Turkish Journal of Mathematics

Volume 44 | Number 1 Article 12

1-1-2020

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 169 – 178
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doi:10.3906/mat-1910-71

Locally d_δ-connected and locally D_δ-compact spaces

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Received: 21.10.2019 • **Accepted/Published Online:** 19.11.2019 • **Final Version:** 20.01.2020

Abstract: The local analogues of the notions of d_δ-connectedness and D_δ-compactness of a topological space are introduced, and are named respectively, locally d_δ-connectedness and locally D_δ-compactness. Several properties including characterization of the concepts are discussed.

Key words: d_δ-separation relative to a space, d_δ-connected relative to a space, D_δ-closed relative to a space, locally d_δ-connected space, locally D_δ-compact space

1. Introduction

In a topological space $(X, \Im)$, a point x is in the d_δ-closure of $S \subset X$, denoted by $[S]_{d_\delta}$, if every regular F_σ-set containing x intersects S. The set S is d_δ-closed if $[S]_{d_\delta} = S$ and the complement of a d_δ-closed set is d_δ-open. We say that S is d_δ-connected relative to X, if there exists no pair P and Q of nonempty subsets of X such that $S = P \cup Q$ with $[P]_{d_\delta} \cap Q = \emptyset$ and $P \cap [Q]_{d_\delta} = \emptyset$, a d_δ-separation relative to X. Moreover, S is D_δ-closed relative to X if every covering of S by regular F_σ-sets in X has a finite subcover, and S is D_δ-compact if S is D_δ-closed relative to itself. These concepts are generalizations of connected and compact sets by d_δ-closure operator, see [4] and [8]. The purpose of this article is to localize the properties of these concepts and observe to what extent the standard results about locally connected and locally compact spaces remain valid. We introduce the concepts of locally d_δ-connected and locally D_δ-compact spaces, and determine their characterizations. Then, we consider the conditions under which these concepts are heritable to subspaces and the class of functions which preserve them. Also, we establish the invariance of these concepts under the formation of product. The article is concluded with a discussion of two useful applications of the concepts, including a form of Poincaré-Volterra Theorem.

We now recall some definitions and notations.

A regular G_δ-set $G \subset X$ is an intersection of a sequence of closed sets whose interior contains G. The complement of a regular G_δ-set is called a regular F_σ-set.

For spaces X and Y, a function $f : X \to Y$ is said to be pseudo-D_δ-supercontinuous if for each $x \in X$ and for each regular F_σ-set V containing $f(x)$ in Y, there exists a regular F_σ-set U containing x such that $f(U) \subset V$ [7].

A space X is said to be D_δ-Hausdorff if for each pair of distinct points x and y of X, there exist disjoint regular F_σ-sets U and V containing x and y, respectively [4].

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2010 *AMS Mathematics Subject Classification:* Primary 54D05, 54D45; Secondery 54C10.

Let B_{d_δ} denote the collection of all regular F_σ-sets in X, which forms a base for a topology $\Im_{d_\delta}$ on X such that $\Im_{d_\delta} \subset \Im$. The space $(X, \Im_{d_\delta})$ is called a D_δ-completely regularization of the space $(X, \Im)$, and is denoted by X^* (see [4]). However, if B_{d_δ} forms a basis for $\Im$ itself, then the space $(X, \Im)$ is said to be a D_δ-completely regular space (see [5]).

For a family of spaces $\{X_\alpha\}_\Lambda$, the product space $\prod_\Lambda X_\alpha$ is said to be Hd_δ-completely regular space if for each $(x_\alpha)_\Lambda \in \prod_\Lambda X_\alpha$ and for each regular F_σ-set F containing $(x_\alpha)_\Lambda$, there exists regular F_σ-set $(\bigcap_\Gamma \pi_i^{-1}(U_i)) \subset F$ containing $(x_\alpha)_\Lambda$, where $\Gamma \subset \Lambda$ is finite and each U_i is a regular F_σ-set containing x_i in X_i for each $i \in \Gamma$ (see[8]).

2. Locally d_δ-connected spaces

We begin this section with the definition of locally d_δ-connected spaces, and explore its basic properties.

Definition 2.1 *A space X is said to be locally d_δ-connected at a point $x \in X$ if for each d_δ-open set U containing x, there exist a regular F_σ-set V containing x and a subset C which is d_δ-connected relative to X such that $x \in V \subset C \subset U$.*

It is clear from the definition that the family of all regular F_σ-sets of a locally d_δ-connected space X, which are d_δ-connected relative to X, forms a basis for its D_δ-completely regularization X^*.

The following example enables us to distinguish the concept of locally d_δ-connected space with that of locally connected space, connected space, and subspace d_δ-connected relative to a space.

Example 2.2 *Consider $[0,1] \subset \mathbb{R}$ and $A = \mathbb{Q} \cap (0,1)$. Let X and Y be the spaces $([0,1], \Im_1)$ and $([0,1], \Im_2)$ respectively, where $\Im_1$ denotes the usual topology and $\Im_2$ denotes the topology generated by $\Im_1 \cup \{A\}$ as a subbase. It follows that the disconnected subspace $S = (0, 1/2) \cup (1/2, 1)$ of Y is a locally d_δ-connected space which is not even locally connected. Although it is not d_δ-connected relative to Y.*

Now, we see a characterization of a locally d_δ-connected space X in terms of d_δ-component of a d_δ-open subset relative to X. Here, a d_δ-component of subset S relative to X is subset $P \subset S$, which is d_δ-connected relative to X, and no d_δ-connected relative to X subset of S properly contains P, see [8].

Theorem 2.3 *A topological space X is locally d_δ-connected if and only if for each d_δ-open set U, d_δ-components of U relative to X are d_δ-open.*

Proof Let J be a d_δ-component of U relative to X. For any $x \in J$, there exist a regular F_σ-set V and a subset C which is d_δ-connected relative to X such that $x \in V \subset C \subset U$, by our hypothesis. Thus, $x \in V \subset C \subset J$ implies that J is d_δ-open.

Conversely, let $U \subset X$ be a d_δ-open set and $x \in U$. By hypothesis, d_δ-component of U relative to X containing x, is d_δ-open. Thus, X is locally d_δ-connected at x. Hence, X is locally d_δ-connected. □

Corollary 2.4 *In a locally d_δ-connected space X, if $x \neq y$ are points lying in different d_δ-components relative to X, then there exists a d_δ-separation (P, Q) relative to X with $x \in P$ and $y \in Q$.*

Proof It is clear from Theorem 2.3 that a d_δ-component relative to X is d_δ-open in X, which is d_δ-closed in X as well, by [8, Theorem 3.5(2)]. Hence, the pair (J_x, J_y) is a d_δ-separation relative to X, where J_x and J_y are d_δ-components relative to X containing x and y, respectively. □

Consequently, for a locally d_δ-connected space X, quasicomponents coincide with d_δ-components relative to X.

Theorem 2.5 *Let X be a connected, locally d_δ-connected space and U be a d_δ-open set in X. If J is a d_δ-component of U relative to X with $X-[J]_{d_\delta}$ is nonempty, then $[J]_{d_\delta}-J$ is nonempty such that the pair $(J, X-[J]_{d_\delta})$ forms a d_δ-separation relative to X.*

Proof Assume that $[J]_{d_\delta}-J$ is empty. Then J is a d_δ-closed set in X. Theorem 2.3 implies that J is d_δ-open as well, which is contrary to the hypothesis. Thus, $[J]_{d_\delta}-J\neq\emptyset$. Using [8, Theorem 3.5(1)], $(J, X-[J]_{d_\delta})$ forms a d_δ-separation relative to X because $J\cup(X-[J]_{d_\delta})=X-([J]_{d_\delta}-J)$ is a d_δ-open set. □

We now introduce the notion of d_δ-boundary of a subset of a space to discuss some useful results from [6].

Definition 2.6 *For a subset S of space X, the d_δ-boundary of S is the set $\partial_{d_\delta}S=[S]_{d_\delta}\cap[X-S]_{d_\delta}$. A point $x\in\partial_{d_\delta}S$ is called a d_δ-boundary point of S.*

Proposition 2.7 *If S is a subset of a locally d_δ-connected space X, then each d_δ-component J of S relative to X satisfies $\partial_{d_\delta}J\subset\partial_{d_\delta}S$.*

Proof Suppose that there is an $x\in\partial_{d_\delta}J$ which is not in $\partial_{d_\delta}S$. Then there exists a regular F_σ-set U_x containing x in X such that either $U_x\subset S$ or $U_x\subset(X-S)$. Since X is locally d_δ-connected, there exist another regular F_σ-set V_x and a subset C_x which is d_δ-connected relative to X, such that $x\in V_x\subset C_x\subset U_x$. As $V_x\cap J\neq\emptyset$, we have $C_x\cap J\neq\emptyset$. Firstly, we assume that $U_x\subset S$. Since J is a d_δ-component of S relative to X, $x\in V_x\subset C_x\subset J$, which gives a contradiction to the fact that $x\in\partial_{d_\delta}J$. On the other hand, if $U_x\subset(X-S)$, then $V_x\cap J=\emptyset$, again contrary to our assumption. □

It follows from Proposition 2.7 that for a d_δ-open set U in a locally d_δ-connected space X, $\partial_{d_\delta}J\cap U=\emptyset$ where J is a d_δ-component of U relative to X. Moreover, if d_δ-open set U is d_δ-connected relative to X, then $\partial_{d_\delta}U$ may fail to be d_δ-connected relative to X or locally d_δ-connected. For instance, consider the d_δ-open subset $U=\mathbb{R}^2-(\{(0,0)\}\cup\{(1/n,0)\mid n=1,2,\ldots\})$ of the locally d_δ-connected space $\mathbb{R}^2$. Then U is d_δ-connected relative to $\mathbb{R}^2$, but $\partial_{d_\delta}U$ is neither d_δ-connected relative to $\mathbb{R}^2$ nor locally d_δ-connected at point $(0,0)$.

Now, we shall see that the property of locally d_δ-connectedness of a space is inherited by its regular F_σ-subset.

Theorem 2.8 *Every regular F_σ-set A in a locally d_δ-connected space X is locally d_δ-connected.*

Proof Let $x\in A$ and U be a regular F_σ-set containing x in A. Since [8, Lemma 3.6] ensures that U is a regular F_σ-set in X, we have a regular F_σ-set V and a subset C which is d_δ-connected relative to X such that $x\in V\subset C\subset U$, by the hypothesis. Hence, V is the required regular F_σ-subset of A contained in subset C, which is d_δ-connected relative to A from [8, Proposition 3.7]. □

In general, even pseudo-D_δ-supercontinuous functions need not preserve locally d_δ-connectedness; we introduce the following functions which do so.

Definition 2.9 *For spaces X and Y, a surjective map $f : X \to Y$ is said to be Hd_δ-quotient provided a subset $S \subset Y$ is d_δ-open in Y if and only if $f^{-1}(S)$ is d_δ-open in X.*

It is clear that every Hd_δ-quotient map is pseudo-D_δ-supercontinuous.

Example 2.10 *In Example 2.2, the identity map $f : X \to Y$ is an Hd_δ-quotient map.*

Theorem 2.11 *Let $f : X \to Y$ be an Hd_δ-quotient map. If X is locally d_δ-connected, then so is Y.*

Proof For $y \in Y$, let $W \subset Y$ be a d_δ-open set containing y. Since f is an Hd_δ-quotient map, $f^{-1}(W) = \bigcup_\Delta U_\alpha$ where each U_α is a regular F_σ-set in X. For each $x \in f^{-1}(y)$, there is an $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Then there exist a regular F_σ-set V_{α_x} and a subset C_{α_x} which is d_δ-connected relative to X such that $x \in V_{\alpha_x} \subset C_{\alpha_x} \subset U_{\alpha_x}$. Now, [8, Theorem 3.18] ensures that $f(C_{\alpha_x})$ is d_δ-connected relative to Y such that $y \in f(C_{\alpha_x}) \subset W$. Let $V = \bigcup\{V_{\alpha_x} \mid x \in f^{-1}(y)\}$ and $C = \bigcup\{C_{\alpha_x} \mid x \in f^{-1}(y)\}$. As f is an Hd_δ-quotient map, subset $f(V)$ is a d_δ-open set containing y in Y, and subset $f(C)$ is d_δ-connected relative to Y being a union of subsets d_δ-connected relative to Y having common point y. Hence, $y \in f(V) \subset f(C) \subset W$. □

Definition 2.12 *[3] A function $f : X \to Y$ is said to be D_δ-supercontinuous at a point $x \in X$ if for each open set U containing $f(x)$, there is a regular F_σ-set V containing x such that $f(V) \subset U$. The function f is said to be D_δ-supercontinuous if it is D_δ-supercontinuous at each $x \in X$.*

Let $f : X \to Y$ be a D_δ-supercontinuous function and S be a subset of X. Let (P, Q) be a separation in Y, that is, P and Q are nonempty subsets of Y, where $\mathrm{cl}(P) \cap Q = \emptyset = P \cap \mathrm{cl}(Q)$ with $f(S) = P \cup Q$. Here, $\mathrm{cl}(P)$ denoted the closure of subset P in space Y. Let $A = S \cap f^{-1}(P)$ and $B = S \cap f^{-1}(Q)$. Then $S = A \cup B$, where A and B are nonempty. As f is D_δ-supercontinuous, it follows from [3, Theorem 3.5] that $[f^{-1}(Q)]_{d_\delta} \subset f^{-1}(\mathrm{cl}(Q))$, which further implies that $A \cap [B]_{d_\delta} = \emptyset$. Similarly, we have $[A]_{d_\delta} \cap B = \emptyset$. Therefore, (A, B) forms a d_δ-separation relative to X with $S = A \cup B$. Hence, the image of a subset which is d_δ-connected relative to X under D_δ-supercontinuous function $f : X \to Y$, is a connected subset of Y.

Definition 2.13 *[3] Let $f : X \to Y$ be a function from space X onto a set Y. The topology on Y for which a subset $S \subset Y$ is open if and only if $f^{-1}(S)$ is d_δ-open in X, is called the D_δ-quotient topology, and the map f is called the D_δ-quotient map.*

It easily follows from Definition 2.12 that a D_δ-quotient map is D_δ-supercontinuous.

Remark 2.14 *It is observed that the space X of all rational numbers with discrete topology is locally d_δ-connected, whereas the space Y, of all rational numbers with the usual topology is not locally d_δ-connected although the identity function $f : X \to Y$ is a D_δ-quotient map.*

Theorem 2.15 *Let $f : X \to Y$ be a D_δ-quotient map. If X is locally d_δ-connected, then Y is locally connected.*

Proof Let $y \in Y$, and let $W \subset Y$ be an open set containing y. Then $f^{-1}(W)$ is a d_δ-open set in X because f is a D_δ-quotient map. By following the similar arguments as in the proof of Theorem 2.11, for each $x \in f^{-1}(y)$, we have a regular F_σ-set V_x and a subset C_x which is d_δ-connected relative to X such that $x \in V_x \subset C_x \subset f^{-1}(W)$. As f is D_δ-supercontinuous, $f(C_x)$ is a connected subset of Y such that $y \in f(V_x) \subset f(C_x) \subset W$. Let $V = \bigcup\{V_x \mid x \in f^{-1}(y)\}$ and $C = \bigcup\{C_x \mid x \in f^{-1}(y)\}$. Since f is a D_δ-quotient map, $f(V)$ is an open neighbourhood of y, and the set $f(C)$ is connected being a union of connected sets having common point y. Hence, $y \in f(V) \subset f(C) \subset W$. □

Next, we note that the concept of locally d_δ-connected space X simplifies into locally connected space when X is D_δ-completely regular.

Theorem 2.16 *The D_δ-completely regularization X^* of a space X is locally connected if and only if X is locally d_δ-connected.*

Proof Assume that the space X^* is locally connected, and let $f : X^* \to X$ be the identity map, where a subset $S \subset X$ is d_δ-open if and only if $f^{-1}(S)$ is open in X^*. Let $x \in X$, and let $U \subset X$ be a d_δ-open subset containing x. Then U is an open set in X^*. Thus, there exist an open subset V and a connected subset C in space X^* such that $x \in V \subset C \subset U$, by hypothesis. Now, V and U are d_δ-open in X, and [8, Theorem 3.25] implies that C is d_δ-connected relative to X such that $x \in V \subset C \subset U$. Hence, X is locally d_δ-connected space. Since $f^{-1} : X \to X^*$ is a D_δ-quotient map, the converse holds from Theorem 2.15. □

We close this section with the extension of locally d_δ-connectedness to arbitrary products which is subsequent to the introduction of the following useful term.

Definition 2.17 *A function $f : X \to Y$ is said to be Hd_δ-open if for each regular F_σ-set U of X, $f(U)$ is a regular F_σ-set in Y.*

Example 2.18 *Consider the space Y of Example 2.2 and the space $Z = [0,1]$ with topology $\Im_K$ generated by the basis with members in the form of U and $U - K$, where U is an Euclidean neighborhood and $K = \{1/n \mid n = 1, 2, \ldots\}$. Then the identity map $f : Y \to Z$ is an Hd_δ-open map.*

It is immediate from the preceding example that an Hd_δ-open map need not be open. However, both the concepts coincide on D_δ-completely regular spaces.

Theorem 2.19 *Let $\{X_\alpha\}_\Lambda$ be a family of spaces such that $\prod_\Lambda X_\alpha$ is Hd_δ-completely regular. Then $\prod_\Lambda X_\alpha$ is locally d_δ-connected if and only if each X_α is locally d_δ-connected, and all but finitely many X_α are also connected.*

Proof Suppose $\prod_\Lambda X_\alpha$ is locally d_δ-connected. Let $x \in X_\beta$, and let $U_\beta \subset X_\beta$ be a regular F_σ-set containing x, for $\beta \in \Lambda$. Then $\pi_\beta^{-1}(U_\beta)$ is a regular F_σ-set in $\prod_\Lambda X_\alpha$ containing $w = (w_\alpha)_\Lambda$ with $w_\beta = x$, where π_β is the projection of $\prod_\Lambda X_\alpha$ onto X_β. By our assumption, there exist a regular F_σ-set $V \subset \prod_\Lambda X_\alpha$ and a subset C which is d_δ-connected relative to $\prod_\Lambda X_\alpha$ such that $w \in V \subset C \subset \pi_\beta^{-1}(U_\beta)$. Since π_β is a surjective Hd_δ-open pseudo-D_δ-supercontinuous function, we have $x \in \pi_\beta(V) \subset \pi_\beta(C) \subset U_\beta$, where the set $\pi_\beta(V)$ is a d_δ-open set in X_β, and $\pi_\beta(C)$ is d_δ-connected relative to X_β from [8, Theorem 3.18]. Additionally, for a regular F_σ-set F containing an element $z = (z_\alpha)_{\alpha \in \Lambda}$ in $\prod_\Lambda X_\alpha$, there exist another regular F_σ-set $\bigcap_\Gamma \pi_i^{-1}(U_i) \subset F$ containing

z and a subset C which is d_δ-connected relative to $\prod_\Lambda X_\alpha$ such that $z \in \bigcap_\Gamma \pi_i^{-1}(U_i) \subset C \subset F$, where $\Gamma \subset \Lambda$ is finite and each U_i is a regular F_σ-set containing z_i in X_i, for each $i \in \Gamma$. Then, for $\beta \in \Lambda - \Gamma$, we have $\pi_\beta(C) = X_\beta$ is connected.

Conversely, let $z = (z_\alpha)_\Lambda \in \prod_\Lambda X_\alpha$, and let $F \subset \prod_\Lambda X_\alpha$ be a regular F_σ-set containing z. Then there is another regular F_σ-set $\bigcap_\Gamma \pi_i^{-1}(U_i) \subset F$ containing z, where $\Gamma \subset \Lambda$ is finite and each U_i is a regular F_σ-set containing z_i in X_i, for each $i \in \Gamma$. By our hypothesis, there are at most finitely many indices $\alpha \in \Lambda - \Gamma$ such that X_α is not connected. Assume that the set of these indices is $\Omega = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. For each $\beta \in \Gamma \cup \Omega$, X_β is locally d_δ-connected. Thus, for regular F_σ-set $U_\beta \subset X_\beta$ containing z_β, there exist a regular F_σ-set V_β and a subset C_β which is d_δ-connected relative to X_β such that $z_\beta \in V_\beta \subset C_\beta \subset U_\beta$. Therefore, $M = \bigcap_{\Gamma\cup\Omega} \pi_\beta^{-1}(V_\beta)$ is a regular F_σ-set in $\prod_\Lambda X_\alpha$ and $N = \bigcap_{\Gamma\cup\Omega} \pi_\beta^{-1}(C_\beta)$ is d_δ-connected relative to $\prod_\Lambda X_\alpha$ from [8, Theorem 3.20], such that $z \in M \subset N \subset F$. Hence, $\prod_\Lambda X_\alpha$ is locally d_δ-connected. □

3. Locally D_δ-compact spaces

In this section, we develop the localized version of the concept of D_δ-compact space and characterize it.

Definition 3.1 *A space X is said to be locally D_δ-compact at $x \in X$ if there exist a regular F_σ-set U containing x and a subset C which is D_δ-closed relative to X such that $x \in U \subset C$. The space X is locally D_δ-compact if it is locally D_δ-compact at each of its points.*

It is clear that a D_δ-compact space is locally D_δ-compact. Besides, it can be seen in the following example that a locally D_δ-compact space may not be D_δ-compact.

Example 3.2 *The subspace $X = [0, 1)$ of the space Z in Example 2.18 is a locally D_δ-compact space, which is neither D_δ-compact nor a subset D_δ-closed relative to Z. Even the space X is not locally compact at 0 because it does not possess any compact neighborhood of 0.*

Next, we have some useful characterizations of locally D_δ-compact spaces.

Theorem 3.3 *Let X be a D_δ-Hausdorff space, then the following statements are equivalent:*

(i) X is locally D_δ-compact.

(ii) For each $x \in X$, there exists a regular F_σ-set U containing x such that $[U]_{d_\delta}$ is D_δ-closed relative to X.

(iii) For each $x \in X$ and each regular F_σ-set V containing x, there exists another regular F_σ-set M such that $x \in M \subset [M]_{d_\delta} \subset V$ and $[M]_{d_\delta}$ is D_δ-closed relative to X.

(iv) For each subset S which is D_δ-closed relative to X and a regular F_σ-set V containing S, there exists another regular F_σ-set M such that $S \subset M \subset [M]_{d_\delta} \subset V$ and $[M]_{d_\delta}$ is D_δ-closed relative to X.

Proof

$(i) \Rightarrow (ii)$: For $x \in X$, there exist a regular F_σ-set U containing x and a subset C which is D_δ-closed relative to X such that $x \in U \subset [U]_{d_\delta} \subset C$ because C is d_δ-closed in the D_δ-Hausdorff space X, [8, Theorem 4.9(3)]. Clearly, it follows from [8, Theorem 4.7] that $[U]_{d_\delta}$ is D_δ-closed relative to X.

$(ii) \Rightarrow (iii)$: For $x \in X$, there exists a regular F_σ-set U containing x such that $[U]_{d_\delta}$ is D_δ-closed relative to X. Then $U \cap V$ is a regular F_σ-set containing x and contained in $[U]_{d_\delta}$. From [8, Theorem 4.22], we have a regular F_σ-set M such that $x \in M \subset [M]_{d_\delta} \subset U \cap V \subset V$. Since $[M]_{d_\delta}$ is a d_δ-closed subset of X contained in $[U]_{d_\delta}$ which is D_δ-closed relative to X, we conclude $[M]_{d_\delta}$ is D_δ-closed relative to X.

$(iii) \Rightarrow (iv)$: For each $s \in S$, there exists a regular F_σ-set M_s such that $s \in M_s \subset [M_s]_{d_\delta} \subset V$ where $[M_s]_{d_\delta}$ is D_δ-closed relative to X. Since S is D_δ-closed relative to X, there is a finite collection $\{M_{s_i} \mid i = 1, 2, \ldots, n\}$ such that $S \subset \bigcup_{i=1}^{n} M_{s_i} = M$. As $[M]_{d_\delta} = \bigcup_{i=1}^{n} [M_{s_i}]_{d_\delta}$ is a finite union of d_δ-closed sets, the subset $[M]_{d_\delta}$ is D_δ-closed relative to X. Clearly, $S \subset M \subset [M]_{d_\delta} \subset V$.

$(iv) \Rightarrow (i)$: Since a point is certainly D_δ-closed relative to X and is contained in the regular F_σ-set X, for each $x \in X$, there is a regular F_σ-set M such that $\{x\} \subset M \subset [M]_{d_\delta}$, where $[M]_{d_\delta}$ is D_δ-closed relative to X. □

Remark 3.4 *Note that if X is a locally D_δ-compact D_δ-Hausdorff space, then the family of all regular F_σ-sets whose d_δ-closure is D_δ-closed relative to X, forms a basis for its D_δ-completely regularization X^*.*

We turn now to introduce the functions that preserve locally D_δ-compact spaces.

Theorem 3.5 *The image of a locally D_δ-compact space under a surjective Hd_δ-open pseudo-D_δ-supercontinuous function is locally D_δ-compact.*

Proof Let $f : X \to Y$ be a surjective Hd_δ-open pseudo-D_δ-supercontinuous function where X is a locally D_δ-compact space. For $y \in Y$, there exist an $x \in X$, a regular F_σ-set $U \subset X$ containing x, and a subset C which is D_δ-closed relative to X such that $x \in U \subset C$ with $f(x) = y$. Now, [8, Theorem 4.9(5)] ensures that $f(C)$ is the required subset which is D_δ-closed relative to Y. Since f is Hd_δ-open, $f(U) \subset Y$ is a d_δ-open set containing y such that $y \in f(U) \subset f(C)$. □

The following theorem can be proved easily, so the proof is omitted.

Theorem 3.6 *A space X is locally D_δ-compact if and only if X^* is locally compact.*

It is clear from the Definition 2.9 that every surjective Hd_δ-open pseudo-D_δ-supercontinuous function is an Hd_δ-quotient map, whereas the following example establishes that the notion of locally D_δ-compact space is not preserved by an Hd_δ-quotient map.

Example 3.7 *Consider $\mathbb{R}$ with the usual topology, and let an equivalence relation be defined on $\mathbb{R}$ as $x \sim y$ if and only if either $x = y$ or $\{x, y\} \subset \mathbb{Z}$. Let Y be the quotient space $\mathbb{R}/\sim$. Now, the quotient map $f : \mathbb{R} \to Y$ defined by $x \mapsto \alpha_x$ is an Hd_δ-quotient map which is not Hd_δ-open, where α_x is the equivalence class of x. However, the D_δ-Hausdorff space Y is not locally D_δ-compact.*

We shall prove that the d_δ-closure of any regular F_σ-set containing α_n is not D_δ-closed relative to Y, where $n \in \mathbb{Z}$ and $\alpha_n \in Y$. It is sufficient to consider a regular F_σ-set containing α_n of the form $f(U)$ where

$$U = \bigcup_{n \in \mathbb{Z}} (n - r_n, n + r_n),$$

with $r_n < \frac{1}{4}$. *It follows that* $[f(U)]_{d_\delta} = f([U]_{d_\delta})$ *with*

$$[U]_{d_\delta} = \bigcup_{n\in\mathbb{Z}} ([n - r_n, n + r_n]) .$$

For each $m \in \mathbb{Z}$, *consider a regular* F_σ*-set in* $\mathbb{R}$ *of the following form*

$$V_m = \left(\bigcup_{n<m} \left(n - r_n - \frac{1}{4}, n + r_n + \frac{1}{4} \right) \right) \bigcup \left(\bigcup_{m<n} (n - r_n, n + r_n) \right).$$

It is clear that the collection $\{f(V_m) \mid m \in \mathbb{Z}\}$ *is a covering of* $[f(U)]_{d_\delta}$ *by regular* F_σ*-sets in* Y*; however, it has no finite subcovering.*

In general, subspace of a locally D_δ-compact space need not be locally D_δ-compact, although d_δ-open subsets inherit the property.

Theorem 3.8 *Every* d_δ*-open subset of a locally* D_δ*-compact* D_δ*-Hausdorff space is locally* D_δ*-compact.*

Proof Let U be a d_δ-open set in the locally D_δ-compact D_δ-Hausdorff space X, and let $x \in U$. Then Theorem 3.3(iii) ensures that there exist regular F_σ-sets V and M in X such that $x \in M \subset [M]_{d_\delta} \subset V \subset U$ where $[M]_{d_\delta}$ is D_δ-closed relative to X. Now, [8, Lemma 3.6] implies that the subset $[M]_{d_\delta}$ is D_δ-closed relative to U because the collection $\{(H_\lambda \cap V)\}_\Lambda$ is a covering of $[M]_{d_\delta}$ by regular F_σ-sets of V, whenever $\{H_\lambda\}_\Lambda$ is a covering of $[M]_{d_\delta}$ by regular F_σ-sets in U. □

Next, we discuss the invariance of locally D_δ-compact spaces under the formation of products. For simplicity, we first treat the case of finite product.

Theorem 3.9 *Let* $\{X_1, X_2, \ldots, X_n\}$ *be a finite family of spaces such that* $\prod_{i=1}^n X_i$ *is* Hd_δ*-completely regular. Then the product space* $\prod_{i=1}^n X_i$ *is locally* D_δ*-compact if and only if each coordinate space* X_i *is locally* D_δ*-compact for* $i = 1, 2, \ldots, n$.

Proof Let $X_1, X_2, \ldots, X_n$ be locally D_δ-compact spaces and $(x_1, x_2, \ldots, x_n) \in \prod_{i=1}^n X_i$. Then there exist a regular F_σ-set $U_i \subset X_i$ containing x_i and a subset C_i which is D_δ-closed relative to X_i for each $i = 1, 2, \ldots, n$. Thus, $\prod_{i=1}^n U_i$ is a regular F_σ-set of $\prod_{i=1}^n X_i$ containing $(x_1, \ldots, x_n)$ and contained in the subset $\prod_{i=1}^n C_i$ which is D_δ-closed relative to $\prod_{i=1}^n X_i$, by [8, Theorem 4.10].

Conversely, if $\prod_{i=1}^n X_i$ is locally D_δ-compact, then $\pi_j(\prod_{i=1}^n X_i) = X_j$ is locally D_δ-compact for each $1 \le j \le n$, because the projection map $\pi_j : \prod_{i=1}^n X_i \to X_j$ is surjective Hd_δ-open pseudo-D_δ-supercontinuous. □

However, the property of locally D_δ-compactness is not transmitted to arbitrary products. In this regard, we have the following.

Theorem 3.10 *Let* $\{X_\alpha\}_\Lambda$ *be an infinite family of spaces such that* $\prod_\Lambda X_\alpha$ *is* Hd_δ*-completely regular. Then the product space* $\prod_\Lambda X_\alpha$ *is locally* D_δ*-compact if and only if each* X_α *is locally* D_δ*-compact, and all but finitely many* X_α *are* D_δ*-compact.*

Proof Let X_α be a locally D_δ-compact space for each $\alpha \in \Lambda$, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the indices such that space X_{α_i} is not D_δ-compact for each $i = 1, 2, \ldots, n$. For $(x_\alpha)_\Lambda \in \prod_\Lambda X_\alpha$, the hypothesis ensures the existence of regular F_σ-set V_{α_i} containing x_{α_i} in X_{α_i} and subset C_{α_i} which is D_δ-closed relative to X_{α_i} such that $x_{\alpha_i} \in V_{\alpha_i} \subset C_{\alpha_i}$ for $i = 1, 2, \ldots, n$. Then $V_{\alpha_1} \times \ldots \times V_{\alpha_n} \times \prod\{X_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \ldots, \alpha_n\}\}$ is a regular F_σ-set in $\prod_\Lambda X_\alpha$ containing $(x_\alpha)_\Lambda$ and contained in $C_{\alpha_1} \times \ldots \times C_{\alpha_n} \times \prod\{X_\alpha \mid \alpha \in \Lambda - \{\alpha_1, \ldots, \alpha_n\}\}$, which is D_δ-closed relative to $\prod_\Lambda X_\alpha$ using [8, Theorem 4.10].

Conversely, if $\prod_\Lambda X_\alpha$ is locally D_δ-compact, we assert that $\pi_\beta(\prod_\Lambda X_\alpha) = X_\beta$ is locally D_δ-compact for each $\beta \in \Lambda$. Now, we have a regular F_σ-set U containing $(x_\alpha)_\Lambda$ in $\prod_\Lambda X_\alpha$ and a subset C which is D_δ-closed relative to $\prod_\Lambda X_\alpha$ such that $(x_\alpha)_\Lambda \in U \subset C$. Since $\prod_\Lambda X_\alpha$ is Hd_δ-completely regular space, there exists another regular F_σ-set $W = \bigcap_\Gamma \pi_i^{-1}(V_i)$ containing $(x_\alpha)_\Lambda$ and contained in U, where $\Gamma \subset \Lambda$ is finite and each V_i is a regular F_σ-set containing x_i in X_i. Also, [8, Theorem 4.9(5)] ensures that $\pi_\alpha(C) = X_\alpha$ is D_δ-compact for $\alpha \notin \Gamma$. □

We end this section with the following which discusses a kind of separation through regular F_σ-sets.

Theorem 3.11 *Let X be a locally D_δ-compact D_δ-Hausdorff space. If C_1 and C_2 are disjoint subsets which are D_δ-closed relative to X, then there exist disjoint regular F_σ-sets U and V containing C_1 and C_2, respectively, such that $[U]_{d_\delta}$ and $[V]_{d_\delta}$ are D_δ-closed relative to X.*

Proof Being a subset D_δ-closed relative to X, C_2 is d_δ-closed subset of D_δ-Hausdorff space X from [8, Theorem 4.9(3)]. Thus, $X - C_2$ is a d_δ-open set containing C_1. By part (iv) of Theorem 3.3, there exists a regular F_σ-set U such that $C_1 \subset U \subset [U]_{d_\delta} \subset X - C_2$ where $[U]_{d_\delta}$ is D_δ-closed relative to X. Similarly, for d_δ-open set $X - [U]_{d_\delta}$ containing C_2, there exists another regular F_σ-set V such that $C_2 \subset V \subset [V]_{d_\delta} \subset X - [U]_{d_\delta}$ where $[V]_{d_\delta}$ is D_δ-closed relative to X. □

4. Consequences of the notions

Now we shall demonstrate two interesting applications of the above studied concepts.

Lemma 4.1 *[2, p.108] If X is a connected space, then for every covering $\mathcal{U}$ of X by open sets and for every pair of elements x and y in X, there exist $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$, called a string from x to y in $\mathcal{U}$ where $x \in U_1$ and $y \in U_n$.*

Note that Lemma 4.1 holds for the covering of X such that d_δ-interiors of its members cover X. Here, d_δ-interior of $A \subset X$ is the union of all regular F_σ-sets of X contained in A.

Theorem 4.2 *Let X be a connected, locally d_δ-connected, locally D_δ-compact D_δ-Hausdorff space, and let $x, y \in X$. Then there is a subset C which is d_δ-connected relative to X and D_δ-closed relative to X containing both x and y.*

Proof For each $x \in X$, there exist a regular F_σ-set U containing x and a subset K which is D_δ-closed relative to X such that $x \in U \subset K$. Since X is locally d_δ-connected, there exist another regular F_σ-set V and a subset C which is d_δ-connected relative to X such that $x \in V \subset C \subset U \subset K$. Then $[C]_{d_\delta}$ is d_δ-connected relative to X and D_δ-closed relative to X such that $x \in [C]_{d_\delta} \subset K$, using [8, Theorem 3.5(2) and Theorem

4.7]. Now, being a connected space, for each pair of elements x and y in X, Lemma 4.1 ensures the existence of finitely many subsets $C_1, C_2, \ldots, C_n$ which are d_δ-connected relative to X and D_δ-closed relative to X, which forms a string from x to y. Hence, $\bigcup_{i=1}^{n} C_i$ is the required subset which is d_δ-connected relative to X and D_δ-closed relative to X, containing x and y. □

We conclude our discussion with the following form of the Poincaré–Volterra Theorem.

Theorem 4.3 *Let X be a connected and locally d_δ-connected space which satisfies that for each $x \in X$ and each d_δ-open set A containing x in X, there exists a regular F_σ-set B such that $x \in B \subset [B]_{d_\delta} \subset A$. Let Y be another space having a countable collection of regular F_σ-sets, which forms a basis for the D_δ-completely regularization Y^* of Y, and let $f : X \to Y$ be a pseudo-D_δ-supercontinuous function such that, for each $y \in Y$, there is a regular F_σ-set H in X with $H \cap f^{-1}(y)$ is a singleton. Now, let $\mathcal{U}$ be a collection of subsets of X whose d_δ-interiors cover X with the following additional properties:*

1. *The restriction function $f|_P$ of f to each $P \in \mathcal{U}$, maps each d_δ-closed set in P to a d_δ-closed set in Y.*
2. *Every $P \in \mathcal{U}$ has a countable subset Q with $[Q]_{d_\delta} = P$.*

Then, the space X is the union of a countable collection of d_δ-open sets, each of which is a subset of a member of $\mathcal{U}$.

Proof Let $\tilde{f} : X^* \to Y^*$ be the function associated with f such that $\tilde{f}(x) = f(x)$ for each $x \in X^*$, where X^* is D_δ-completely regularization of X. Thus, [8, Corollary 3.9, Theorem 3.24] and Theorem 2.16 implies that spaces X^* and Y^* satisfy all the conditions of [1, Theorem I, p.114], which provides the required countable collection of d_δ-open sets. □

Acknowledgements

First author is financially supported by Council of Scientific and Industrial Research, India
(Ref. No. 09/045(1664)/2019-EMR-I).

References

[1] Bourbaki N. Elements of Mathematics, General topology Part 1. Paris, France: Hermann, 1966.

[2] Hocking JG, Young GS. Topology. London, UK: Addison-Wesley Publishing Co., Inc., 1961.

[3] Kohli JK, Singh D. D_δ-supercontinuous functions. Indian Journal of Pure and Applied Mathematics 2003; 34 (7): 1089-1100.

[4] Kohli JK, Singh D. Between compactness and quasicompactness. Acta Mathematica Hungarica 2005; 106 (4): 317-329.

[5] Kohli JK, Singh D. Between regularity and complete regularity and a factorization of complete regularity. Studii şi Cercetări Ştiinţifice. Seria: Matematică. Universitatea din Bacău 2007; 17: 125-134.

[6] Singh TB. Elements of topology. Boca Raton, FL, USA: CRC Press, 2013.

[7] Singh D. Pseudo strongly θ-continuous functions. Scientific Studies and Research. Series Mathematics and Informatics 2013; 23 (2): 127-140.

[8] Singh D, Mathur H. Generalizations of connected and compact sets by d_δ-closure operator. Bulletin of the Belgian Mathematical Society. Simon Stevin 2019; 26 (2): 255-273.