frac{h_{110}}{\alpha A^3 (s-1)^2} (l_j s_{k|0} + l_k s_{j|0}) + \frac{h_{111}}{A^3 (s-1)^2} (s_{j|0} b_k + s_{k|0} b_j),
 \end{aligned}$$

$A = 1 + B - s - s^2$ and h_i ($i = 1, 2, \dots, 111$) are the polynomials of s and B . Putting (5.3) in (3.4) and multiplying the result with $A^6 (s-1)^4 \alpha^4$ implies that

$$H_{jk} A^6 \alpha^4 (s-1)^4 - \frac{n+1}{2} \theta F_{y^j y^k} A^6 \alpha^4 (s-1)^4 = 0. \tag{5.3}$$

Since $\theta F_{y^j y^k} A^6 \alpha^4 (s-1)^4 \equiv 0, \text{ mod}(A)$, then (5.3) is equal to

$$\begin{aligned}
 & \Big[h_{49} (s-1) \alpha s_0 r_{00} + h_{51} (s-1)^4 r_{00}^2 + h_{56} \alpha^2 s_0^2 \Big] (l_j b_k + l_k b_j) + \Big[h_1 (s-1) \alpha s_0 r_{00} + h_8 \alpha^2 s_0^2 \\
 & + h_3 (s-1)^4 r_{00}^2 \Big] b_j b_k + \Big[h_{15} (s-1) \alpha s_0 r_{00} + h_{17} (s-1)^4 r_{00}^2 + h_{22} \alpha^2 s_0^2 \Big] l_j l_k \equiv 0, \text{ mod}(A). \tag{5.4}
 \end{aligned}$$

Multiplying (5.4) with $b^j b^k$ yields $I_1 r_{00} s_0 \alpha + I_2 r_{00}^2 + I_3 s_0^2 \alpha^2 \equiv 0, \text{ mod}(A)$, where I_i , ($i = 1, 2, 3$) are polynomials of variations s and B . Put $I_1 \equiv h_1$, $I_2 \equiv h_2$ and $I_3 \equiv h_3 \text{ mod}(A)$. Then, we get (5.1). \square

Proof of Theorem 1.9: Let β be a closed 1-form on M . By Lemma 5.1, we get $h_2 r_{00}^2 \equiv 0, \text{ mod}(A)$, where $I_2 \equiv h_2, \text{ mod}(A)$, and h_2 is a polynomial of s and B and of degree 1 in s . By Lemma 3.1, β is Killing. Putting it in (5.3) yields $\mathbf{H} = 0$. The converse is trivial. In this case, it follows that β is parallel with respect to α . Then, F reduces to a Berwald metric.

6. Proof of Theorem 1.11

In this section, we are going to prove Theorem 1.11. For this aim, we need the following.

Lemma 6.1 *Let $F = \alpha\varphi(s)$, $s = \beta/\alpha$, be an exponential metric on an n -dimensional manifold M , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M . Suppose that F is an Einstein metric. Then the following holds*

$$k_1 r_{00} s_0 \alpha + k_2 r_{00}^2 + k_3 s_0^2 \alpha^2 \equiv 0, \quad \text{mod}(-s^2 - s + B + 1), \tag{6.1}$$

where k_j , ($j = 1, 2, 3$), are polynomials of variations B and s .

Proof For the exponential metric $\varphi(s) = e^s$, we have

$$R_m^m = {}^\alpha R_m^m + T_m^m = \mathbf{Ric}(x)F^2, \tag{6.2}$$

where

$$\begin{aligned} T_m^m := & \left[(n-1) \frac{c_1}{A^3} + \frac{c_2}{A^4} \right] \frac{r_{00}^2}{\alpha^2} + \frac{1}{\alpha} \left[\left[(n-1) \frac{c_3}{A^3 D} + \frac{c_4}{A^4 D} \right] r_{00} s_0 + \left[(n-1) \frac{c_5}{A^2} + \frac{c_6}{A^3} \right] r_{00} r_0 \right. \\ & + \left. \left[(n-1) \frac{c_7}{A} + \frac{c_8}{A^2} \right] r_{00|0} \right] + \left[\left[(n-1) \frac{c_9}{A^3 D^3} + \frac{c_{10}}{A^4 D^3} \right] s_0^2 + \frac{c_{11}}{A^2} (r r_{00} - r_0^2) \right. \\ & + \left. \left[(n-1) \frac{c_{12}}{A^2 D} + \frac{c_{13}}{A^3 D} \right] r_0 s_0 + \frac{c_{14}}{A} (r_{00} r_m^m - r_{0m} r_0^m + r_{00|m} b^m - r_{0m|0} b^m) \right. \\ & + \left. \left[(n-1) \frac{c_{15}}{A D} + \frac{c_{16}}{A^2 D} \right] r_{0m} s_0^m + \left[(n-1) \frac{c_{17}}{A D} + \frac{c_{18}}{A^2 D} \right] s_{0|0} + \frac{c_{19}}{D^3} s_{0m} s_0^m \right] \\ & + \left[\frac{c_{20}}{A^2 D} r s_0 + \left[(n-1) \frac{c_{21}}{A D^2} + \frac{c_{22}}{A^2 D^2} \right] s_m s_0^m + \frac{c_{23}}{A D} (3s_m r_0^m - 2s_0 r_m^m + 2r_m s_0^m \right. \\ & \left. - 2s_{0|m} b^m + s_{m|0} b^m) + \frac{c_{24}}{D} s_{0|m}^m \right] \alpha + \left[\frac{c_{25}}{A D^2} s_m s^m + \frac{c_{26}}{D^2} s_m^i s_i^m \right] \alpha^2, \tag{6.3} \end{aligned}$$

$A = 1 + B - s - s^2$, $D = s - 1$ and c_i , ($i = 1, \dots, 26$), are polynomials of variations s and B (see [2]). Putting T_m^m into (6.2) and multiplying the result with $A^4 D^3 \alpha^2$ implies that

$${}^\alpha R_m^m A^4 D^3 \alpha^2 + T_m^m A^4 D^3 \alpha^2 - \mathbf{Ric}(x)F^2 A^4 D^3 \alpha^2 = 0.$$

${}^\alpha R_m^m - \mathbf{Ric}(x)F^2$ is a polynomial of s and B . Thus,

$${}^\alpha R_m^m A^4 D^3 \alpha^2 - \mathbf{Ric}(x)F^2 A^4 D^3 \alpha^2 \equiv 0, \quad \text{mod}(A).$$

Then $T_m^m A^4 D^3 \alpha^2 \equiv 0, \text{ mod}(A)$. By (6.3), we get $r_{00} s_0 \alpha c_4 D^2 + r_{00}^2 c_2 D^3 + s_0^2 \alpha^2 c_{10} \equiv 0, \text{ mod}(A)$. Put

$$c_4 D^2 \equiv h_1 \quad \text{and} \quad c_2 D^3 \equiv h_2 \quad \text{and} \quad c_{10} \equiv h_3, \quad \text{mod}(A).$$

Then, we get (6.1). □

Proof of Theorem 1.11: By Lemma 6.1, we have (6.1). Let $\beta = b_i(x)y^i$ be a closed 1-form. Then $k_2 r_{00}^2 \equiv 0, \text{ mod}(A)$, where $I_2 \equiv k_2 \text{ mod}(A)$, and k_2 is a polynomial of variations s and B . By Lemma 3.1, β is a Killing 1-form. It follows that β is parallel with respect to α . In this case, F reduces to a Berwald metric.

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