

1-1-2020

Weighted composition operators from the Bloch space to nth weighted-typespaces

EBRAHIM ABBASI

SONGXIAO LI

HAMID VAEZI

Follow this and additional works at: <https://dctubitak.researchcommons.org/math>



Part of the [Mathematics Commons](#)

Recommended Citation




ABBASI, EBRAHIM; LI, SONGXIAO; and VAEZI, HAMID (2020) "Weighted composition operators from the Bloch space to nth weighted-typespaces," *Turkish Journal of Mathematics*: Vol. 44: No. 1, Article 7.

<https://doi.org/10.3906/mat-1907-34>

Available at: <https://dctubitak.researchcommons.org/math/vol44/iss1/7>

This Article is brought to you for free and open access by TÜBİTAK Academic Journals. It has been accepted for inclusion in Turkish Journal of Mathematics by an authorized editor of TÜBİTAK Academic Journals.

Weighted composition operators from the Bloch space to n th weighted-type spaces

Ebrahim ABBASI¹, Songxiao LI^{2,*}, Hamid VAEZI³

¹Department of Mathematics, Mahabad Branch, Islamic Azad University, Mahabad, Iran

²Institute of Fundamental and Frontier Sciences,

University of Electronic Science and Technology of China, Chengdu, Sichuan, P.R. China

³Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Received: 09.07.2019

Accepted/Published Online: 02.11.2019

Final Version: 20.01.2020

Abstract: In this work, we characterize the boundedness of weighted composition operators from the Bloch space and the little Bloch space to n th weighted-type spaces. Some estimates for the essential norm of these operators are also given. As a corollary, we obtain some characterizations for the compactness of weighted composition operators from the Bloch space and the little Bloch space to n th weighted-type spaces.

Key words: Weighted composition operator, Bloch space, essential norm, n th weighted-type space

1. Introduction

Let \mathbb{D} be the open unit disc in the complex plane \mathbb{C} , $H(\mathbb{D})$ the set of all analytic functions on \mathbb{D} , and $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on \mathbb{D} with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the set of all $f \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty.$$

The Bloch space \mathcal{B} is a Banach space with the above norm $\|\cdot\|_{\mathcal{B}}$. The little Bloch space \mathcal{B}_0 consists of all $f \in H(\mathbb{D})$ such that $\lim_{|z| \rightarrow 1} (1 - |z|^2) |f'(z)| = 0$. It is well known that \mathcal{B}_0 is the closure of polynomials in \mathcal{B} .

Let μ be a weight, which means that μ is a positive and continuous function on \mathbb{D} . Let $n \in \mathbb{N}$, the set of all positive integers. The n th weighted-type space, denoted by $\mathcal{W}_\mu^n = \mathcal{W}_\mu^n(\mathbb{D})$, is the set of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{W}_\mu^n} = \sum_{i=0}^{n-1} |f^{(i)}(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)| < \infty.$$

It is easy to check that \mathcal{W}_μ^n is a Banach space with the above norm. We refer the interested reader to [16–18] for the space \mathcal{W}_μ^n . When $n = 1$, the space \mathcal{W}_μ^1 is called the Bloch-type space. Let $\beta > 0$ and $\mu(z) = (1 - |z|^2)^\beta$. The space \mathcal{W}_μ^1 coincides with the Bloch-type space \mathcal{B}^β . In particular, \mathcal{B}^1 is the classical Bloch space \mathcal{B} .

*Correspondence: jyulsx@163.com

2010 AMS Mathematics Subject Classification: 47B33, 30H99

The space \mathcal{W}_μ^2 is also called the Zygmund-type space. For more information about Bloch-type spaces and Zygmund-type spaces, see [2, 7, 9, 21, 22].

Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the set of all analytic self-maps of \mathbb{D} . The weighted composition operator, denoted by C_φ^g , induced by φ and g is defined as follows.

$$(C_\varphi^g f)(z) = g(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

When $g \equiv 1$, the operator C_φ^g is denoted by C_φ and called the composition operator. If $\varphi(z) = z$, then C_φ^g is called the multiplication operator and denoted by M_g . Interested readers can refer to [5] for the theory of composition operators and weighted composition operators.

For any $\varphi \in S(\mathbb{D})$, it is widely known that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is bounded. The compactness of $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ was investigated in [12]. Wulan et al. [19] showed that $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ is compact if and only if $\lim_{k \rightarrow \infty} \|\varphi^k\|_{\mathcal{B}} = 0$. Zhao in [20] characterized the essential norm of $C_\varphi : \mathcal{B} \rightarrow \mathcal{B}$ by using $\|\varphi^k\|_{\mathcal{B}}$. The operator $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{B}$ was studied in [14, 15]. In [3], Colonna investigated $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{B}$ by using $\|g\varphi^k\|_{\mathcal{B}}$. See [6, 8, 10, 11, 13, 23, 24] for more results of the operator $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{B}$.

Motivated by [10, 19], in this paper, we give some characterizations for the boundedness of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$. Moreover, we give some estimates for the essential norm of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$. By applying these estimates, some characterizations for the compactness of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ are obtained.

Throughout this paper, we will use the notation $A \preceq B$ if there exists a constant $C > 0$ such that $A \leq CB$. In particular, if $A \preceq B$ and $B \preceq A$, then we write $A \approx B$ and say that A and B are comparable.

2. Boundedness

To investigate the boundedness of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$, we need to state some lemmas. The next lemma was proved in [21].

Lemma 2.1 For any $f \in \mathcal{B}$ and $n \in \mathbb{N}$,

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| \approx \sum_{i=0}^n |f^{(i)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n+1} |f^{(n+1)}(z)|.$$

Lemma 2.2 ([11]) Let $f \in \mathcal{B}$. Then,

$$|f(z)| \leq \frac{1}{\log 2} \|f\|_{\mathcal{B}} \log \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

For any $a \in \mathbb{D}$ and $i \in \{1, 2, \dots, n\}$, set

$$f_{i,a}(z) = \left(\frac{1 - |a|^2}{1 - \bar{a}z} \right)^i, \quad z \in \mathbb{D}. \tag{2.1}$$

It is clear that $f_{i,a} \in \mathcal{B}$ for each $i \in \{1, 2, \dots, n\}$. From Lemma 2.3 of [1], we have the following Lemma.

Lemma 2.3 For any $0 \neq a \in \mathbb{D}$ and $i \in \{1, \dots, n\}$, there exist functions $v_{i,a} \in \mathcal{B}$ such that

$$v_{i,a}^{(k)}(a) = \begin{cases} \frac{\bar{a}^i}{(1 - |a|^2)^i}, & k = i, \\ 0, & k \neq i. \end{cases}$$

For $n, k \in \mathbb{N}_0$, the set of all nonnegative integers, with $k \leq n$, the partial Bell polynomials are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

where the sum taken over all $j_1, j_2, \dots, j_{n-k+1}$ such that

$$j_1 + j_2 + \dots + j_{n-k+1} = k, \quad j_1 + 2j_2 + \dots + (n-k+1)j_{n-k+1} = n.$$

where $j_1, j_2, \dots, j_{n-k+1} \in \mathbb{N}_0$. For more information about Bell polynomials, see [4, p 134].

The proof of the following lemma is similar to the proof of Lemma 4 in [17], so we omit the details.

Lemma 2.4 *Let $f, h, g \in H(\mathbb{D})$. Then for any $n \in \mathbb{N}_0$,*

$$(C_h^g f)^{(n)}(z) = \sum_{i=0}^n f^{(i)}(h(z)) \sum_{l=i}^n \binom{n}{l} g^{(n-l)}(z) B_{l,i}(h'(z), \dots, h^{(l-i+1)}(z)).$$

Lemma 2.5 *Let $n \in \mathbb{N}$ and $\varphi \in S(\mathbb{D})$. Then for any $a \in \mathbb{D}$, there exists a function $u_a \in \mathcal{B}_0$ such that*

$$u_a(\varphi(a)) = \log \frac{2}{1 - |\varphi(a)|^2} \text{ and } u'_a(\varphi(a)) = u''_a(\varphi(a)) = \dots = u_a^{(n)}(\varphi(a)) = 0.$$

Proof If $\varphi(a) = 0$, then $u_a(z) = \log 2$. For any $a \in \mathbb{D}$ with $\varphi(a) \neq 0$, set

$$g_{a,k}(z) = (n+k) \log \frac{2}{1 - \varphi(a)z} - \frac{(\log \frac{2}{1 - \varphi(a)z})^{n+k}}{(\log \frac{2}{1 - |\varphi(a)|^2})^{n+k-1}}, \quad k \in \{1, 2, \dots, n\}.$$

Then $g_{a,k} \in \mathcal{B}$ and $\lim_{|z| \rightarrow 1} (1 - |z|^2) |g'_{a,k}(z)| = 0$. So, $g_{a,k} \in \mathcal{B}_0$. Now we calculate $g_{a,k}^{(s)}(\varphi(a))$ for $s \in \{2, \dots, n\}$.

Set

$$g(z) = 1, \quad h(z) = \log \frac{2}{1 - \varphi(a)z} \quad \text{and} \quad f(z) = (n+k)z - \frac{z^{n+k}}{\gamma^{n+k-1}}$$

in Lemma 2.4, where $\gamma = \log \frac{2}{1 - |\varphi(a)|^2}$. We get

$$\begin{aligned} g_{a,k}^{(s)}(\varphi(a)) &= (f(h))^{(s)}(\varphi(a)) \\ &= \sum_{i=1}^s f^{(i)}(h(\varphi(a))) B_{s,i}(h'(\varphi(a)), \dots, h^{(s-i+1)}(\varphi(a))). \end{aligned} \tag{2.2}$$

Also $f'(h(\varphi(a))) = g'_{a,k}(\varphi(a)) = 0$ and for $i \in \{2, \dots, s\}$,

$$f^{(i)}(h(\varphi(a))) = -p_i^{n+k} \times \gamma^{1-i}, \tag{2.3}$$

where $p_i^{n+k} = \frac{(n+k)!}{(n+k-i)!}$. After a calculation, we have

$$h^{(i)}(\varphi(a)) = \frac{(i-1)! \overline{\varphi(a)}^i}{(1 - |\varphi(a)|^2)^i} = (i-1)! (h'(\varphi(a)))^i.$$

Hence,

$$\begin{aligned}
 B_{s,i}(h'(\varphi(a)), \dots, h^{(s-i+1)}(\varphi(a))) &= \sum \frac{s!}{j_1! j_2! \dots j_{s-i+1}!} \left(\frac{h'(\varphi(a))}{1!}\right)^{j_1} \left(\frac{h''(\varphi(a))}{2!}\right)^{j_2} \dots \left(\frac{h^{(s-i+1)}(\varphi(a))}{(s-i+1)!}\right)^{j_{s-i+1}} \\
 &= \sum \frac{s!}{j_1! \dots j_{s-i+1}! 1^{j_1} \dots (s-i+1)^{j_{s-i+1}}} (h'(\varphi(a)))^{j_1 + \dots + (s-i+1)j_{s-i+1}} \\
 &= \underbrace{\sum \frac{s!}{j_1! j_2! \dots j_{s-i+1}! 1^{j_1} 2^{j_2} \dots (s-i+1)^{j_{s-i+1}}}}_{b_i^s} \times \frac{\overline{\varphi(a)}^s}{(1 - |\varphi(a)|^2)^s}. \tag{2.4}
 \end{aligned}$$

So, from (2.2)–(2.4), for any $s \in \{2, \dots, n\}$, we get

$$g_{a,k}^{(s)}(\varphi(a)) = -\frac{\overline{\varphi(a)}^s}{(1 - |\varphi(a)|^2)^s} \sum_{i=2}^s \gamma^{1-i} b_i^s p_i^{n+k}.$$

Now for any $a \in \mathbb{D}$ with $\varphi(a) \neq 0$ and coefficients c_1, c_2, \dots, c_n , we set

$$d_{a,c_1,c_2,\dots,c_n}(z) = \sum_{k=1}^n c_k g_{a,k}(z).$$

It is clear that $d_{a,c_1,c_2,\dots,c_n} \in \mathcal{B}_0$. We consider the system of linear equations

$$\begin{aligned}
 d_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) &= \gamma \sum_{k=1}^n (n+k-1)c_k = \gamma \\
 d'_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) &= \sum_{k=1}^n c_k \times \underbrace{g'_{a,k}(\varphi(a))}_0 = 0 \\
 d''_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) &= -\frac{\overline{\varphi(a)}^2}{(1-|\varphi(a)|^2)^2} \gamma^{-1} \sum_{k=1}^n b_2^2 p_2^{n+k} c_k = 0 \\
 \dots &= \dots = \dots \\
 d^{(s)}_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) &= -\frac{\overline{\varphi(a)}^s}{(1-|\varphi(a)|^2)^s} \sum_{k=1}^n \left(\sum_{i=2}^s \gamma^{1-i} b_i^s p_i^{n+k}\right) c_k = 0 \\
 \dots &= \dots = \dots \\
 d^{(n)}_{a,c_1,c_2,\dots,c_{n+1}}(\varphi(a)) &= -\frac{\overline{\varphi(a)}^n}{(1-|\varphi(a)|^2)^n} \sum_{k=1}^n \left(\sum_{i=2}^n \gamma^{1-i} b_i^n p_i^{n+k}\right) c_k = 0.
 \end{aligned} \tag{2.5}$$

Similar to the proof of Lemma 2.3 in [1], we see that the system (2.5) has a unique solution and the solution is independent of choice a and $\varphi(a)$. If c_1, c_2, \dots, c_n is that solution, we get $u_a(z) = d_{a,c_1,c_2,\dots,c_n}(z)$, as desired.

□

Let $\varphi \in S(\mathbb{D})$, $i, n \in \mathbb{N}_0$ and $i \leq n$. For simplicity, we set

$$I_i^n(z) := \sum_{l=i}^n \binom{n}{l} g^{(n-l)}(z) B_{l,i}(\varphi'(z), \varphi''(z), \dots, \varphi^{(l-i+1)}(z)). \tag{2.6}$$

Theorem 2.6 *Let $n \in \mathbb{N}$, μ be a weight, $g \in \mathcal{W}_\mu^n$ and $\varphi \in S(\mathbb{D})$. Then the following statements are equivalent.*

(a) $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is bounded.

(b) $C_\varphi^g : \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n$ is bounded.

(c) $\sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty$ and $\sup_{j \geq 0} \|g\varphi^j\|_{\mathcal{W}_\mu^n} < \infty$.

$$(d) \sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty \text{ and for each } i \in \{1, \dots, n\},$$

$$\sup_{a \in \mathbb{D}} \|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n} < \infty, \quad \sup_{z \in \mathbb{D}} \mu(z) |I_i^n(z)| < \infty,$$

where $f_{i,a}$ are defined in (2.1).

$$(e) \sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty \text{ and for each } i \in \{1, \dots, n\},$$

$$\sup_{z \in \mathbb{D}} \frac{\mu(z) |I_i^n(z)|}{(1-|\varphi(z)|^2)^i} < \infty.$$

Proof (a) \Rightarrow (b) It is obvious.

(b) \Rightarrow (c) For any $w \in \mathbb{D}$, let u_w be the function defined in Lemma 2.5. Then $L = \sup_{w \in \mathbb{D}} \|u_w\|_{\mathcal{B}} < \infty$. Using Lemmas 2.4 and 2.5, we get

$$\begin{aligned} \mu(w) |g^{(n)}(w)| \log \frac{2}{1-|\varphi(w)|^2} &= \mu(w) |g^{(n)}(w)| |u_w(\varphi(w))| \\ &= \mu(w) |(C_\varphi^g u_w)^{(n)}(\varphi(w))| \leq \|C_\varphi^g u_w\|_{\mathcal{W}_\mu^n} \leq L \|C_\varphi^g\|_{\mathcal{W}_\mu^n} < \infty. \end{aligned}$$

Thus, $\sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} < \infty$. Since the sequence $\{z^j\}_0^\infty$ is bounded in \mathcal{B}_0 (see [11]), we get $\sup_{j \geq 0} \|g\varphi^j\|_{\mathcal{W}_\mu^n} < \infty$ by the boundedness of $C_\varphi^g : \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n$.

(c) \Rightarrow (d) It follows from Theorem 3.1 in [1].

(d) \Rightarrow (e) It also follows from Theorem 3.1 in [1].

(e) \Rightarrow (a) For any $f \in \mathcal{B}$, from Lemmas 2.1, 2.2 and 2.4, we get

$$\mu(z) |(C_\varphi^g f)^{(n)}(z)| \leq \frac{1}{\log 2} \|f\|_{\mathcal{B}} \sup_{z \in \mathbb{D}} \mu(z) |g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} + C \|f\|_{\mathcal{B}} \sum_{i=1}^n \sup_{z \in \mathbb{D}} \frac{\mu(z) |I_i^n(z)|}{(1-|\varphi(z)|^2)^i}$$

and for $j = 0, 1, \dots, n-1$,

$$|(C_\varphi^g f)^{(j)}(0)| \leq \frac{1}{\log 2} |g(0)| \|f\|_{\mathcal{B}} \log \frac{2}{1-|\varphi(0)|^2} + C \frac{\|f\|_{\mathcal{B}}}{\mu(0)} \sum_{i=1}^j \frac{\mu(0) |I_i^j(0)|}{(1-|\varphi(0)|^2)^i}.$$

Thus, $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is bounded by the assumed condition. The proof is complete. \square

3. Essential norm

In this section we give some estimates for $\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n}$, the essential norm of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$. Recall that

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} = \inf \{ \|C_\varphi^g - K\|_{\mathcal{B} \rightarrow \mathcal{W}_\mu^n} : K \text{ is compact} \}.$$

We begin with the following lemma.

Lemma 3.1 *Let $n \in \mathbb{N}$, $\varphi \in S(\mathbb{D})$ and $\{a_i\}$ be a sequence in \mathbb{D} such that $|\varphi(a_i)| \rightarrow 1$ as $i \rightarrow \infty$. Then there exists a bounded sequence $\{h_i\}$ in \mathcal{B}_0 such that, $\{h_i\}$ converge to 0 uniformly on compact subsets of \mathbb{D} and*

$$h_i(\varphi(a_i)) = \log \frac{2}{1-|\varphi(a_i)|^2}, \quad h_i'(\varphi(a_i)) = h_i''(\varphi(a_i)) = \dots = h_i^{(n)}(\varphi(a_i)) = 0.$$

Proof The proof is similar to the proof of Lemma 2.5. Hence, we omit the details. □

Theorem 3.2 Let $n \in \mathbb{N}$, μ be a weight, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is bounded. Then

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n \approx \max\{B_i\}_{i=0}^n \approx \|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \approx \max\left\{\limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0\right\},$$

where

$$A_0 = B_0 = \limsup_{|\varphi(z)| \rightarrow 1} \mu(z) |g^{(n)}(z)| \log \frac{2}{1 - |\varphi(z)|^2},$$

$$A_i = \limsup_{|a| \rightarrow 1} \|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n}, \quad B_i = \limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z) |I_i^n(z)|}{(1 - |\varphi(z)|^2)^i}, \quad i \in \{1, \dots, n\},$$

and $f_{i,a}$ are defined in (2.1).

Proof First we show that

$$\|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \succeq A_0 = B_0. \tag{3.1}$$

Let $\{z_i\}$ be a sequence in \mathbb{D} such that $|\varphi(z_i)| \rightarrow 1$ as $i \rightarrow \infty$. We assume that for any i , $\varphi(z_i) \neq 0$. Let $\{h_i\}$ be the sequence defined in Lemma 4.1. Then for any compact operator $K : \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n$, $\lim_{i \rightarrow \infty} \|Kh_i\|_{\mathcal{W}_\mu^n} = 0$. Thus,

$$\|(C_\varphi^g - K)h_i\|_{\mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \geq \limsup_{i \rightarrow \infty} \|C_\varphi^g h_i\|_{\mathcal{W}_\mu^n} - \limsup_{i \rightarrow \infty} \|Kh_i\|_{\mathcal{W}_\mu^n} \geq \limsup_{i \rightarrow \infty} \mu(z_i) |g^{(n)}(z_i)| \log \frac{2}{1 - |\varphi(z_i)|^2},$$

which implies the desired result.

From Theorem 4.2 in [1], we have

$$\|C_\varphi^g\|_{e, H^\infty \rightarrow \mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \|C_\varphi^g\|_{e, H^\infty \rightarrow \mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=1}^n.$$

So,

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=1}^n.$$

From the last inequality and (3.1) we obtain

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \succeq \max\{A_i\}_{i=0}^n \quad \text{and} \quad \|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \succeq \max\{B_i\}_{i=0}^n. \tag{3.2}$$

Next, we show that

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \max\{A_i\}_{i=0}^n \quad \text{and} \quad \|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \max\{B_i\}_{i=0}^n.$$

For $r \in [0, 1)$, let $K_r f(z) = f_r(z) = f(rz)$. Then $K_r : \mathcal{B} \rightarrow \mathcal{B}$ is compact with $\|K_r\| \leq 1$. It is obvious that f_r uniformly converge to f on compact subsets of \mathbb{D} as $r \rightarrow 1$. Let $\{r_j\} \subset (0, 1)$ such that $r_j \rightarrow 1$ as $j \rightarrow \infty$. Then for any $j \in \mathbb{N}$, $C_\varphi^g K_{r_j} : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is compact. Thus,

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \leq \limsup_{j \rightarrow \infty} \|C_\varphi^g - C_\varphi^g K_{r_j}\|. \tag{3.3}$$

Hence, we only need to show that

$$\limsup_{j \rightarrow \infty} \|C_\varphi^g - C_\varphi^g K_{r_j}\|_{\mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \min\{\max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n\}.$$

For any $f \in \mathcal{B}$ such that $\|f\|_{\mathcal{B}} \leq 1$,

$$\begin{aligned} \|(C_\varphi^g - C_\varphi^g K_{r_j})f\|_{\mathcal{W}_\mu^n} &= \sum_{t=0}^{n-1} \left| \sum_{k=0}^t (f - f_{r_j})^{(k)}(\varphi(0)) I_k^t(0) \right| + \sup_{z \in \mathbb{D}} \mu(z) \left| \sum_{k=0}^n (f - f_{r_j})^{(k)}(\varphi(z)) I_k^n(z) \right| \\ &\leq \underbrace{\sum_{t=0}^{n-1} \left| \sum_{k=0}^t (f - f_{r_j})^{(k)}(\varphi(0)) I_k^t(0) \right|}_{H_1} + \underbrace{\sup_{|\varphi(z)| \leq r_N} \mu(z) \left| \sum_{k=0}^n (f - f_{r_j})^{(k)}(\varphi(z)) I_k^n(z) \right|}_{H_2} \\ &\quad + \underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) \left| \sum_{k=0}^n (f - f_{r_j})^{(k)}(\varphi(z)) I_k^n(z) \right|}_{H_3}, \end{aligned} \tag{3.4}$$

where $N \in \mathbb{N}$ such that $r_j \geq \frac{2}{3}$ for all $j \geq N$. Since for any $s \in \mathbb{N}_0$, $(f - f_{r_j})^{(s)}$ converges to zero uniformly on compact subsets of \mathbb{D} as $j \rightarrow \infty$, by using Theorem 2.6 (d), we obtain

$$\limsup_{j \rightarrow \infty} H_1 = 0 \quad \text{and} \quad \limsup_{j \rightarrow \infty} H_2 = 0. \tag{3.5}$$

In addition,

$$H_3 \leq \sum_{k=0}^n \left[\underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) |f^{(k)}(\varphi(z))| |I_k^n(z)|}_{M_k} + \underbrace{\sup_{|\varphi(z)| > r_N} \mu(z) |r_j^k f^{(k)}(r_j \varphi(z))| |I_k^n(z)|}_{N_k} \right]. \tag{3.6}$$

First we estimate M_0 and N_0 . From Lemma 2.2,

$$M_0 = \sup_{|\varphi(z)| > r_N} \mu(z) |f(\varphi(z))| |g^{(n)}(z)| \leq \sup_{|\varphi(z)| > r_N} \mu(z) |g^{(n)}(z)| \frac{1}{\log 2} \|f\|_{\mathcal{B}} \log \frac{2}{1 - |\varphi(z)|^2} \preceq A_0 = B_0. \tag{3.7}$$

Similarly,

$$N_0 \preceq A_0 = B_0. \tag{3.8}$$

For $k \in \{1, \dots, n\}$, by (2.6) and Lemmas 2.1, 2.3 and 2.4,

$$\begin{aligned} M_k &= \sup_{|\varphi(z)| > r_N} \mu(z) \frac{(1 - |\varphi(z)|^2)^k |f^{(k)}(\varphi(z))| |\varphi(z)|^k |I_k^n(z)|}{|\varphi(z)|^k (1 - |\varphi(z)|^2)^k} \\ &\preceq \|f\|_{\mathcal{B}} \sup_{|\varphi(z)| > r_N} \|C_\varphi^g v_{k, \varphi(z)}\|_{\mathcal{W}_\mu^n} \preceq \sum_{j=1}^n |c_j^k| \sup_{|a| > r_N} \|C_\varphi^g f_{j,a}\|_{\mathcal{W}_\mu^n}. \end{aligned} \tag{3.9}$$

Taking the above limit as $N \rightarrow \infty$, we obtain

$$\limsup_{j \rightarrow \infty} M_k \preceq \underbrace{\sum_{i=1}^n \limsup_{|a| \rightarrow 1} \|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n}}_{A_i} \preceq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \limsup_{j \rightarrow \infty} M_k \preceq B_k. \tag{3.10}$$

Similarly,

$$\limsup_{j \rightarrow \infty} N_k \preceq \max\{A_i\}_{i=1}^n \quad \text{and} \quad \limsup_{j \rightarrow \infty} N_k \preceq B_k. \quad (3.11)$$

Thus, from (3.4)–(3.8), (3.10), and (3.11), we obtain

$$\limsup_{j \rightarrow \infty} \|C_\varphi^g - C_\varphi^g K_{r_j}\|_{\mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \min \left\{ \max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n \right\}.$$

Hence, from (3.3),

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \min \left\{ \max\{A_i\}_{i=0}^n, \max\{B_i\}_{i=0}^n \right\}.$$

Therefore,

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n \approx \max\{B_i\}_{i=0}^n.$$

Finally, we show that

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \approx \|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \approx \max \left\{ \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \right\}. \quad (3.12)$$

Let j be any positive integer and $h_j(z) = z^j$. Then $h_j \in \mathcal{B}_0$, $\|h_j\|_{\mathcal{B}} \approx 1$ and $\{h_j\}_{j \in \mathbb{N}}$ converges to 0 weakly (see [11]). Hence, for any compact operator K from \mathcal{B}_0 into \mathcal{W}_μ^n , we have $\lim_{j \rightarrow \infty} \|Kh_j\|_{\mathcal{W}_\mu^n} = 0$. Thus,

$$\|C_\varphi^g - K\|_{\mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \succeq \limsup_{j \rightarrow \infty} \|(C_\varphi^g - K)h_j\|_{\mathcal{W}_\mu^n} \geq \limsup_{j \rightarrow \infty} \|C_\varphi^g h_j\|_{\mathcal{W}_\mu^n} - \limsup_{j \rightarrow \infty} \|Kh_j\|_{\mathcal{W}_\mu^n} = \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}.$$

Hence, $\|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \succeq \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}$, which together with (3.1) imply

$$\|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \succeq \max \left\{ \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \right\}. \quad (3.13)$$

From the proof of Theorem 4.3 in [1], we have that $\max\{A_i\}_{i=1}^n \preceq \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}$; hence,

$$\max\{A_i\}_{i=0}^n \preceq \max \left\{ \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \right\}.$$

Since $\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \approx \max\{A_i\}_{i=0}^n$, we obtain

$$\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} \preceq \max \left\{ \limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n}, A_0 \right\}. \quad (3.14)$$

Since $\|C_\varphi^g\|_{e, \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n} \leq \|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n}$, from (3.13) and (3.14), we get (3.12). The proof is complete. \square

It is well known that $\|C_\varphi^g\|_{e, \mathcal{B} \rightarrow \mathcal{W}_\mu^n} = 0$ if and only if $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is a compact operator. Hence, we have the following result.

Corollary 3.3 *Let $n \in \mathbb{N}$, μ be a weight, $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is bounded. Then the following statements are equivalent.*

- (a) $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{W}_\mu^n$ is compact.
- (b) $C_\varphi^g : \mathcal{B}_0 \rightarrow \mathcal{W}_\mu^n$ is compact.
- (c) $\limsup_{j \rightarrow \infty} \|g\varphi^j\|_{\mathcal{W}_\mu^n} = 0$ and $\limsup_{|\varphi(z)| \rightarrow 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0$.
- (d) For $i \in \{1, \dots, n\}$, $\limsup_{|a| \rightarrow 1} \|C_\varphi^g f_{i,a}\|_{\mathcal{W}_\mu^n} = 0$ and $\limsup_{|\varphi(z)| \rightarrow 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0$.
- (e) For $i \in \{1, \dots, n\}$, $\limsup_{|\varphi(z)| \rightarrow 1} \frac{\mu(z)|I_i^n(z)|}{(1-|\varphi(z)|^2)^i} = 0$ and $\limsup_{|\varphi(z)| \rightarrow 1} \mu(z)|g^{(n)}(z)| \log \frac{2}{1-|\varphi(z)|^2} = 0$.

Remark 3.4 Putting $n = 1$ and $\mu(z) = (1 - |z|^2)^\beta$ in Theorem 2.6 and Corollary 3.3, we get some characterizations for the boundedness and compactness of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ (see Theorems 2.1 and 3.1 in [15]). Moreover, we obtain some estimates for the essential norm of $C_\varphi^g : \mathcal{B} \rightarrow \mathcal{B}^\beta$ (see [8, 10, 11]).

Acknowledgments

The corresponding author was funded by The Science and Technology Development Fund, Macau SAR (File no. 186/2017/A3) and NNSF of China (No.11720101003).

References

- [1] Abbasi E, Vaezi H, Li S. Essential norm of weighted composition operators from H^∞ to n th weighted type spaces. *Mediterranean Journal of Mathematics* 2019; 16 (5): 133. doi: 10.1007/s00009-019-1409-8
- [2] Choe B, Koo H, Smith W. Composition operators on small spaces. *Integral Equations and Operator Theory* 2006; 56: 357-380. doi: 10.1007/s00020-006-1420-x
- [3] Colonna F. New criteria for boundedness and compactness of weighted composition operators mapping into the Bloch space. *Central European Journal of Mathematics* 2013; 11: 55-73. doi: 10.2478/s11533-012-0097-4
- [4] Comtet L. *Advanced Combinatorics*. Dordrecht, the Netherlands: D. Reidel, 1974.
- [5] Cowen CC, MacCluer BD. *Composition Operators on Spaces of Analytic Functions*. Studies in Advanced Mathematics. Boca Raton, CA, USA: CRC Press 1995.
- [6] Hu Q, Li S, Wulan H. New essential norm estimates of weighted composition operators from H^∞ into the Bloch space. *Complex Variables and Elliptic Equations* 2017; 62: 600-615. doi: 10.1080/17476933.2016.1234462
- [7] Hu Q, Zhu X. A new characterization of differences of generalized weighted composition operators from the Bloch space into weighted-type spaces. *Mathematical Inequalities and Applications* 2018; 21 (1): 63-76. doi: 10.7153/mia-2018-21-05
- [8] Hyvärinen O, Lindström M. Estimates of essential norms of weighted composition operators between Bloch-type spaces. *Journal of Mathematical Analysis and Applications* 2012; 393: 38-44. doi: 10.1016/j.jmaa.2012.03.059
- [9] Li S, Stević S. Weighted differentiation composition operators from the logarithmic Bloch space to the weighted-type space. *Analele Stiintifice ale Universitatii Ovidius Constanta* 2016; 24 (3): 223-240. doi: 10.1515/auom-2016-0056
- [10] Liu X, Li S. Norm and essential norm of a weighted composition operator on the Bloch space. *Integral Equations and Operator Theory* 2017; 87: 309-325. doi: 10.1007/s00020-017-2349-y
- [11] Maccluer BD, Zhao R. Essential norms of weighted composition operators between Bloch-type spaces. *Rocky Mountain Journal of Mathematics* 2003; 33: 1437-1458. doi:10.1216/rmjm/1181075473

- [12] Madigan K, Matheson A. Compact composition operators on the Bloch space. *Transactions of the American Mathematical Society* 1995; 347: 2679-2687. doi: 10.2307/2154848
- [13] Manhas J, Zhao R. New estimates of essential norms of weighted composition operators between Bloch-type spaces. *Journal of Mathematical Analysis and Applications* 2012; 389: 32-47. doi: 10.1016/j.jmaa.2011.11.039
- [14] Ohno S, Zhao R. Weighted composition operators on the Bloch space. *Bulletin of the Australian Mathematical Society* 2011; 63: 177-185. doi: 10.1017/S0004972700019250
- [15] Ohno S, Storethoff K, Zhao, R. Weighted composition operators between Bloch-type spaces. *Rocky Mountain Journal of Mathematics* 2003; 33: 191-215. doi:10.1216/rmj/1181069993
- [16] Stević S. Composition operators from the weighted Bergman space to the n th weighted spaces on the unit disc. *Discrete Dynamics in Nature and Society* 2009; 2009: 11p. doi:10.1155/2009/742019
- [17] Stević S. Weighted differentiation composition operators from H^∞ and Bloch spaces to n th weighted-type spaces on the unit disk. *Applied Mathematics and Computation* 2010; 216: 3634-3641. doi: 10.1016/j.amc.2010.05.014
- [18] Stević S. Essential norm of some extensions of the generalized composition operators between k -th weighted-type spaces. *Journal of Inequalities and Applications* 2017; 2017: 220. doi: 10.1186/s13660-017-1493-x
- [19] Wulan H, Zheng D, Zhu K. Compact composition operators on BMOA and the Bloch space. *Proceedings of the American Mathematical Society* 2009; 137: 3861-3868. doi: 10.1090/S0002-9939-09-09961-4
- [20] Zhao R. Essential norms of composition operators between Bloch type spaces. *Proceedings of the American Mathematical Society* 2010; 138: 2537-2546. doi: 10.1090/S0002-9939-10-10285-8
- [21] Zhu K. Bloch type spaces of analytic functions. *Rocky Mountain Journal of Mathematics* 1993; 23: 1143-1177. doi:10.1216/rmj/1181072549
- [22] Zhu K. *Spaces of Holomorphic Functions in the Unit Ball*. Graduate Texts in Mathematics, 226: Springer, 2005.
- [23] Zhu X. Generalized weighted composition operators on Bloch-type spaces. *Journal of Inequalities and Applications* 2015; 2015: 59-68. doi: 10.1186/s13660-015-0580-0
- [24] Zhu X. Essential norm of generalized weighted composition operators on Bloch-type spaces. *Applied Mathematics and Computation* 2016; 274: 133-142. doi: 10.1016/j.amc.2015.10.061