Turkish Journal of Mathematics

Volume 44 | Number 1 Article 6

1-1-2020

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Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2020) 44: 98 – 107
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doi:10.3906/mat-1908-66

Coefficient estimation of a certain subclass of bi-close-to-convex functions analytic in the exterior of the unit disc

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Received: 21.08.2019 • **Accepted/Published Online:** 01.11.2019 • **Final Version:** 20.01.2020

Abstract: In this paper, we introduce two new subclasses of biunivalent functions analytic in the exterior of the unit disc. The bounds obtained for the $zero^{th}$, first and second coefficient improves upon earlier known results. The results are obtained by refining the well-known estimates for the initial coefficients of the Carth$\acute{e}$odory functions.

Key words: Analytic function, analytic continuation, univalent functions, biunivalent functions, coefficient bounds

1. Introduction

Let Σ denote the class of functions $f(z)$ of the following form:

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{1.1}$$

which are analytic and univalent in the exterior of the unit disk, written by

$$\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}. \tag{1.2}$$

The function $f \in \Sigma$ has a compositional inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (w \in \text{range of } g).$$

The inverse of the function f is represented by the following series:

$$f^{-1}(w) = g(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} \quad (M < |w| < \infty, M > 1). \tag{1.3}$$

By substituting $w = f(z)$ in the above series (1.3) the coefficients $B_n s$ of f^{-1} can be expressed in terms of the coefficients $b_n s$ of $f(z)$. Thus, for initial values of n we have

$$B_0 = -b_0,\ B_1 = -b_1,\ B_2 = -(b_2 + b_0 b_1),\ B_3 = -(b_3 + 2b_0 b_2 + b_0^2 b_1 + b_1^2) \tag{1.4}$$

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2010 *AMS Mathematics Subject Classification:* 30C45

and so on. In two recent interesting papers [10, 17], the coefficient estimate problem for the functions f and f^{-1} have been discussed when f is a member of the class Σ in addition to being a starlike function. We say that the function $f \in \Sigma$ is biunivalent in Δ if $f^{-1}(w)$ has analytic continuation to Δ. The subclass of Σ consisting of biunivalent functions shall be denoted here by $\mathfrak{B}$. The study of biunivalent analytic functions on the unit disc was initiated by Lewin [12]. Prominent work on biunivalent functions can be found in [1–3, 11–13, 18–21]. Recently, Srivastava et al. [16] discussed afresh the old problem of finding coefficient bounds for biunivalent analytic functions. At present there is renewed interest for research on this topic. There are about more than one hundred follow up research papers on the estimates of the initial coefficients of functions which belong to different subclasses of biunivalent functions which are analytic in the interior or exterior of the unit disc.

The objective of the present paper is to introduce two new subclasses of biunivalent functions analytic in the exterior of the unit disc and to find estimates on the coefficients $|b_0|, |b_1|$, and $|b_2|$ for the functions in these newly introduced subclasses of Σ. We now define the following.

Definition 1.1 *A function $f(z)$ given by the series (1.1) is said to be in the class $\mathfrak{B}\mathcal{K}_\lambda^\alpha$ if the following conditions are satisfied:*

$$f \in \sigma, \quad \Re\left\{\frac{zf'(z)}{\lambda z + (1-\lambda)f(z)}\right\} \leq \frac{\alpha\pi}{2} \quad (z \in \Delta), \tag{1.5}$$

and

$$\Re\left\{\frac{wg'(w)}{\lambda w + (1-\lambda)g(w)}\right\} \leq \frac{\alpha\pi}{2} \quad (w \in \Delta), \tag{1.6}$$

where g is the analytic continuation of f^{-1} to Δ.

Definition 1.2 *A function $f(z)$ given by the series (1.1) is said to be in the class $\mathfrak{B}\mathcal{K}_\lambda(\beta)$ if the following conditions are satisfied:*

$$f \in \sigma, \quad \Re\left\{\frac{zf'(z)}{\lambda z + (1-\lambda)f(z)}\right\} > \beta \quad (z \in \Delta), \tag{1.7}$$

and

$$\Re\left\{\frac{wg'(w)}{\lambda w + (1-\lambda)g(w)}\right\} > \beta \quad (w \in \Delta), \tag{1.8}$$

where g is the analytic continuation of f^{-1} to Δ.

In the particular case $\lambda = 0$ the class $\mathfrak{B}\mathcal{K}_\lambda(\beta)$ reduces to the class $\mathfrak{B}\mathcal{K}(\beta)$ consisting of bistarlike functions order β in the exterior of the unit disc. We shall also need the Carathéodory class $\mathcal{P}$ [5] consisting of analytic functions $p : \mathbb{U} \to \mathbb{C}$ satisfying $\Re(p(z)) > 0$ and $p(0) = 1$. Recently, Halim et al. [6] obtained bounds for $|b_0|$ and $|b_1|$ for the $f \in \mathfrak{B}\mathcal{K}(\beta)$ $0 \leq \beta < 1$. These bounds were improved in [7].

In the present paper, the method is suitably modified to find the estimates on $|b_0|$, $|b_1|$, and $|b_2|$ for the function class $\mathfrak{B}\mathcal{K}_\lambda^\alpha$ and $\mathfrak{B}\mathcal{K}_\lambda(\beta)$. The bounds obtained for $|b_0|$ and $|b_1|$ for $f \in \mathfrak{B}\mathcal{K}^\alpha$ $(0 < \alpha \leq 1)$ are improved as obtained earlier in [6, 14, 15]. The estimates in [7] for $|b_0|$ and $|b_1|$ for the function class $\mathfrak{B}\mathcal{K}(\beta)$ follows as a particular case of our results. The methods adopted and developed in this paper are applicable for finding improved coefficient estimates for the several subclasses of biunivalent functions studied in the literature, for example in [4, 8, 9, 14, 15].

2. Coefficient bounds for the function class $\mathfrak{B}\mathcal{K}_\lambda^\alpha$ and $\mathfrak{B}\mathcal{K}_\lambda(\beta)$

We state and prove the following:

Theorem 2.1 *Let the function $f(z)$ given by the series (1.1) be in the class $\mathfrak{B}\mathcal{K}_\lambda^\alpha$ $(0 < \alpha \leq 1,\ \lambda \geq 0)$. Then*

$$|b_0| \leq \frac{\sqrt{2}\alpha}{|1-\lambda|\sqrt{1+\alpha}} \qquad \lambda \neq 1, \tag{2.1}$$

$$|b_1| \leq \frac{2\alpha}{|2-\lambda|} \qquad \lambda \neq 2, \tag{2.2}$$

and

$$|b_2| \leq \frac{2\alpha}{|3-\lambda|} \begin{cases} 1 + \frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0 < \alpha \leq 1,\ 0 \leq \lambda \leq 1) \\ \frac{\alpha(3-2\lambda+\lambda^2)-2(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)-2(2-3\lambda+\lambda^2)} + \frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0 < \alpha \leq 1,\ 1 \leq \lambda \leq 2) \\ \frac{\alpha(3-2\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)} + \frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0 < \alpha \leq 1,\ \lambda \geq 2, \lambda \neq 3)\,. \end{cases} \tag{2.3}$$

Proof Let $f(z)$ be a member of the class $\mathfrak{B}\mathcal{K}_\lambda^\alpha$ $(0 < \alpha \leq 1,\ \lambda \geq 0)$. Then by Definition 1.1, we have the following

$$\frac{zf'(z)}{\lambda z + (1-\lambda)f(z)} = [p(z)]^\alpha, \tag{2.4}$$

and

$$\frac{wg'(w)}{\lambda w + (1-\lambda)g(w)} = [q(w)]^\alpha, \tag{2.5}$$

where $p(z)$

$$p(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \ldots\ldots\ldots \quad (z \in \Delta), \tag{2.6}$$

and

$$q(w) = 1 + \frac{l_1}{w} + \frac{l_2}{w^2} + \frac{l_3}{w^3} + \ldots\ldots\ldots \quad (w \in \Delta). \tag{2.7}$$

are functions with positive real part in Δ.

Now, equating the coefficients of $\frac{zf'(z)}{\lambda z+(1-\lambda)f(z)}$ with the coefficients $[p(z)]^\alpha$ we get the following:

$$-(1-\lambda)b_0 = \alpha c_1, \tag{2.8}$$

$$-(2-\lambda)b_1 + (1-\lambda)^2 b_0^2 = \alpha c_2 + \frac{1}{2}\alpha(\alpha-1)c_1^2, \tag{2.9}$$

$$-(3-\lambda)b_2 + (1-\lambda)(3-\lambda)b_0 b_1 - (1-\lambda)^3 b_0^3 = \alpha c_3 + \alpha(\alpha-1)c_1 c_2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3. \tag{2.10}$$

Similarly, a comparison of the coefficients of both sides of (2.5) yields:

$$(1-\lambda)b_0 = \alpha l_1, \tag{2.11}$$

$$(2-\lambda)b_1+(1-\lambda)^2b_0^2=\alpha l_2+\frac{1}{2}\alpha(\alpha-1)l_1^2, \tag{2.12}$$

and

$$(3-\lambda)b_2+[(3-\lambda)+(3-2\lambda)(1-\lambda)]b_0b_1+(1-\lambda)^3b_0^3=\alpha l_3+\alpha(\alpha-1)l_1l_2+\frac{1}{6}\alpha(\alpha-1)(\alpha-2)l_1^3. \tag{2.13}$$

Equations (2.8) and (2.11) give the following relation between c_1 and l_1:

$$c_1^2=l_1^2. \tag{2.14}$$

We shall first obtain a refined estimate for $|c_1|$ for our future use. For this purpose we add (2.9) with (2.12) and using $c_1^2=l_1^2$ from (2.14) we get the following:

$$2(1-\lambda)^2b_0^2=\alpha(c_2+l_2)+\alpha(\alpha-1)c_1^2. \tag{2.15}$$

Substituting $b_0=-\frac{\alpha}{1-\lambda}c_1$ from (2.8) in the above relation, we get after simplification

$$c_1^2=\frac{c_2+l_2}{1+\alpha}. \tag{2.16}$$

Now using well-known estimates $|c_2|\leq 2$ and $|l_2|\leq 2$ we get

$$|c_1|\leq\frac{2}{\sqrt{1+\alpha}}. \tag{2.17}$$

In (2.15) we replace $c_1^2=\frac{c_2+l_2}{1+\alpha}$ from (2.16) and get the following

$$b_0^2=\frac{\alpha^2}{2(1-\lambda)^2(1+\alpha)}(c_2+l_2). \tag{2.18}$$

Therefore, on appropriately using the well-known estimate $|c_2|\leq 2$ and $|l_2|\leq 2$, the relation (2.18) yields our assertion (2.1) for $|b_0|$.

To find bounds on $|b_1|$, we subtract (2.12) from (2.9) and use the relation $c_1^2=l_1^2$ and get

$$-2(2-\lambda)b_1=\alpha(c_2-l_2). \tag{2.19}$$

Similar to our considerations for $|b_0|$, in this case we make use of the estimates $|c_2|\leq 2$ and $|l_2|\leq 2$ and obtain our claimed estimate at (2.2) on $|b_1|$.

We, next, derive a relation between $c_1(c_2-l_2)$ and c_3+l_3 for our use. For this purpose we add (2.13) and (2.10) and then use the relation $c_1^2=l_1^2$. After simplification we have

$$(9-10\lambda+3\lambda^2)b_0b_1=\alpha(c_3+l_3)+\alpha(\alpha-1)c_1(c_2-l_2). \tag{2.20}$$

Next, by substituting $b_1=-\frac{\alpha(c_2-l_2)}{2(2-\lambda)}$ from (2.19) and $b_0=-\frac{\alpha c_1}{1-\lambda}$ the above equation (2.20) takes the form

$$c_1(c_2-l_2)=\mu_1(c_3+l_3), \tag{2.21}$$

where $\mu_1 = \frac{2(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda++\lambda^2)+2(2-3\lambda+\lambda^2)}$. We notice that $0 < \mu_1 \leq 1$ for every $\lambda \geq 0$.

We next find bounds for $|b_2|$. For this purpose we subtract (2.13) from (2.10), use the relation $c_1^2 = l_1^2$ and get the following after simplification

$$\begin{aligned}
-2(3-\lambda)b_2 &= (3-2\lambda+\lambda^2)b_0b_1 + 2(1-\lambda)^3b_0^3 + \alpha(c_3-l_3) \\
&\quad + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3 \\
&= (9-10\lambda+3\lambda^2)b_0b_1 - (6-8\lambda+2\lambda^2)b_0b_1 + 2(1-\lambda)^3b_0^3 \\
&\quad + \alpha(c_3-l_3) + \alpha(\alpha-1)c_1(c_2+l_2) + \frac{1}{3}\alpha(\alpha-1)(\alpha-2)c_1^3. \qquad (2.22)
\end{aligned}$$

We replace $(9-10\lambda+3\lambda^2)b_0b_1$ with $\alpha(c_3+l_3)+\alpha(\alpha-1)c_1(c_2-l_2)$ from (2.20), $b_0 = -\frac{\alpha c_1}{1-\lambda}$ and $b_1 = -\frac{\alpha(c_2-l_2)}{2(2-\lambda)}$ in Equation (2.22). We, thus, express b_2 in terms of the coefficients of $p(z)$ and $q(w)$ as follows

$$\begin{aligned}
-2(3-\lambda)b_2 &= 2\alpha c_3 - \frac{\alpha\{(1-\lambda)\alpha + (2-3\lambda+\lambda^2)\}}{(2-3\lambda+\lambda^2)}c_1(c_2-l_2) \\
&\quad + \alpha(\alpha-1)c_1(c_2+l_2) - \frac{\alpha(5\alpha^2+3\alpha-2)}{3}c_1c_1^2. \qquad (2.23)
\end{aligned}$$

We substitute $c_1(c_2-l_2)$ with $\mu_1(c_3+l_3)$ from (2.21) and c_1^2 by $\frac{c_2+l_2}{1+\alpha}$ from (2.16) in (2.23), which after rearrangement and simplification gives

$$\begin{aligned}
-2(3-\lambda)b_2 &= 2\alpha\left(\frac{\alpha(4-3\lambda+\lambda^2)+(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)}\right)c_3 \\
&\quad - 2\alpha\left(\frac{\alpha(1-\lambda)+(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)}\right)l_3 \\
&\quad + \frac{\alpha(1+2\alpha)}{3}c_1(c_2+l_2). \qquad (2.24)
\end{aligned}$$

Now we apply triangle inequality in (2.24) and refined estimate for $|c_1|$ from (2.17). This gives

$$\begin{aligned}
|b_2| \leq \frac{\alpha}{|3-\lambda|}&\left\{\frac{\alpha(4-3\lambda+\lambda^2)+\left|2-3\lambda+\lambda^2\right|}{\alpha(5-4\lambda+\lambda^2)+2\left|2-3\lambda+\lambda^2\right|}|c_3|\right. \\
&+ \frac{\alpha\left|1-\lambda\right|+\left|2-3\lambda+\lambda^2\right|}{\alpha(5-4\lambda+\lambda^2)+2\left|2-3\lambda+\lambda^2\right|}|l_3| \\
&\left.+ \frac{(1+2\alpha)}{3\sqrt{1+\alpha}}|c_2+l_2|\right\}.
\end{aligned}$$

Next using the usual bounds $|c_n| \le 2\ (n=2,3), |l_n| \le 2\ (n=2,3)$ we have

$$|b_2| \le \frac{2\alpha}{|3-\lambda|}\left\{\frac{\alpha(4-3\lambda+\lambda^2)+\left|2-3\lambda+\lambda^2\right|}{\alpha(5-4\lambda+\lambda^2)+2\left|2-3\lambda+\lambda^2\right|}\right.$$
$$+\frac{\alpha\left|1-\lambda\right|+\left|2-3\lambda+\lambda^2\right|}{\alpha(5-4\lambda+\lambda^2)+2\left|2-3\lambda+\lambda^2\right|}$$
$$\left.+\frac{2(1+2\alpha)}{3\sqrt{1+\alpha}}\right\}.$$

The above expression simplifies to the following form for different range of λ.

$$|b_2| \le \frac{2\alpha}{|3-\lambda|}\begin{cases} 1+\frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0<\alpha\le 1,\ 0\le\lambda\le 1) \\ \frac{\alpha(3-2\lambda+\lambda^2)-2(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)-2(2-3\lambda+\lambda^2)}+\frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0<\alpha\le 1,\ 1\le\lambda\le 2) \\ \frac{\alpha(3-2\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)}{\alpha(5-4\lambda+\lambda^2)+2(2-3\lambda+\lambda^2)}+\frac{2(1+2\alpha)}{3\sqrt{1+\alpha}} & (0<\alpha\le 1,\ \lambda\ge 2, \lambda\ne 3)\,. \end{cases}$$

This is precisely the estimate of (2.3). The proof Theorem 2.1 is completed. □

Taking $\lambda = 0$ in Theorem 2.1 we get the following results.

Corollary 2.2 *For the function $f \in \mathfrak{B}\mathcal{K}^{\alpha}$ represented by the series (1.1) then*

$$|b_0| \le \frac{\sqrt{2}\alpha}{\sqrt{1+\alpha}},$$

$$|b_1| \le \alpha,$$

and

$$|b_2| \le \frac{2\alpha}{3}\left(1+\frac{2(1+2\alpha)}{3\sqrt{1+\alpha}}\right).$$

Theorem 2.3 *Let the function $f(z)$ given by the series (1.1) be in the class $\mathfrak{B}\mathcal{K}_{\lambda}(\beta)$ $(0 \le \beta < 1,\ \lambda \ge 0)$. Then*

$$|b_0| \le \begin{cases} \frac{\sqrt{2(1-\beta)}}{|1-\lambda|} & \left(0\le\beta\le\frac{1}{2}, \lambda\ne 1\right) \\ \frac{2(1-\beta)}{|1-\lambda|} & \left(\frac{1}{2}\le\beta<1, \lambda\ne 1\right) \end{cases} \tag{2.25}$$

$$|b_1| \le \frac{2(1-\beta)}{|2-\lambda|} \quad (\lambda \ne 2)\,, \tag{2.26}$$

and

$$|b_2| \leq \frac{2(1-\beta)}{|3-\lambda|}\begin{cases} 1+\sqrt{2(1-\beta)} & \left(0 \leq \beta \leq \frac{1}{2}, 0 \leq \lambda \leq 1 \ or \ \lambda > 3\right) \\ 1+4(1-\beta)^2 & \left(\frac{1}{2} \leq \beta < 1, 0 \leq \lambda \leq 1 \ or \ \lambda > 3\right) \\ \frac{(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2}+\sqrt{2(1-\beta)} & \left(0 \leq \beta \leq \frac{1}{2}, 1 \leq \lambda < 3\right) \\ \frac{(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2}+4(1-\beta)^2 & \left(\frac{1}{2} \leq \beta \leq 1, 1 \leq \lambda < 3\right). \end{cases} \tag{2.27}$$

Proof Let $f(z)$ be a member of the class $\mathfrak{B}\mathcal{K}_\lambda(\beta)$ $(0 \leq \beta < 1,\ \lambda \geq 0)$. Then by Definition 1.2, we have the following

$$\frac{zf'(z)}{\lambda z+(1-\lambda)f(z)} = \beta + (1-\beta)p(z), \tag{2.28}$$

and

$$\frac{wg'(w)}{\lambda w+(1-\lambda)g(w)} = \beta + (1-\beta)q(w). \tag{2.29}$$

where $p(z)$ and $q(w)$ are given by the Equations (2.6) and (2.7) respectively.

Now, equating the coefficients of $\frac{zf'(z)}{\lambda z+(1-\lambda)f(z)}$ with the coefficients of $\beta+(1-\beta)p(z)$, we get the following

$$-(1-\lambda)b_0 = (1-\beta)c_1, \tag{2.30}$$

$$-(2-\lambda)b_1 + (1-\lambda)^2 b_0^2 = (1-\beta)c_2, \tag{2.31}$$

$$-(3-\lambda)b_2 + (1-\lambda)(3-\lambda)b_0 b_1 - (1-\lambda)^3 b_0^3 = (1-\beta)c_3. \tag{2.32}$$

Similarly, a comparison of the coefficients of both sides of (2.29) yields:

$$(1-\lambda)b_0 = (1-\beta)l_1, \tag{2.33}$$

$$(2-\lambda)b_1 + (1-\lambda)^2 b_0^2 = (1-\beta)l_2, \tag{2.34}$$

and

$$(3-\lambda)b_2 + [(3-\lambda)+(3-2\lambda)(1-\lambda)]b_0 b_1 + (1-\lambda)^3 b_0^3 = (1-\beta)l_3. \tag{2.35}$$

Equations (2.30) and (2.33) give the following relation between c_1 and l_1:

$$c_1^2 = l_1^2. \tag{2.36}$$

We shall first obtain a refined estimate for $|c_1|$ for our future use. For this purpose we add (2.31) with (2.34)

$$2(1-\lambda)^2 b_0^2 = (1-\beta)(c_2+l_2). \tag{2.37}$$

Substituting $b_0 = -\frac{1-\beta}{1-\lambda}c_1$ from (2.30), in the above relation we get the following

$$c_1^2 = \frac{c_2+l_2}{2(1-\beta)}. \tag{2.38}$$

Now using the well-known estimates $|c_2| \leq 2$ and $|l_2| \leq 2$ we get

$$|c_1| \leq \begin{cases} \sqrt{\frac{2}{1-\beta}} & \left(0 \leq \beta \leq \frac{1}{2}\right) \\ 2 & \left(\frac{1}{2} \leq \beta < 1\right). \end{cases} \tag{2.39}$$

The relation (2.38) also yields

$$|c_2 + l_2| \leq \begin{cases} 4 & \left(0 \leq \beta \leq \frac{1}{2}\right) \\ 8(1-\beta) & \left(\frac{1}{2} \leq \beta < 1\right). \end{cases} \tag{2.40}$$

An application of (2.39) in (2.30) gives the following

$$|b_0| \leq \begin{cases} \frac{\sqrt{2(1-\beta)}}{|1-\lambda|} & \left(0 \leq \beta \leq \frac{1}{2}, \lambda \neq 1\right) \\ \\ \frac{2(1-\beta)}{|1-\lambda|} & \left(\frac{1}{2} \leq \beta < 1, \lambda \neq 1\right) \end{cases} \tag{2.41}$$

This is precisely our assertion at (2.25).
To find bounds on $|b_1|$ we subtract (2.34) from (2.31) and get

$$2(\lambda - 2)b_1 = (1-\beta)(c_2 - l_2). \tag{2.42}$$

We, use the estimate $|c_2| \leq 2$ and $|l_2| \leq 2$ and obtain our claimed estimate at (2.26) on $|b_1|$. That is

$$|b_1| \leq \frac{2(1-\beta)}{|2-\lambda|} \quad (\lambda \neq 2). \tag{2.43}$$

In order to find a bound for $|b_2|$ we subtract (2.35) from (2.32) and get

$$-2(3-\lambda)b_2 = (3 - 2\lambda + \lambda^2)b_0 b_1 + 2(1-\lambda)^3 b_0^3 + (1-\beta)(c_3 - l_3). \tag{2.44}$$

We shall express b_2 in terms of the first three coefficients of the functions p and q. For this purpose we add (2.35) with (2.32) and get the following after simplification

$$(9 - 10\lambda + 3\lambda^2)b_0 b_1 = (1-\beta)(c_3 + l_3). \tag{2.45}$$

We replace $b_0 b_1$ with $\frac{1-\beta}{9-10\lambda+3\lambda^2}(c_3 + l_3)$ from (2.45) and b_0 by $-\frac{1-\beta}{1-\lambda}c_1$ in (2.44). We thus have the following after simplification

$$-2(3-\lambda)b_2 = (1-\beta)(c_3 + l_3) + \frac{(1-\beta)(3 - 2\lambda + \lambda^2)}{9 - 10\lambda + 3\lambda^2}(c_3 + l_3) - 2(1-\beta)^3 c_1 c_1^2. \tag{2.46}$$

We substitute $c_1^2 = \frac{c_2 + l_2}{2(1-\beta)}$ from (2.38) in (2.46), which after rearrangement and simplification gives

$$-2(3-\lambda)b_2 = (1-\beta)\left\{\frac{4(3 - 3\lambda + \lambda^2)}{9 - 10\lambda + 3\lambda^2}c_3 - \frac{2(3 - 4\lambda + \lambda^2)}{9 - 10\lambda + 3\lambda^2}l_3 - (1-\beta)c_1(c_2 + l_2)\right\}.$$

Now, we apply triangle inequality in (2.3), and use the refined estimate for $|c_1|$ in the range $0 \le \beta \le \frac{1}{2}$ from (2.39) and refined bound for $|c_2 + l_2|$ in the range $\frac{1}{2} \le \beta < 1$ from (2.40). This gives, for $\lambda \ne 3$

$$|b_2| \le \frac{(1-\beta)}{2|3-\lambda|} \begin{cases} \frac{4(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2}|c_3| + \frac{2|3-4\lambda+\lambda^2|}{9-10\lambda+3\lambda^2}|l_3| + \sqrt{2(1-\beta)}|c_2+l_2| & \left(0 \le \beta \le \frac{1}{2}\right) \\ \frac{4(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2}|c_3| + \frac{2|3-4\lambda+\lambda^2|}{9-10\lambda+3\lambda^2}|l_3| + 8(1-\beta)^2|c_1| & \left(\frac{1}{2} \le \beta < 1\right). \end{cases}$$

Next, using the usual bounds $|c_n| \le 2$ $(n = 1, 2, 3)$ and $|l_3| \le 2$ we have

$$|b_2| \le \frac{2(1-\beta)}{|3-\lambda|} \begin{cases} \frac{2(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2} + \frac{|3-4\lambda+\lambda^2|}{9-10\lambda+3\lambda^2} + \sqrt{2(1-\beta)} & \left(0 \le \beta \le \frac{1}{2}\right) \\ \frac{2(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2} + \frac{|3-4\lambda+\lambda^2|}{9-10\lambda+3\lambda^2} + 4(1-\beta)^2 & \left(\frac{1}{2} \le \beta < 1\right). \end{cases}$$

The above expression simplifies to the following form for different range of λ.

$$|b_2| \le \frac{2(1-\beta)}{|3-\lambda|} \begin{cases} 1 + \sqrt{2(1-\beta)} & \left(0 \le \beta \le \frac{1}{2}, 0 \le \lambda \le 1 \text{ or } \lambda > 3\right) \\ 1 + 4(1-\beta)^2 & \left(\frac{1}{2} \le \beta < 1, 0 \le \lambda \le 1 \text{ or } \lambda > 3\right) \\ \frac{(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2} + \sqrt{2(1-\beta)} & \left(0 \le \beta \le \frac{1}{2}, 1 \le \lambda < 3\right) \\ \frac{(3-3\lambda+\lambda^2)}{9-10\lambda+3\lambda^2} + 4(1-\beta)^2 & \left(\frac{1}{2} \le \beta \le 1, 1 \le \lambda < 3\right). \end{cases}$$

This is precisely our estimate at (2.27). The proof Theorem 2.3 is completed.

Remark 2.4 *For the function $h \in \mathfrak{BK}(\beta)$ represented by the series (1.1), the bounds $|b_0|$ and $|b_1|$ are given by*

$$|b_0| \le \begin{cases} \sqrt{2(1-\beta)} & 0 \le \beta \le \frac{1}{2} \\ 2(1-\beta) & \frac{1}{2} \le \beta < 1, \end{cases}$$

$$|b_1| \le 1 - \beta \qquad (0 \le \beta < 1),$$

and

$$|b_2| \le \frac{2(1-\beta)}{3} \begin{cases} 1 + \sqrt{2(1-\beta)} & 0 \le \beta \le \frac{1}{2} \\ 1 + 4(1-\beta)^2 & \frac{1}{2} \le \beta < 1. \end{cases}$$

These are particular cases of our result in (2.25) and (2.26) when $\lambda = 0$. The bounds for the zeroth and the first coefficients are previously known in [7] and the bound on the second coefficient is new.

□

References

[1] Brannan DA, Cluine JG. Aspects of contemporary complex analysis. In: Proceedings of the NATO Advanced Study Institute Held at the University of Durham. New York, NY, USA: Academic Press, 1980.

[2] Brannan DA, Taha TS. On some classes of biunivalent functions. In: Mazhar SM, Hamoui A, Faour NS (editors). KFAS Proceedings Series, Vol. 3. Oxford, UK: Pergamon Press, 1988, pp. 53-60.

[3] Bshouty D, Hengartner W, Schober G. Estimate for the Koebe constant and the second coefficient for some classes of univalent functions. Canadian Journal of Mathematics 1980; 32 (6): 1311-1324. https://doi.org/10.4153/CJM-1980-101-9.

[4] Bulut S, Magesh N, Balaji K. Faber polynomial coefficient estimates for certain subclasses of meromorphic bi-univalent functions. Comptes Rendus Mathematique 2015; 353: 113-116. doi: 10.1016/j.crma.2014.10.019

[5] Duren PL. Univalent functions. In: Grundlheren der Mathematischen Wissenschaften, Band 259. New York, NY, USA: Springer-Verlag, 1983.

[6] Halim S, Hamidi SG, Ravichandran V. Coefficient estimates for meromorphic bi-univalent functions. arXiv:1108.4089

[7] Hamidi SG, Halim S, Jahangiri JM. Faber polynomial Coefficient estimates for meromorphic bi-univalent functions. International Journal of Mathematics and Mathematical Sciences 2013; Article ID 498159. doi: 10.1155/2013/498159.

[8] Hamidi SG, Jahangiri JM. Faber polynomial coefficient estimates for bi-univalent functions defined by subordination. Bulletin of Iranian Mathematical Society Soc. 2015; 41 (5): 1103-1119.

[9] Hamidi SG, Janani T, Murugusundaramoorthy G, Jahangiri JM. Coefficient estimates for certain classes of meromorphic bi-univalent functions. Comptes Rendus Mathematique 2014; 352 (4): 277-282. doi: 10.1016/j.crma.2014.01.010

[10] Kapoor GP, Mishra AK. Coefficient estimates for inverses of starlike functions of positive order. Journal of Mathematical Analysis and Application 2007; 329 (2): 922-934. doi: 10.1016/j.jmaa.2006.07.020

[11] Kedzierawski AW. Some remarks on bi-univalent functions. Annales Universitatis Mariae Curie-Sklodowska, sectio A. 1985; 39: 77-81.

[12] Lewin M. On the coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society 1967; 18: 63-68. doi: 10.1090/S0002-9939-1967-0206255-1

[13] Natanyahu E. The minimal distance of image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Archive for Rational Mechanics and Analysis 1969; 32: 100-112.

[14] Orhan H, Magesh N, Balaji VK. Initial coefficients bounds for certain subclasses of meromorphic bi-univalent functions. Asian-European Journal of Mathematics 2014; 07 (01). doi: 10.1142/S1793557114500053

[15] Panigrahi T. Coefficient bounds for certain subclasses of Meromorphic and bi-univalent functions. Bulletin of the Korean Mathematical Society 2013; 50 (5): 1531-1538. doi: 10.4134/BKMS.2013.50.5.1531

[16] Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Applied Mathematics Letters 2010; 23: 1188-1192. doi: 10.1016/j.aml.2010.05.009

[17] Srivastava HM, Mishra AK, Kund SN. Coefficient estimates for the inverses of starlike functions represented by symmetric gap series. Pan-American Mathematical Journal 2011: 21 (4); 105-123.

[18] Styer D, Wright DJ. Results on bi-univalent functions. Proceedings of the American Mathematical Society 1981; 82: 243-248. doi: 10.1090/S0002-9939-1981-0609659-5

[19] Suffridge TJ. A coefficient problem for a class of univalent functions. The Michigan Mathematical Journal 1969; 16: 33-42. doi: 10.1307/mmj/1029000163

[20] Tan DL. Coefficient estimates for bi-univalent functions. Chinese Annals of Mathematics Ser. A. 1984; 5: 559-568.

[21] Taha TS. Topics in univalent function theory. PhD, University of London, UK, 1981.