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Conceptions on topological transitivity in products and symmetric products

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Abstract: Having a finite number of topological spaces $X_i$ and functions $f_i : X_i \to X_i$, and considering one of the following classes of functions: exact, transitive, strongly transitive, totally transitive, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, mixing, weakly mixing, mild mixing, chaotic, exactly Devaney chaotic, minimal, backward minimal, totally minimal, $TT_{++}$, scattering, Touhey or an $F$-system, in this paper, we study dynamical behaviors of the systems $(X_i, f_i)$, $(\prod_{i=1}^m X_i, \prod_{i=1}^m f_i)$, $(\mathcal{F}_n(\prod_{i=1}^m X_i), \mathcal{F}_n(\prod_{i=1}^m f_i))$, and $(\mathcal{F}_n(X_i), \mathcal{F}_n(f_i))$.

Key words: Topological transitivity, symmetric products, dynamical systems

1. Introduction

Given a topological space $X$ and a positive integer $n$, we consider the $n$-fold symmetric product of $X$, $\mathcal{F}_n(X)$, consisting of all nonempty subsets of $X$ with at most $n$ points [7]. A function $f : X \to X$ induces a map on $\mathcal{F}_n(X)$ denoted by $\mathcal{F}_n(f) : \mathcal{F}_n(X) \to \mathcal{F}_n(X)$ and defined by $\mathcal{F}_n(f)(A) = f(A)$, for each $A \in \mathcal{F}_n(X)$ [3]. Thereby, the discrete dynamical system $(X, f)$ induces the discrete dynamical system $(\mathcal{F}_n(X), \mathcal{F}_n(f))$.

Let $X_1, \ldots, X_m$ be topological spaces, with $m \geq 2$ and for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function. We define the function $\prod_{i=1}^m f_i : \prod_{i=1}^m X_i \to \prod_{i=1}^m X_i$ by $\prod_{i=1}^m f_i((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m))$, for each $(x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$. This function is called product function. In this way, we can analyze the relationships between the dynamical of the systems (1) $(\mathcal{F}_n(\prod_{i=1}^m X_i), \mathcal{F}_n(\prod_{i=1}^m f_i))$; (2) $(\mathcal{F}_n(X_i), \mathcal{F}_n(f_i))$, for each $i \in \{1, \ldots, m\}$; (3) $(\prod_{i=1}^m X_i, \prod_{i=1}^m f_i)$ and (4) $(X_i, f_i)$, for each $i \in \{1, \ldots, m\}$. Hou et al. [11] considered two compact metric spaces without isolated points $X$ and $Y$, and two continuous functions $f : X \to X$ and $g : Y \to Y$, and they showed the following result: if $f$ and $g$ are sensitive functions, then the function $2^{f \times g} : 2^{X \times Y} \to 2^{X \times Y}$ is sensitive. Later, Degirmenci and Kocak [8] considered two metric spaces, $X$ and $Y$, and two functions $f : X \to X$ and $g : Y \to Y$ (not necessarily continuous) and they analyzed the relationship between $f$, $g$ and $f \times g$ when any of them is a chaotic function. In particular, they proved the following result: if $f$ is continuous and chaotic, and $g$ is chaotic and mixing (not necessarily continuous), then $f \times g$ is chaotic. Later, Wu and Zhu [21] proved that for each integer $m \geq 2$, if $\prod_{i=1}^m f_i$ is chaotic in the sense of Devaney, then for each $i \in \{1, \ldots, m\}$, $f_i$ is also chaotic in the sense of Devaney. Moreover, they proved that if $\prod_{i=1}^m f_i$ is transitive, then, for each $i \in \{1, \ldots, m\}$, $f_i$ is transitive. The converse problem is not true in general. In [21], Wu and Zhu considered metric spaces without isolated points and continuous functions. Moreover, Li and Zhou

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[13] analyzed the relationships between \( f, g \) and \( f \times g \) when any of these are: topologically transitive, topologically weakly mixing, syndetically transitive, cofinitely sensitive, multisensitive and ergodically sensitive, always considering metric spaces and functions not necessarily continuous. Wu et al. [20] studied the \( \mathcal{F} \)-sensitivity and the multisensitivity of the dynamical system \((2^X \times Y, 2^{f \times g})\), when \( X \) and \( Y \) are both compact metric spaces. Recently, Mangang [6] studied the Li-Yorke chaos of the product dynamical system \((\prod_{i=1}^{m} X_i, \prod_{i=1}^{m} f_i)\) when each dynamical system \((X_i, f_i)\) has the property. In particular, he proved that \((X, f)\) and \((Y, g)\) are two exact dynamical systems if and only if the product dynamical system \((X \times Y, f \times g)\) is exact. In this last paper, \( X \) and \( Y \) are compact metric spaces and \( f \) and \( g \) are continuous functions. In order to make a contribution to this line of investigation, let \( \mathcal{M} \) be one of the following classes of functions: exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, orbit-transitive, strictly orbit-transitive, \( \omega \)-transitive, \( T T_{++} \), mild mixing, exactly Devaney chaotic, backward minimal, totally minimal, scattering, Touhey or an \( F \)-system, in this paper we study the relationships between the following four statements:

1. For each \( i \in \{1, \ldots, m\} \), \( f_i \in \mathcal{M} \).
2. \( \prod_{i=1}^{m} f_i \in \mathcal{M} \).
3. \( \mathcal{F}_n(\prod_{i=1}^{m} f_i) \in \mathcal{M} \).
4. For each \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \in \mathcal{M} \).

It is important to emphasize that in the aforementioned articles, the authors work with compact metric spaces or without isolated points metric spaces and continuous function. In this paper, we are going to answer similar questions that we can find in [6, 8, 11, 13, 20, 21], considering topological spaces and functions not necessarily continuous.

2. Definitions and notations
Throughout this paper, \( m \) is an integer greater than one. A set is said to be nondegenerate if it has more than one point. A (discrete) dynamical system is a pair \((X, f)\), where \( X \) is a nondegenerate topological space and \( f : X \to X \) is a function, \( X \) is called the phase space. Let \( X \) be a topological space and let \( A \) be a subset of \( X \), \( \text{cl}_X(A) \) denotes the closure of the set \( A \) in \( X \). The symbols \( \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{N} \) denote the set of integers, the set of nonnegative integers and the set of positive integers, respectively. Given a finite collection of topological spaces \( X_1, \ldots, X_m \), the Cartesian product of these topological spaces is denoted by \( \prod_{i=1}^{m} X_i \). This space is considered with the product topology [16, p. 86]. On the other hand, given a finite collection of functions, \( f_1 : X_1 \to X_1, \ldots, f_m : X_m \to X_m \) (not necessarily continuous), we define the product function \( \prod_{i=1}^{m} f_i : \prod_{i=1}^{m} X_i \to \prod_{i=1}^{m} X_i \) by \( \prod_{i=1}^{m} f_i((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m)) \), for each \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i \). Particularly, if \( X \) is a topological space and \( f : X \to X \) is a function, the Cartesian product of \( X \) with itself \( m \) times is denoted by \( X^m \) and the Cartesian product of \( f \) with itself \( m \) times is denoted by \( f^\times m \).

Given a dynamical system \((X, f)\), for each \( k \in \mathbb{N} \), the \( k \)th iteration of \( f \) is defined as repeated composition of \( f \) with itself \( k \) times and is denoted by \( f^k \). This is, \( f^k = f \circ f^{k-1} \), where \( f^1 = f \) and \( f^0 = \text{id}_X \), the identity function on \( X \). For a subset \( A \) of \( X \) and \( k \in \mathbb{Z} \), we denote by \( f^k(A) \) the image of \( A \) under \( f^k \), when \( k \geq 0 \), and the preimage under \( f^{|k|} \) when \( k < 0 \). If \( z \in X \), \( f^{-k}(z) \) denotes the set \( f^{-k}(\{z\}) \), for each \( k > 0 \).
Let \((X, f)\) be a dynamical system and let \(x \in X\). The orbit of \(x\) under \(f\) is the set \(O(x, f) = \{f^k(x) \mid k \in \mathbb{Z}_+\}\). A point \(x\) of \(X\) is a transitive point of the function \(f\) if the set \(O(x, f)\) is dense in \(X\). The set of transitive points of \(f\) is denoted by \(\text{trans}(f)\). The point \(x\) is a fixed point of \(f\) if \(f(x) = x\). The point \(x\) is a periodic point of \(f\) if there exists \(k \in \mathbb{N}\) such that \(f^k(x) = x\). The set of periodic points of \(f\) is denoted by \(\text{Per}(f)\). If \(k = \min\{l \in \mathbb{N} \mid f^l(x) = x\}\), we say that \(k\) is the period of \(x\) under \(f\). A point \(y\) in \(X\) is an \(\omega\)-limit point of \(x\) under \(f\) if for any \(k \in \mathbb{N}\) and for any open subset \(U\) of \(X\) such that \(y \in U\), there exists a positive integer \(l \geq k\) such that \(f^l(x) \in U\). The set of \(\omega\)-limit points of \(x\) under \(f\), is denoted by \(\omega(x, f)\) and is called \(\omega\)-limit set of \(x\). Given a subset \(A\) of \(X\), we say that \(A\) is +invariant under \(f\) if \(f(A) \subseteq A\), \(A\) is −invariant under \(f\) if \(f^{-1}(A) \subseteq A\) and \(A\) is invariant under \(f\) if \(f(A) = A\). A topological space \(X\) is +invariant over open subsets under \(f\), if for each open subset \(U\) of \(X\), \(U\) is +invariant under \(f\). For subsets \(A\) and \(B\) of \(X\), it is defined the following subset of \(\mathbb{Z}\), \(n_f(A, B) = \{k \in \mathbb{Z}_+ \mid A \cap f^{-k}(B) \neq \emptyset\}\). A topological space \(X\) is pseudoregular if for any nonempty open subset \(U\) of \(X\), there exists a nonempty open subset \(V\) of \(X\) such that \(\text{cl}_X(V) \subseteq U\) \([15]\). Let \(X\) be a topological space, let \(B\) be a subset of \(X\) and let \(b \in B\). We say that \(b\) is an isolated point of \(B\) if there exists an open subset \(U\) of \(X\) such that \(U \cap B = \{b\}\). Denote by \(\text{IP}(B)\) the set of isolated points in \(B\). A point \(x\) of \(X\) is a quasiisolated point of \(X\) if there exists a dense subset \(B\) of \(X\) such that \(x \in B\) and \(x\) is an isolated point of \(B\) \([15]\). A topological space is perfect if it does not have isolated points. The following definitions can be found in \([1, 8, 15]\).

Let \(X\) be a topological space and let \(f : X \to X\) be a function. Then \(f\) is:

- **Exact**, if for each nonempty open subset \(U\) of \(X\), there exists \(k \in \mathbb{N}\) such that \(f^k(U) = X\).
- **Mixing**, if for each pair of nonempty open subsets \(U\) and \(V\) of \(X\), there exists \(N \in \mathbb{N}\) such that \(f^k(U) \cap V \neq \emptyset\), for all \(k \geq N\).
- **Transitive**, if for every pair of nonempty open subsets \(U\) and \(V\) of \(X\), there exists \(k \in \mathbb{N}\) such that \(f^k(U) \cap V \neq \emptyset\) (equivalently, for each pair of nonempty open subsets \(U\) and \(V\) of \(X\), \(n_f(U, V) \setminus \{0\} \neq \emptyset\)).
- **Weakly mixing**, if \(f^s\) is transitive.
- **Totally transitive**, if \(f^s\) is transitive, for all \(s \in \mathbb{N}\).
- **Strongly transitive**, if for each nonempty open subset \(U\) of \(X\), there exists \(s \in \mathbb{N}\) such that \(X = \bigcup_{k=0}^s f^k(U)\).
- **Chaotic**, if it is transitive and \(\text{Per}(f)\) is dense in \(X\).
- **Minimal**, if for each nonempty closed subset \(A\) of \(X\) which is +invariant under \(f\), we have \(A = X\).
- **Orbit-transitive**, if there exists \(x \in X\) such that \(\text{cl}_X(O(x, f)) = X\).
- **Strictly orbit-transitive**, if there exists a point \(x\) in \(X\) such that \(\text{cl}_X(O(f(x), f)) = X\).
- **\(\omega\)-transitive**, if there exists \(x \in X\) such that \(\omega(x, f) = X\).
- **\(TT_{++}\)**, if for any pair of nonempty open subsets \(U\) and \(V\) of \(X\), the set \(n_f(U, V)\) is infinite.
- **Mild mixing**, if for any transitive function \(f_1 : X_1 \to X_1\), the function \(f \times f_1\) is transitive.
• Exactly Devaney chaotic, if $f$ is exact and $\text{Per}(f)$ is dense in $X$.

• Backward minimal, if the subset $\{y \in X : f^n(y) = x, \text{ for some } n \in \mathbb{N}\}$ is dense in $X$, for every $x \in X$.

• Totally minimal, if $f^s$ is minimal for all $s \in \mathbb{N}$.

• Scattering, if for any minimal function, $f_1 : X_1 \rightarrow X_1$, the function $f \times f_1$ is transitive.

• Touhey, if for every pair of nonempty open subsets $U$ and $V$ of $X$, there exist a periodic point $x \in U$ and $k \in \mathbb{Z}_+$ such that $f^k(x) \in V$.

• An $F$-system, if $f$ is totally transitive and $\text{Per}(f)$ is dense in $X$.

In the diagram of Figure, we put the inclusions between some of these classes of functions for the general case, that is to say, when $X$ is a topological space and $f : X \rightarrow X$ is a function. For the proofs of these inclusions see, for instance, [1, 5, 15].

When we add properties to the phase space or to the function in Figure, we obtain other relationships, namely: Let $X$ be a topological space and let $f : X \rightarrow X$ be a function. If $X$ is a Hausdorff and compact topological space, and $f$ is a surjective continuous function, we have that if $f$ is scattering, then $f$ is totally transitive [2, Theorem 2.9]. Moreover, if $f$ is a continuous function, it follows that if $f$ is chaotic, then $f$ is Touhey [18, Proposition 2.6].

Hyperspace theory started in early 1900, with the work of Hausdorff [9] and Vietoris [19]. Nowadays hyperspaces are widely studied, mainly in continuum theory, see [12, 14, 17].

Given a topological space $(X, \tau)$ and a positive integer $n$, we define the $n$-fold symmetric product of $X$ by:

$$\mathcal{F}_n(X) = \{A \subseteq X \mid A \neq \emptyset \text{ and } A \text{ has at most } n \text{ elements}\}.$$
This set, equipped with the Vietoris topology [17], is called a hyperspace. Next we describe this topology.

Let \((X, \tau)\) be a topological space. Given a finite collection of nonempty subsets \(U_1, \ldots, U_k\) of \(X\), we denote by \(\langle U_1, \ldots, U_k \rangle\) the subset of \(\mathcal{F}_n(X)\):

\[
\left\{ A \in \mathcal{F}_n(X) \mid A \subseteq \bigcup_{i=1}^k U_i \quad \text{and} \quad A \cap U_i \neq \emptyset, \; \text{for each} \; i \in \{1, \ldots, k\} \right\}.
\]

The family:

\[
\mathcal{B} = \{\langle U_1, \ldots, U_k \rangle \mid U_i \in \tau, \; \text{for each} \; i \in \{1, \ldots, k\} \text{and} \; k \in \mathbb{N}\}
\]

forms a basis for a topology on \(\mathcal{F}_n(X)\) which is denoted by \(\tau_V\) and called the Vietoris topology.

### 3. Preliminary results

Let \(X_1, \ldots, X_m\) be topological spaces and for each \(i \in \{1, \ldots, m\}\) let \(f_i : X_i \to X_i\) be a function. In this section, we present some topological and dynamical properties of the space \(\prod_{i=1}^m X_i\). Moreover, we review the basic results that we need to know about the function \(\prod_{i=1}^m f_i\).

**Remark 3.1** Let \(X_1, \ldots, X_m\) be topological spaces, for each \(i \in \{1, \ldots, m\}\), let \(U_i, V_i\) be nonempty subsets of \(X_i\), for each \(i \in \{1, \ldots, m\}\), let \(f_i : X_i \to X_i\) be a function and let \(k \in \mathbb{N}\). Then the following hold:

1. \((\prod_{i=1}^m f_i)^k = \prod_{i=1}^m f_i^k\).
2. \([\mathcal{F}_n(\prod_{i=1}^m f_i)]^k = \mathcal{F}_n(\prod_{i=1}^m f_i^k)\).
3. If \((\prod_{i=1}^m f_i)^k(\prod_{i=1}^m U_i) = \prod_{i=1}^m V_i\), then, for each \(i \in \{1, \ldots, m\}\), \(f_i^k(U_i) = V_i\).

**Lemma 3.2** Let \(X_1, \ldots, X_m\) be topological spaces, for each \(i \in \{1, \ldots, m\}\), let \(U_i\) be a nonempty subset of \(X_i\), let \(x_i \in X_i\) and let \(f_i : X_i \to X_i\) be a function. If for each \(i \in \{1, \ldots, m\}\), \(X_i\) is +invariant over open subsets under \(f_i\) and, for each \(i \in \{1, \ldots, m\}\), there exists \(k_i \in \mathbb{N}\) such that \(f_i^{k_i}(x_i) \in U_i\), then, for \(k = \max\{k_1, \ldots, k_m\}\), it follows that, for each \(i \in \{1, \ldots, m\}\), \(f_i^k(x_i) \in U_i\).

**Proof** Suppose that, for each \(i \in \{1, \ldots, m\}\), \(X_i\) is +invariant over open subsets under \(f_i\) and that there exists \(k_i \in \mathbb{N}\) such that \(f_i^{k_i}(x_i) \in U_i\). Let \(k = \max\{k_1, \ldots, k_m\}\). It follows that, for each \(i \in \{1, \ldots, m\}\), there exists \(l_i \in \mathbb{Z}_+\) such that \(k = k_i + l_i\). Thus, for each \(i \in \{1, \ldots, m\}\), \(f_i^k(x_i) = f_i^{k_i+l_i}(x_i) = f_i^{k_i}(f_i^{l_i}(x_i))\). Consequently, for each \(i \in \{1, \ldots, m\}\), \(f_i^k(x_i) \in f_i^{l_i}(U_i)\). By hypothesis, since, for each \(i \in \{1, \ldots, m\}\), \(U_i\) is +invariant under \(f_i\), we have that, for each \(i \in \{1, \ldots, m\}\), \(f_i^{l_i}(x_i) \in U_i\).

**Theorem 3.3** Let \(X_1, \ldots, X_m\) be topological spaces, for each \(i \in \{1, \ldots, m\}\), let \(f_i : X_i \to X_i\) be a function, and let \((x_1, \ldots, x_m) \in \prod_{i=1}^m X_i\). Then the following hold:

1. If \((x_1, \ldots, x_m)\) is a transitive point of \(\prod_{i=1}^m f_i\), then, for each \(i \in \{1, \ldots, m\}\), \(x_i\) is a transitive point of \(f_i\).
2. If \(\omega((x_1, \ldots, x_m), \prod_{i=1}^m f_i) = \prod_{i=1}^m X_i\), then, for each \(i \in \{1, \ldots, m\}\), \(\omega(x_i, f_i) = X_i\).
3. For each \(i \in \{1, \ldots, m\}\), \(x_i\) is an isolated point in \(X_i\) if and only if \((x_1, \ldots, x_m)\) is an isolated point in \(\prod_{i=1}^{m} X_i\).

4. For each \(i \in \{1, \ldots, m\}\), \(x_i\) is a periodic point of \(f_i\) if and only if \((x_1, \ldots, x_m)\) is a periodic point of \(\prod_{i=1}^{m} f_i\).

**Proof** Suppose that \(\text{cl}_{\prod_{i=1}^{m} X_i}(O((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i\). Let \(i_0 \in \{1, \ldots, m\}\) and let \(U_{i_0}\) be a nonempty open subset of \(X_{i_0}\). For each \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(V_i = X_i\) and \(V_{i_0} = U_{i_0}\). It follows that \(\prod_{i=1}^{m} V_i\) is a nonempty open subset of \(\prod_{i=1}^{m} X_i\). By hypothesis, \(O((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) \cap (\prod_{i=1}^{m} V_i) \neq \emptyset\). Then, there exists \(k \in \mathbb{N}\) such that \((\prod_{i=1}^{m} f_i)^k((x_1, \ldots, x_m)) \in \prod_{i=1}^{m} V_i\). By Remark 3.1, part (1), \((\prod_{i=1}^{m} f_i)^k((x_1, \ldots, x_m)) = (f_1^k(x_1), \ldots, f_m^k(x_m)), f_i^k(x_{i_0}) \in U_{i_0}\). Therefore, \(U_{i_0} \cap O(x_{i_0}, f_{i_0}) \neq \emptyset\) and \(\text{cl}_{X_{i_0}}(O(x_{i_0}, f_{i_0})) = X_{i_0}\).

Suppose that \(\omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i\). Let \(i_0 \in \{1, \ldots, m\}\), let \(y_{i_0} \in X_{i_0}\), let \(k \in \mathbb{N}\), let \(U_{i_0}\) be an open subset of \(X_{i_0}\) such that \(y_{i_0} \in U_{i_0}\) and for each \(j \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(y_j \in X_j\). Moreover, for each \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), we put \(V_i = X_i\) and \(V_{i_0} = U_{i_0}\). It follows that \(\prod_{i=1}^{m} V_i\) is a nonempty open subset of \(\prod_{i=1}^{m} X_i\) such that \((y_1, \ldots, y_m) \in \prod_{i=1}^{m} V_i\). Thus, by hypothesis, there exists \(l \in \mathbb{N}\) with \(l \geq k\) such that \((\prod_{i=1}^{m} f_i)^l((x_1, \ldots, x_m)) \in \prod_{i=1}^{m} V_i\). By Remark 3.1, part (1), we have that \(f_{i_0}^l(x_{i_0}) \in U_{i_0}\). Therefore, \(y_{i_0} \in \omega(x_{i_0}, f_{i_0})\). Consequently, \(X_{i_0} = \omega(x_{i_0}, f_{i_0})\).

Suppose that \((x_1, \ldots, x_m)\) is an isolated point in \(\prod_{i=1}^{m} X_i\). Then there exists an open subset \(U\) of \(\prod_{i=1}^{m} X_i\) such that \((\prod_{i=1}^{m} X_i) \cap U = \{(x_1, \ldots, x_m)\}\). Even more, for each \(i \in \{1, \ldots, m\}\), there exists a nonempty open subset \(U_i \subseteq X_i\) such that \((\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} X_i) = \{(x_1, \ldots, x_m)\}\). Observe that, for each \(i \in \{1, \ldots, m\}\), \(U_i \cap X_i = \{x_i\}\). Consequently, for each \(i \in \{1, \ldots, m\}\), \(x_i\) is an isolated point in \(X_i\).

Now suppose that, for each \(i \in \{1, \ldots, m\}\), \(x_i\) is an isolated point in \(X_i\). Then, for each \(i \in \{1, \ldots, m\}\), there exists an open subset \(U_i \subseteq X_i\) such that \(U_i \cap X_i = \{x_i\}\). Note that, \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} U_i\) and \((\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} X_i) = \{(x_1, \ldots, x_m)\}\). Thus, \((x_1, \ldots, x_m)\) is an isolated point in \(\prod_{i=1}^{m} X_i\).

Suppose that, for each \(i \in \{1, \ldots, m\}\), \(x_i\) is a periodic point of \(f_i\). Thus, for each \(i \in \{1, \ldots, m\}\), there exists \(k_i \in \mathbb{N}\) such that \(f_i^{k_i}(x_i) = x_i\). Let \(k = k_1 \cdots k_m\). It follows that, for each \(i \in \{1, \ldots, m\}\), \(f_i^k(x_i) = x_i\). Hence, \((f_1^k(x_1), \ldots, f_m^k(x_m)) = (x_1, \ldots, x_m)\). By Remark 3.1, part (1), \((\prod_{i=1}^{m} f_i)^k((x_1, \ldots, x_m)) = (x_1, \ldots, x_m)\). Therefore, \((x_1, \ldots, x_m)\) is a periodic point of \(\prod_{i=1}^{m} f_i\).

Now, suppose that \((x_1, \ldots, x_m)\) is a periodic point of \(\prod_{i=1}^{m} f_i\). Then, there exists \(k \in \mathbb{N}\) such that \((\prod_{i=1}^{m} f_i)^k((x_1, \ldots, x_m)) = (x_1, \ldots, x_m)\). Thus, by Remark 3.1, part (1), for each \(i \in \{1, \ldots, m\}\), \(f_i^k(x_i) = x_i\). Therefore, for each \(i \in \{1, \ldots, m\}\), \(x_i\) is a periodic point of \(f_i\).

As a consequence of Theorem 3.3, we have the following:

**Remark 3.4** Let \(X_1, \ldots, X_m\) be topological spaces and, for each \(i \in \{1, \ldots, m\}\), let \(f_i : X_i \to X_i\) be a function. Then the following hold:

1. \(\text{trans}(\prod_{i=1}^{m} f_i) \subseteq \prod_{i=1}^{m} \text{trans}(f_i)\).

2. \(\omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) \subseteq \prod_{i=1}^{m} \omega(x_i, f_i)\).
3. $IP(\prod_{i=1}^{m} X_i) = \prod_{i=1}^{m} IP(X_i)$.

4. $Per(\prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} Per(f_i)$.

**Corollary 3.5** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then the following hold:

1. $cl_{\prod_{i=1}^{m} X_i} (\text{trans}(\prod_{i=1}^{m} f_i)) \subseteq \prod_{i=1}^{m} cl_{X_i} (\text{trans}(f_i))$.

2. $cl_{\prod_{i=1}^{m} X_i} (\prod_{i=1}^{m} Per(f_i)) = \prod_{i=1}^{m} cl_{X_i} (Per(f_i))$.

In Example 3.6 we show that the converse of Theorem 3.3, parts (1) and (2), are not true in general.

**Example 3.6** Let $X = \{1, 2\}$ topologized with $\tau = \emptyset, X, \{1\}$ and let $f : X \to X$ be a function given by $f(1) = 2$ and $f(2) = 1$. Note that

1. $cl_X (O(1, f)) = X$ and $cl_X (O(2, f)) = X$. However, $O((1, 2), f \times f) \cap (\{1\} \times \{1\}) = \emptyset$. Consequently, $cl_X \times X (O((1, 2), f \times f)) \neq X \times X$.

2. $\omega(1, f) = X$ and $\omega(2, f) = X$. However, $\omega((1, 2), f \times f) \neq X \times X$.

There exist conditions that make the converse of Theorem 3.3, parts (1) and (2) true. One of these conditions is given in Theorem 3.7.

**Theorem 3.7** Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $x_i \in X_i$, and let $f_i : X_i \to X_i$ be a function. Then the following hold:

1. If, for each $i \in \{1, \ldots, m\}$, $\omega(x_i, f_i) = X_i$ and $X_i$ is $+$invariant over open subsets under $f_i$, then $\omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i$.

2. If, for each $i \in \{1, \ldots, m\}$, $cl_{X_i} (O(x_i, f_i)) = X_i$ and $X_i$ is $+$invariant over open subsets under $f_i$, then:

$$cl_{\prod_{i=1}^{m} X_i} \left( O \left( (x_1, \ldots, x_m), \prod_{i=1}^{m} f_i \right) \right) = \prod_{i=1}^{m} X_i.$$

**Proof** Suppose that, for each $i \in \{1, \ldots, m\}$, $\omega(x_i, f_i) = X_i$ and that $X_i$ is $+$invariant over open subsets under $f_i$. Let $(y_1, \ldots, y_m) \in \prod_{i=1}^{m} X_i$, let $k \in \mathbb{N}$ and let $U$ be an open subset of $\prod_{i=1}^{m} X_i$ such that $(y_1, \ldots, y_m) \in U$. Then, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $U_i$ of $X_i$, such that $(y_1, \ldots, y_m) \in \prod_{i=1}^{m} U_i \subseteq U$. By hypothesis, for each $i \in \{1, \ldots, m\}$, there exists $l_i \in \mathbb{N}$ such that $l_i \geq k$ and $f_i^{l_i}(x_i) \in U_i$. For each $i \in \{1, \ldots, m\}$, let $l = \max\{l_1, \ldots, l_m\}$. By Lemma 3.2, for each $i \in \{1, \ldots, m\}$, we have that $f_i^{l}(x_i) \in U_i$. Thus, $(\prod_{i=1}^{m} f_i)^{(l)}((x_1, \ldots, x_m)) \in U$. Also note that $l \geq k$. Therefore, $(x_1, \ldots, x_m) \in \omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)$ and $\omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i$.

Now suppose that, for each $i \in \{1, \ldots, m\}$, $cl_{X_i} (O(x_i, f_i)) = X_i$ and that $X_i$ is $+$invariant over open subsets under $f_i$. Let $U$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \ldots, m\}$,
there exists a nonempty open subset $U_i$ of $X_i$ such that $\prod_{i=1}^{m} U_i \subseteq U$. By hypothesis, for each $i \in \{1, \ldots, m\}$, $\mathcal{O}(x_i, f_i) \cap U_i \neq \emptyset$. It follows that, for each $i \in \{1, \ldots, m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(x_i) \in U_i$. Let $k = \max\{k_1, \ldots, k_m\}$. By Lemma 3.2, for each $i \in \{1, \ldots, m\}$, we have that $f_i^k(x_i) \in U_i$. Consequently, $(\prod_{i=1}^{m} f_i)^k((x_1, \ldots, x_m)) = (f_1^k(x_1), \ldots, f_m^k(x_m)) \in \prod_{i=1}^{m} U_i$. Hence, $\mathcal{O}((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) \cap U \neq \emptyset$. Therefore, $\mathcal{C}(\prod_{i=1}^{m} x_i, \mathcal{O}((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$.

As a consequence of Theorem 3.3, part (3), we have:

**Corollary 3.8** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then $\prod_{i=1}^{m} X_i$ is perfect if and only if, for each $i \in \{1, \ldots, m\}$, $X_i$ is perfect.

**Theorem 3.9** Let $X_1, \ldots, X_m$ be topological spaces. Then $\prod_{i=1}^{m} X_i$ is pseudoregular if and only if, for each $i \in \{1, \ldots, m\}$, $X_i$ is pseudoregular.

**Proof** Suppose that $\prod_{i=1}^{m} X_i$ is pseudoregular. Let $i_0 \in \{1, \ldots, m\}$ and let $U_{i_0}$ be a nonempty open subset of $X_{i_0}$. For each $i \in \{1, \ldots, m\}\{i_0\}$, let $V_i = X_i$ and let $V_{i_0} = U_{i_0}$. Thus, $\prod_{i=1}^{m} V_i$ is a nonempty open subset of $\prod_{i=1}^{m} X_i$. Since $\prod_{i=1}^{m} X_i$ is pseudoregular, there exists a nonempty open subset $V$ of $\prod_{i=1}^{m} X_i$ such that $\mathcal{C}(\prod_{i=1}^{m} x_i, V) \subseteq \prod_{i=1}^{m} V_i$. Moreover, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $V_i' \subseteq X_i$ such that $\prod_{i=1}^{m} V_i' \subseteq V$. Consequently, $\mathcal{C}(\prod_{i=1}^{m} x_i, (\prod_{i=1}^{m} V_i')) \subseteq \prod_{i=1}^{m} V_i$. Then $\mathcal{C}(X_{i_0}, (V_{i_0})) \subseteq U_{i_0}$. Therefore, $X_{i_0}$ is pseudoregular. Because $i_0 \in \{1, \ldots, m\}$ is arbitrary, we have that, for each $i \in \{1, \ldots, m\}$, $X_i$ is pseudoregular.

Now suppose that, for each $i \in \{1, \ldots, m\}$, $X_i$ is pseudoregular. Let $U$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $U_i$ of $X_i$ such that $\prod_{i=1}^{m} U_i \subseteq U$. Since, for each $i \in \{1, \ldots, m\}$, $X_i$ is pseudoregular, we have that, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $V_i$ of $X_i$ such that $\mathcal{C}(X_i, V_i) \subseteq U_i$. Hence, $\prod_{i=1}^{m} \mathcal{C}(X_i, V_i) \subseteq \prod_{i=1}^{m} U_i$. On the other hand, since $\mathcal{C}(\prod_{i=1}^{m} x_i, (\prod_{i=1}^{m} V_i)) \subseteq \prod_{i=1}^{m} \mathcal{C}(X_i, V_i)$, we have that $\mathcal{C}(\prod_{i=1}^{m} x_i, (\prod_{i=1}^{m} V_i)) \subseteq U$. Therefore, $\prod_{i=1}^{m} X_i$ is pseudoregular.

**Proposition 3.10** Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $U_i$ be an open subset of $X_i$, and let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1, \ldots, m\}$, $U_i$ is $+\text{invariant}$ under $f_i$ if and only if $\prod_{i=1}^{m} U_i$ is $+\text{invariant}$ under $\prod_{i=1}^{m} f_i$.

**Proof** Suppose that, for each $i \in \{1, \ldots, m\}$, $U_i$ is $+\text{invariant}$ under $f_i$. Let $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} f_i(\prod_{i=1}^{m} U_i)$. Then there exists $(x_1, \ldots, x_m) \in \prod_{i=1}^{m} U_i$ such that $\prod_{i=1}^{m} f_i((x_1, \ldots, x_m)) = (a_1, \ldots, a_m)$. It follows that, for each $i \in \{1, \ldots, m\}$, $f_i(x_i) = a_i$. Then, for each $i \in \{1, \ldots, m\}$, $a_i \in f_i(U_i)$. Since, for each $i \in \{1, \ldots, m\}$, $U_i$ is $+\text{invariant}$ under $f_i$, we have that, for each $i \in \{1, \ldots, m\}$, $a_i \in U_i$. Therefore, $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i$. Consequently, $\prod_{i=1}^{m} U_i$ is $+\text{invariant}$ under $\prod_{i=1}^{m} f_i$.

Now suppose that $\prod_{i=1}^{m} U_i$ is $+\text{invariant}$ under $\prod_{i=1}^{m} f_i$. Let $i_0 \in \{1, \ldots, m\}$ and let $x_{i_0} \in f_{i_0}(U_{i_0})$. Then there exists $u_{i_0} \in U_{i_0}$ such that $f_{i_0}(u_{i_0}) = x_{i_0}$. For each $j \in \{1, \ldots, m\}\{i_0\}$, let $u_j \in U_j$. Next, $(u_1, \ldots, u_m) \in \prod_{i=1}^{m} U_i$. Since $\prod_{i=1}^{m} U_i$ is $+\text{invariant}$ under $\prod_{i=1}^{m} f_i$, we have that $\prod_{i=1}^{m} f_i((u_1, \ldots, u_m)) = (f_1(u_1), \ldots, f_m(u_m)) \in \prod_{i=1}^{m} U_i$. Thus, $x_{i_0} = f_{i_0}(u_{i_0}) \in U_{i_0}$. Therefore, $f_{i_0}(U_{i_0}) \subseteq U_{i_0}$.

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Proposition 3.11 Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If, for each $i \in \{1, \ldots, m\}$, $U_i \subseteq X_i$ is $-$ invariant under $f_i$, then $\prod_{i=1}^{m} U_i$ is $-$ invariant under $\prod_{i=1}^{m} f_i$.

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $U_i$ is $-$ invariant under $f_i$. We show that $(\prod_{i=1}^{m} f_i)^{-1}(\prod_{i=1}^{m} U_i) \subseteq \prod_{i=1}^{m} U_i$. Let $(a_1, \ldots, a_m) \in (\prod_{i=1}^{m} f_i)^{-1}(\prod_{i=1}^{m} U_i)$. We have that $\prod_{i=1}^{m} f_i((a_1, \ldots, a_m)) \in \prod_{i=1}^{m} U_i$. It follows that, for each $i \in \{1, \ldots, m\}$, $f_i(a_i) \in U_i$. Thus, for each $i \in \{1, \ldots, m\}$, $a_i \in f_i^{-1}(U_i)$. Since, for each $i \in \{1, \ldots, m\}$, $U_i$ is $-$ invariant under $f_i$, we obtain that, for each $i \in \{1, \ldots, m\}$, $a_i \in U_i$. Consequently, $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i$. Therefore, $\prod_{i=1}^{m} U_i$ is $-$ invariant under $\prod_{i=1}^{m} f_i$.

The converse of Proposition 3.11 is not true in general.

Example 3.12 Let $X = \{1, 2, 3, 4\}$ be a set topologized with $\{X, \emptyset, \{1, 2\}\}$, and let $f : X \to X$ be a function given by $f(x) = 1$, for each $x \in X$. Let $A = \{1\} \times \{2, 3, 4\}$. Note that $(f \times f)^{-1}(A) = \emptyset$. Thus, $(f \times f)^{-1}(A) \subseteq A$. Then $A$ is $-$ invariant under $f \times f$. On the other hand, $f^{-1}(\{1\}) = X$. It follows that, $f^{-1}(\{1\}) \not\subseteq \{1\}$. Consequently, $\{1\}$ it is not $-$ invariant under $f$.

Theorem 3.13 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $U_i$ be an open subset of $X_i$, and let $f_i : X_i \to X_i$ be a surjective function. Then $\prod_{i=1}^{m} U_i$ is $-$ invariant under $\prod_{i=1}^{m} f_i$ if and only if, for each $i \in \{1, \ldots, m\}$, $U_i$ is $-$ invariant under $f_i$.

Proof Suppose that $\prod_{i=1}^{m} U_i$ is $-$ invariant under $\prod_{i=1}^{m} f_i$. Let $i_0 \in \{1, \ldots, m\}$ and let $a_{i_0} \in f_{i_0}^{-1}(U_{i_0})$. Thus, $f_{i_0}(a_{i_0}) \in U_{i_0}$. On the other hand, since, for each $j \in \{1, \ldots, m\}$, $f_j$ is surjective, we have that, for each $j \in \{1, \ldots, m\}$, $f_j^{-1}(U_j) \neq \emptyset$. Then, for each $j \in \{1, \ldots, m\}\{i_0\}$, we can take $a_j \in f_j^{-1}(U_j)$. Hence, for each $j \in \{1, \ldots, m\}$, $f_j(a_j) \in U_j$. It follows that $(f_1(a_1), \ldots, f_m(a_m)) \in \prod_{i=1}^{m} U_i$. Thus, $\prod_{i=1}^{m} f_i((a_1, \ldots, a_m)) \in \prod_{i=1}^{m} U_i$. Then $(a_1, \ldots, a_m) \in (\prod_{i=1}^{m} f_i)^{-1}(\prod_{i=1}^{m} U_i)$. By hypothesis, since $\prod_{i=1}^{m} U_i$ is $-$ invariant, $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i$. Hence, $a_{i_0} \in U_{i_0}$. Therefore, $U_{i_0}$ is $-$ invariant.

The converse implication follows from Proposition 3.11.

Theorem 3.14 Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1, \ldots, m\}$, $X_i$ is $+$ invariant over open subsets under $f_i$ if and only if $\prod_{i=1}^{m} X_i$ is $+$ invariant over open subsets under $\prod_{i=1}^{m} f_i$.

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $X_i$ is $+$ invariant over open subsets under $f_i$. Let $U$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$ and let $(x_1, \ldots, x_m) \in \prod_{i=1}^{m} f_i(U)$. Then there exists $(a_1, \ldots, a_m) \in U$ such that $\prod_{i=1}^{m} f_i((a_1, \ldots, a_m)) = (x_1, \ldots, x_m)$. It follows that, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $U_i$ of $X_i$ such that $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i \subseteq U$. By hypothesis and Proposition 3.10, $\prod_{i=1}^{m} f_i(\prod_{i=1}^{m} U_i) \subseteq \prod_{i=1}^{m} U_i$. Thus, $(x_1, \ldots, x_m) \in \prod_{i=1}^{m} U_i \subseteq U$. Therefore, $U$ is $+$ invariant under $\prod_{i=1}^{m} f_i$. Because $U$ is arbitrary, we have that $\prod_{i=1}^{m} X_i$ is $+$ invariant over open subsets under $\prod_{i=1}^{m} f_i$.

Now, suppose that $\prod_{i=1}^{m} X_i$ is $+$ invariant over open subsets under $\prod_{i=1}^{m} f_i$. Let $i_0 \in \{1, \ldots, m\}$, let $U_{i_0}$ be an open subset of $X_{i_0}$ and, for each $i \in \{1, \ldots, m\}\{i_0\}$, let $V_i = X_i$ and $V_{i_0} = U_{i_0}$. Then $\prod_{i=1}^{m} V_i$ is
a nonempty open subset of $\prod_{i=1}^{m} X_i$. Since $\prod_{i=1}^{m} X_i$ is $+\text{invariant}$ over open subsets under $\prod_{i=1}^{m} f_i$, we have that $\prod_{i=1}^{m} V_i$ is $+\text{invariant}$ under $\prod_{i=1}^{m} f_i$. Then, by Proposition 3.10, $U_{io}$ is $+\text{invariant}$ under $f_{io}$. □

**Theorem 3.15** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then $\text{Per}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$ if and only if, for each $i \in \{1, \ldots, m\}$, $\text{Per}(f_i)$ is dense in $X_i$.

**Proof** Suppose that $\text{Per}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$. Thus, $\text{cl}_{\prod_{i=1}^{m} X_i}(\text{Per}(\prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. Then, by Corollary 3.5, part (2), $\prod_{i=1}^{m} \text{cl}_{X_i}(\text{Per}(f_i)) = \prod_{i=1}^{m} X_i$. Consequently, for each $i \in \{1, \ldots, m\}$, $\text{cl}_{X_i}(\text{Per}(f_i)) = X_i$. Therefore, for each $i \in \{1, \ldots, m\}$, $\text{Per}(f_i)$ is dense in $X_i$.

Now suppose that, for each $i \in \{1, \ldots, m\}$, $\text{Per}(f_i)$ is dense in $X_i$. In consequence, we have that, $\prod_{i=1}^{m} \text{cl}_{X_i}(\text{Per}(f_i)) = \prod_{i=1}^{m} X_i$. On the other hand, by Remark 3.4 and Corollary 3.5, part (2), we have that $\prod_{i=1}^{m} \text{cl}_{X_i}(\text{Per}(f_i)) = \text{cl}_{\prod_{i=1}^{m} X_i}(\text{Per}(\prod_{i=1}^{m} f_i)) = \text{cl}_{\prod_{i=1}^{m} X_i}(\text{Per}(\prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. It follows that $\text{cl}_{\prod_{i=1}^{m} X_i}(\text{Per}(\prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. Therefore, $\text{Per}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$. □

**Proposition 3.16** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If $\text{trans}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$, then, for each $i \in \{1, \ldots, m\}$, $\text{trans}(f_i)$ is dense in $X_i$.

**Proof** Suppose that $\text{trans}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$. Hence, $\text{cl}_{\prod_{i=1}^{m} X_i}(\text{trans}(\prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. Thus, by Corollary 3.5, part (1), $\prod_{i=1}^{m} X_i \subseteq \prod_{i=1}^{m} \text{cl}_{X_i}(\text{trans}(f_i))$. Consequently, for each $i \in \{1, \ldots, m\}$, $X_i \subseteq \text{cl}_{X_i}(\text{trans}(f_i))$. Therefore, for each $i \in \{1, \ldots, m\}$, $\text{trans}(f_i)$ is dense in $X_i$. □

The converse of Proposition 3.16 is not true in general.

**Example 3.17** Let $X = \{1, 2\}$ be a set topologized with $\tau = \{\emptyset, X, \{1\}, \{2\}\}$, and let $f : X \to X$ be a function given by $f(1) = 2$ and $f(2) = 1$. Note that

1. $\mathcal{O}(1, f) = \{1, 2\}$ is dense in $X$ and $\mathcal{O}(2, f) = \{2, 1\}$ is dense in $X$. Thus, $\text{trans}(f)$ is dense in $X$.
2. $\text{trans}(f \times f) = \emptyset$.

**Theorem 3.18** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If, for each $i \in \{1, \ldots, m\}$, $\text{trans}(f_i)$ is dense in $X_i$ and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\text{trans}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$.

**Proof** Suppose that, for each $i \in \{1, \ldots, m\}$, $\text{trans}(f_i)$ is dense in $X_i$ and that $X_i$ is $+\text{invariant}$ over open subsets under $f_i$. Let $U$ be a nonempty open subset of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exists a nonempty open subset $U_i$ of $X_i$ such that $\prod_{i=1}^{m} U_i \subseteq U$. By hypothesis, for each $i \in \{1, \ldots, m\}$, $U_i \cap \text{trans}(f_i) \neq \emptyset$. Consequently, for each $i \in \{1, \ldots, m\}$, there exists $x_i \in U_i$ such that $x_i$ is a transitive point of $f_i$. Since, for each $i \in \{1, \ldots, m\}$, $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, by Theorem 3.7, part (2), we have that $(x_1, \ldots, x_m)$ is a transitive point of $\prod_{i=1}^{m} f_i$. Even more, $(x_1, \ldots, x_m) \in U$. Therefore, $\text{trans}(\prod_{i=1}^{m} f_i)$ is dense in $\prod_{i=1}^{m} X_i$. □
Lemma 3.19 Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. If $j \in \{1, \ldots, m\}$, let $U_j, V_j$ be two nonempty open subsets of $X_j$ and, for each $i \in \{1, \ldots, m\}\{j\}$, we put $U_i = X_i$ and $V_i = X_i$, then $n_{i=1}^m f_i (\prod_{i=1}^m U_i, \prod_{i=1}^m V_i) \subseteq n_{i=1}^m (U_j, V_j)$. 

Proof Let $k \in \prod_{i=1}^m f_i (\prod_{i=1}^m U_i, \prod_{i=1}^m V_i)$. Then $\prod_{i=1}^m U_i \cap (\prod_{i=1}^m f_i)_{-k} (\prod_{i=1}^m V_i) \neq \emptyset$. Let $(y_1, \ldots, y_m) \in (\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m f_i)_{-k} (\prod_{i=1}^m V_i)$. It follows that $\prod_{i=1}^m f_i ((y_1, \ldots, y_m)) \in \prod_{i=1}^m V_i$. Then, by Remark 3.1, part (1), we have that $(f_i^k(y_1), \ldots, f_i^k(y_m)) \in \prod_{i=1}^m V_i$. Consequently, $y_j \in U_j \cap f_j^{-k}(V_j)$. Then, $k \in n_{j=1}^m f_j (U_j, V_j)$. Thus, $n_{i=1}^m f_i (\prod_{i=1}^m U_i, \prod_{i=1}^m V_i) \subseteq n_{j=1}^m (U_j, V_j)$. □

4. Dynamic properties of product functions

Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. In this section, we present the relationships that exist between the functions $\prod_{i=1}^m f_i$ and $f_i$ for each $i \in \{1, \ldots, m\}$, when any of them is exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, $TT_{++}$, mild mixing, exactly Devaney chaotic, backward minimal, totally minimal, scattering, Touhey or an $F$-system.

Theorem 4.1 Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, $TT_{++}$, backward minimal, Touhey, an $F$-system, scattering or mild mixing. If $\prod_{i=1}^m f_i \in \mathcal{M}$, then, for each $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$.

Proof Suppose that $\prod_{i=1}^m f_i$ is transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $U_{i_0}, V_{i_0}$ be nonempty open subsets of $X_{i_0}$. For every $i \in \{1, \ldots, m\}\{i_0\}$, let $U_i = X_i$ and let $V_i = X_i$. Then $\prod_{i=1}^m U_i$ and $\prod_{i=1}^m V_i$ are nonempty open subsets of $\prod_{i=1}^m X_i$. Since $\prod_{i=1}^m f_i$ is transitive, there exists $k \in \mathbb{N}$ such that $\prod_{i=1}^m f_i^{k} (\prod_{i=1}^m U_i) \cap (\prod_{i=1}^m V_i) \neq \emptyset$. Let $(u_1, \ldots, u_m) \in \prod_{i=1}^m U_i$ such that $\prod_{i=1}^m f_i^{k} ((u_1, \ldots, u_m)) \in \prod_{i=1}^m V_i$. Thus, by Remark 3.1, part (1), we have that $f_{i_0}^{k}(u_{i_0}) \in V_{i_0}$. Therefore, $f_{i_0}^k(u_{i_0}) \in f_{i_0}^k(U_{i_0}) \cap V_{i_0}$, $f_{i_0}^k(U_{i_0}) \cap V_{i_0} \neq \emptyset$ and $f_{i_0}$ is transitive.

Suppose that $\prod_{i=1}^m f_i$ is weakly mixing. Let $i_0 \in \{1, \ldots, m\}$ and let $\mathcal{U}, \mathcal{V}$ be nonempty open subsets of $X_{i_0} \times X_{i_0}$. Then there exist nonempty open subsets $U_{i_0}^1, U_{i_0}^2, V_{i_0}^1$ and $V_{i_0}^2$ of $X_{i_0}$ such that $U_{i_0}^1 \times U_{i_0}^2 \subseteq \mathcal{U}$ and $V_{i_0}^1 \times V_{i_0}^2 \subseteq \mathcal{V}$. For each $i \in \{1, \ldots, m\}\{i_0\}$, let $U_i^1 = U_i^2 = V_i^1 = V_i^2 = X_i$. Hence, $(\prod_{i=1}^m U_i^1) \times (\prod_{i=1}^m U_i^2)$ and $(\prod_{i=1}^m V_i^1) \times (\prod_{i=1}^m V_i^2)$ are nonempty open subsets of $(\prod_{i=1}^m X_i) \times (\prod_{i=1}^m X_i)$. By hypothesis, there exists $((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \in (\prod_{i=1}^m U_i^1) \times (\prod_{i=1}^m U_i^2)$ and $k \in \mathbb{N}$ such that $((\prod_{i=1}^m f_i) \times (\prod_{i=1}^m f_i))^{k} ((a_1, \ldots, a_m), (b_1, \ldots, b_m)) \in (\prod_{i=1}^m V_i^1) \times (\prod_{i=1}^m V_i^2)$. Then by Remark 3.1, part (1), $(f_{i_0} \times f_{i_0})^{k} ((a_{i_0}, b_{i_0})) \in V_{i_0}^1 \times V_{i_0}^2$. Even more, $(a_{i_0}, b_{i_0}) \in U_{i_0}^1 \times U_{i_0}^2$. Therefore, $(f_{i_0} \times f_{i_0})^{k} (\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and hence $f_{i_0}^{k \times 2}$ is transitive. Finally, $f_{i_0}$ is weakly mixing.

Suppose that $\prod_{i=1}^m f_i$ is totally transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $s \in \mathbb{N}$. By hypothesis, $(\prod_{i=1}^m f_i)^s$ is transitive. By Remark 3.1, part (1), $\prod_{i=1}^m f_i^s$ is transitive. Thus, by the first paragraph of the proof of this theorem, we have that $f_{i_0}^s$ is transitive. Therefore, $f_{i_0}$ is totally transitive.

Suppose that $\prod_{i=1}^m f_i$ is strongly transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $U_{i_0}$ be a nonempty open subset
of \(X_{i_0}\). For every \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(U_i = X_i\). Then \(\prod_{i=1}^{m} U_i\) is a nonempty open subset of \(\prod_{i=1}^{m} X_i\). By hypothesis, there exists \(s \in \mathbb{N}\) such that \(\prod_{i=1}^{m} X_i = \bigcup_{k=0}^{s} (\prod_{i=1}^{m} f_i)(\prod_{i=1}^{m} U_i)\). Let \(x_{i_0} \in X_{i_0}\) and, for each \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(x_i \in X_i\). Then there exists \(k_1 \in \{0, \ldots, s\}\) such that \((x_1, \ldots, x_m) \in (\prod_{i=1}^{m} f_i)(\prod_{i=1}^{m} U_i)\). Thus, by Remark 3.1, part (1), we have that \(x_{i_0} \in f_{i_0}^{k_1}(U_{i_0})\). Therefore, \(X_{i_0} = \bigcup_{k=0}^{s} f_{i_0}^{k}(U_{i_0})\) and hence \(f_{i_0}\) is strongly transitive.

Suppose that \(\prod_{i=1}^{m} f_i\) is chaotic. By the first paragraph of the proof of this theorem, for all \(i \in \{1, \ldots, m\}\), \(f_i\) is transitive. Moreover, by Theorem 3.15, for every \(i \in \{1, \ldots, m\}\), \(\text{Per}(f_i)\) is dense in \(X_i\). Thus, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is chaotic.

Suppose that \(\prod_{i=1}^{m} f_i\) is orbit-transitive. Consequently, there exists \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i\) such that \(\text{cl}(\prod_{i=1}^{m} X_i, (\mathcal{O}(\prod_{i=1}^{m} f_i))(\prod_{i=1}^{m} f_i))) = \prod_{i=1}^{m} X_i\). Thus, by Theorem 3.3, part (1), for every \(i \in \{1, \ldots, m\}\), we have that \(\text{cl}(X_i, (\mathcal{O}(x_i, f_i))) = X_i\). Thus, for all \(i \in \{1, \ldots, m\}\), \(f_i\) is orbit-transitive.

Suppose that \(\prod_{i=1}^{m} f_i\) is strictly orbit-transitive. Consequently, there exists \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i\) such that \(\text{cl}(\prod_{i=1}^{m} X_i, (\mathcal{O}(\prod_{i=1}^{m} f_i)((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i))) = \prod_{i=1}^{m} X_i\). Therefore, by Theorem 3.3, part (1), for every \(i \in \{1, \ldots, m\}\), \(\text{cl}(X_i, (\mathcal{O}(x_i, f_i))) = X_i\) and hence, for all \(i \in \{1, \ldots, m\}\), \(f_i\) is strictly orbit-transitive.

Suppose that \(\prod_{i=1}^{m} f_i\) is \(\omega\)-transitive. Consequently, there exists \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i\) such that \(\omega((x_1, \ldots, x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i\). Thus, by Theorem 3.3, part (2), for each \(i \in \{1, \ldots, m\}\), \(\omega(x_i, f_i) = X_i\). Therefore, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is \(\omega\)-transitive.

Suppose that \(\prod_{i=1}^{m} f_i\) is \(TT_{++}\). Let \(i_0 \in \{1, \ldots, m\}\) and let \(U_{i_0}, V_{i_0}\) be nonempty open subsets of \(X_{i_0}\). For every \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(U_i = X_i\) and \(V_i = X_i\). Then by Lemma 3.19, \(n_{\prod_{i=1}^{m} f_i}(\prod_{i=1}^{m} U_i, \prod_{i=1}^{m} V_i) \subseteq n_{f_{i_0}}(U_{i_0}, V_{i_0})\). Moreover, by hypothesis, \(n_{\prod_{i=1}^{m} f_i}(\prod_{i=1}^{m} U_i, \prod_{i=1}^{m} V_i)\) is infinite. Therefore, \(n_{f_{i_0}}(U_{i_0}, V_{i_0})\) is infinite and hence \(f_{i_0}\) is \(TT_{++}\).

Suppose that \(\prod_{i=1}^{m} f_i\) is backward minimal. Let \(i_0 \in \{1, \ldots, m\}\), let \(x_{i_0} \in X_{i_0}\) and let \(U_{i_0}\) be a nonempty open subset of \(X_{i_0}\). For each \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(U_i = X_i\) and let \(x_i \in X_i\). Then \(\prod_{i=1}^{m} U_i\) is a nonempty open subset of \(\prod_{i=1}^{m} X_i\). By hypothesis, we deduce that \(\{A \in \prod_{i=1}^{m} X_i : (\prod_{i=1}^{m} f_i)^l(A) = (x_1, \ldots, x_m), \text{ for some } l \in \mathbb{N}\} \cap \prod_{i=1}^{m} U_i \neq \emptyset\). Let \((u_1, \ldots, u_m) \in \prod_{i=1}^{m} U_i\) and let \(l \in \mathbb{N}\) such that \((\prod_{i=1}^{m} f_i)^l((u_1, \ldots, u_m)) = (x_1, \ldots, x_m)\). It follows that, \(x_{i_0} \in \{y \in X_{i_0} : f_{i_0}^l(y) = x_{i_0}, \text{ for some } l \in \mathbb{N}\} \cap \prod_{i=1}^{m} V_i \neq \emptyset\). Thus, the set \(\{y \in X_{i_0} : f_{i_0}^l(y) = x_{i_0}, \text{ for some } l \in \mathbb{N}\}\) is dense in \(X_{i_0}\). Since \(x_{i_0} \in X_{i_0}\) is arbitrary, we have that \(f_{i_0}\) is backward minimal.

Suppose that \(\prod_{i=1}^{m} f_i\) is Touhey. Let \(i_0 \in \{1, \ldots, m\}\) and let \(U_{i_0}, V_{i_0}\) be nonempty open subsets of \(X_{i_0}\). For each \(i \in \{1, \ldots, m\}\setminus\{i_0\}\), let \(U_i = X_i\) and \(V_i = X_i\). Then \(\prod_{i=1}^{m} U_i\) and \(\prod_{i=1}^{m} V_i\) are nonempty open subsets of \(\prod_{i=1}^{m} X_i\). By hypothesis, there exist a periodic point \((x_1, x_m) \in \prod_{i=1}^{m} U_i\) and \(k \in \mathbb{Z}_+\) such that \((\prod_{i=1}^{m} f_i)^k((x_1, x_m)) \in \prod_{i=1}^{m} V_i\). By Theorem 3.3, part (4), \(x_{i_0}\) is a periodic point of \(f_{i_0}\) such that \(x_{i_0} \in U_{i_0}\) and by Remark 3.1, part (1), \(f_{i_0}^k(x_{i_0}) \in V_{i_0}\). Therefore, \(f_{i_0}\) is Touhey.

Suppose that \(\prod_{i=1}^{m} f_i\) is an \(F\)-system. Thus, \(\prod_{i=1}^{m} f_i\) is totally transitive and \(\text{Per}(\prod_{i=1}^{m} f_i)\) is dense in \(\prod_{i=1}^{m} X_i\). By the third paragraph of this proof, we have that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is totally transitive. Moreover, by Theorem 3.15, for each \(i \in \{1, \ldots, m\}\), \(\text{Per}(f_i)\) is dense in \(X_i\). Therefore, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is an \(F\)-system.
Suppose that \( \prod_{i=1}^{m} f_i \) is scattering. Let \( i_0 \in \{1, \ldots, m\} \), let \( Y \) be a topological space and let \( g : Y \to Y \) be a minimal function. Let \( \mathcal{U} \) and \( \mathcal{V} \) be nonempty open subsets of \( X_{i_0} \times Y \). Then, there exist nonempty open subsets \( U_{i_0}^1, U_{i_0}^2 \) of \( X_{i_0} \) and nonempty open subsets \( V_1, V_2 \) of \( Y \) such that \( U_{i_0}^1 \times V_1 \subseteq \mathcal{U} \) and \( U_{i_0}^2 \times V_2 \subseteq \mathcal{V} \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \), let \( U_i^1 = U_i^2 = X_i \). Thus, \( \prod_{i=1}^{m} U_i^1 \) and \( \prod_{i=1}^{m} U_i^2 \) are nonempty open subsets of \( \prod_{i=1}^{m} X_i \). By hypothesis, there exist \( ((u_{i_1}, \ldots, u_{i_m}), v_1) \in (\prod_{i=1}^{m} U_i^1) \times V_1 \) and \( k \in \mathbb{N} \) such that \( ((\prod_{i=1}^{m} f_i) \times g)^k((u_{i_1}, \ldots, u_{i_m}), v_1) \in (\prod_{i=1}^{m} U_i^2) \times V_2 \). It follows that \( (u_{i_0}, v_1) \in U_{i_0}^1 \times V_1 \) and by Remark 3.1, part (1), \( (f_{i_0} \times g)^k((u_{i_0}, v_1)) \in U_{i_0}^2 \times V_2 \). Therefore, \( (f_{i_0} \times g)^k(\mathcal{U}) \cap V \neq \emptyset \) and hence \( f_{i_0} \) is scattering.

The proof for mild mixing is similar to that given for scattering.

The converse of Theorem 4.1 is not true in general. Let us see a partial example of this in the following:

**Example 4.2** Let \( f : [0, 2] \to [0, 2] \) be a function given by:

\[
 f(x) = \begin{cases} 
 2x + 1, & 0 \leq x \leq \frac{1}{2}, \\
 2x + 3, & \frac{1}{2} < x \leq 1, \\
 -x + 2, & 1 < x \leq 2.
\end{cases}
\]

In [8, Example 1], it is proved that \( f \) is a chaotic function. Moreover, it is proved that \( f \times f : [0, 2] \times [0, 2] \to [0, 2] \times [0, 2] \) is not transitive and, therefore, it is not chaotic. Furthermore, in [1, 15], it is proved that for continua and continuous functions, the notions: transitive, orbit-transitive, strictly orbit-transitive, \( \omega \)-transitive and \( \mathcal{TT} \) are equivalent. Therefore, the converse of Theorem 4.1, for all these classes of functions are not true in general.

**Theorem 4.3** Let \( X_1, \ldots, X_m \) be topological spaces and, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function. Then, for each \( i \in \{1, \ldots, m\} \), \( f_i \) is exact if and only if \( \prod_{i=1}^{m} f_i \) is exact.

**Proof** Suppose that \( \prod_{i=1}^{m} f_i \) is exact. Let \( i_0 \in \{1, \ldots, m\} \) and let \( U_{i_0} \) be a nonempty open subset of \( X_{i_0} \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \), let \( U_i = X_i \). Then \( \prod_{i=1}^{m} U_i \) is an open subset of \( \prod_{i=1}^{m} X_i \). By hypothesis, there exists \( k \in \mathbb{N} \) such that \( (\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) = \prod_{i=1}^{m} X_i \). By Remark 3.1, part (3), \( f_{i_0}^k(U_{i_0}) = X_{i_0} \). Thus, \( f_{i_0} \) is exact.

Now, suppose that, for each \( i \in \{1, \ldots, m\} \), \( f_i \) is exact. Let \( \mathcal{U} \) be a nonempty open subset of \( \prod_{i=1}^{m} X_i \). Then, for each \( i \in \{1, \ldots, m\} \), there exists a nonempty open subset \( U_i \) of \( X_i \) such that \( \prod_{i=1}^{m} U_i \subseteq \mathcal{U} \). By hypothesis, for each \( i \in \{1, \ldots, m\} \), there exists \( k_i \in \mathbb{N} \) such that \( f_i^{k_i}(U_i) = X_i \). On the other hand, by the diagram on Figure, we have that, for each \( i \in \{1, \ldots, m\} \), \( f_i \) is surjective. Then, for each \( i \in \{1, \ldots, m\} \) and for each \( l \in \mathbb{N} \), \( f_i^l(X_i) = X_i \). Let \( k = \max\{k_1, \ldots, k_m\} \). It follows that, for each \( i \in \{1, \ldots, m\} \), there exists \( l_i \in \mathbb{Z}^+ \) such that \( k = k_i + l_i \). Thus, for each \( i \in \{1, \ldots, m\} \), \( f_i^k(U_i) = f_i^{k_i + l_i}(U_i) = f_i^{l_i}(f_i^{k_i}(U_i)) = f_i^{l_i}(X_i) = X_i \). Consequently, by Remark 3.1, part (1), \( (\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) = \prod_{i=1}^{m} f_i^k(U_i) = \prod_{i=1}^{m} X_i \). Therefore, \( (\prod_{i=1}^{m} f_i)^k(\mathcal{U}) = \prod_{i=1}^{m} X_i \) and \( \prod_{i=1}^{m} f_i \) is exact.

**Theorem 4.4** Let \( X_1, \ldots, X_m \) be topological spaces and, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function. Then \( \prod_{i=1}^{m} f_i \) is mixing if and only if, for each \( i \in \{1, \ldots, m\} \), \( f_i \) is mixing.
Thus, by Theorem 5.3.3. Theorem 4.6.

**Proof** Suppose that $\prod_{i=1}^{m} f_i$ is mixing. Let $i_0 \in \{1, \ldots, m\}$ and let $U_{i_0}$, $V_{i_0}$ be two nonempty open subsets of $X_{i_0}$. For each $i \in \{1, \ldots, m\} \setminus \{i_0\}$, we put $U_i = X_i$ and $V_i = X_i$. It follows that $\prod_{i=1}^{m} U_i$ and $\prod_{i=1}^{m} V_i$ are nonempty open subsets of $\prod_{i=1}^{m} X_i$. Since $\prod_{i=1}^{m} f_i$ is mixing, there exists $N \in \mathbb{N}$ such that $(\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} V_i) \neq \emptyset$, for each $k \geq N$. Let $k \geq N$ and let $(a_1, \ldots, v_{i_0}, \ldots, a_m) \in ((\prod_{i=1}^{m} f_i)^k(\prod_{i=1}^{m} U_i) \cap (\prod_{i=1}^{m} V_i))$. Then there exists $(x_1, \ldots, u_{i_0}, \ldots, x_m) \in \prod_{i=1}^{m} U_i$ such that $(\prod_{i=1}^{m} f_i)^k((x_1, \ldots, u_{i_0}, \ldots, x_m)) = (a_1, \ldots, v_{i_0}, \ldots, a_m)$.

Thus, $f_{i_0}^k(u_{i_0}) = v_{i_0}$. Thereby, $v_{i_0} \in f_{i_0}^k(U_{i_0}) \cap V_{i_0}$. Consequently, $f_{i_0}^k(U_{i_0}) \cap V_{i_0} \neq \emptyset$, for each $k \geq N$. Therefore, $f_{i_0}$ is mixing.

Now, suppose that, for each $i \in \{1, \ldots, m\}$, $f_i$ is mixing. Let $U$ and $V$ be two nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exist nonempty open subsets $U_i$ and $V_i$ of $X_i$, such that $\prod_{i=1}^{m} U_i \subseteq U$ and $\prod_{i=1}^{m} V_i \subseteq V$. Since $f_i$ is mixing, for each $i \in \{1, \ldots, m\}$, there exists $N_i \in \mathbb{N}$ such that $f_i^k(U_i) \cap V_i \neq \emptyset$, for each $k \geq N_i$. Let $N = \max\{N_1, \ldots, N_m\}$ and let $l \geq N$. Thus, by hypothesis $f_i^l(U_i) \cap V_i \neq \emptyset$. For each $i \in \{1, \ldots, m\}$, let $a_i \in U_i$ be such that $f_i^l(a_i) \in V_i$. Then $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i$ and $(\prod_{i=1}^{m} f_i)^l(a_1, \ldots, a_m) \in \prod_{i=1}^{m} V_i$. Hence, $(\prod_{i=1}^{m} f_i)^l(a_1, \ldots, a_m) \in ((\prod_{i=1}^{m} f_i)^l(\prod_{i=1}^{m} U_i)) \cap (\prod_{i=1}^{m} V_i)$. Hence, for each $k \geq N$, $((\prod_{i=1}^{m} f_i)^l(\prod_{i=1}^{m} U_i)) \cap (\prod_{i=1}^{m} V_i) \neq \emptyset$. Therefore, $\prod_{i=1}^{m} f_i$ is mixing.

By Theorems 3.15 and 4.3, we have the following result.

**Proposition 4.5** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Then, for each $i \in \{1, \ldots, m\}$, $f_i$ is exactly Devaney chaotic if and only if $\prod_{i=1}^{m} f_i$ is exactly Devaney chaotic.

**Theorem 4.6** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function. If $\prod_{i=1}^{m} f_i$ is minimal, then, for each $i \in \{1, \ldots, m\}$, $f_i$ is minimal.

**Proof** Let $i_0 \in \{1, \ldots, m\}$. Since $f_{i_0}$ is continuous, it is enough to show that, for each $x \in X_{i_0}$, $\text{cl}_{X_{i_0}}(\mathcal{O}(x, f_{i_0})) = X_{i_0}$. Let $x \in X_{i_0}$, for each $i \in \{1, \ldots, m\} \setminus \{i_0\}$, let $x_i \in X_i$ and let $x_{i_0} = x$. Then, $(x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i$. Since, for each $i \in \{1, \ldots, m\}$, $f_i$ is continuous, we have that, $\prod_{i=1}^{m} f_i$ is a continuous and continuous function. Thus, we have that $\text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i$. Later, by Theorem 3.3, part (1), for each $i \in \{1, \ldots, m\}$, $\text{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i$. In particular, $\text{cl}_{X_{i_0}}(\mathcal{O}(x, f_{i_0})) = X_{i_0}$. Considering that $x \in X_{i_0}$ is arbitrary, by [15, Proposition 6.2], $f_{i_0}$ is minimal.

**Corollary 4.7** Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function. If $\prod_{i=1}^{m} f_i$ is totally minimal, then, for each $i \in \{1, \ldots, m\}$, $f_i$ is totally minimal.

**Proof** Let $s \in \mathbb{N}$. By hypothesis, $(\prod_{i=1}^{m} f_i)^s$ is minimal. Then, by Remark 3.1, part (1), $\prod_{i=1}^{m} f_i^s$ is minimal. Thus, by Theorem 4.6, for each $i \in \{1, \ldots, m\}$, $f_i^s$ is minimal.
Lemma 4.8 Let $X_1, \ldots, X_{m+1}$ be topological spaces and, for each $i \in \{1, \ldots, m+1\}$, let $f_i : X_i \to X_i$ be a function. If, for each $i \in \{1, \ldots, m\}$, $X_i$ is +invariant over open subsets under $f_i$ and $f_i \times f_{m+1}$ is transitive, then $(\prod_{i=1}^{m} f_i) \times f_{m+1}$ is transitive.

**Proof** Suppose that, for each $i \in \{1, \ldots, m\}$, $X_i$ is +invariant over open subsets under $f_i$ and $f_i \times f_{m+1}$ is transitive. Let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $(\prod_{i=1}^{m} X_i) \times X_{m+1}$. It follows that, there exist nonempty open subsets $\mathcal{U}_i$ and $\mathcal{V}_i$ of $\prod_{i=1}^{m} X_i$ and there exist nonempty open subsets $V_1$ and $V_2$ of $X_{m+1}$ such that, $\mathcal{U}_i \times V_1 \subseteq \mathcal{U}$ and $\mathcal{U}_i \times V_2 \subseteq \mathcal{V}$. Hence, for each $i \in \{1, \ldots, m\}$, there exist nonempty open subsets $U_i^1, U_i^2$ of $X_i$ such that $\prod_{i=1}^{m} U_i^1 \subseteq \mathcal{U}_i$ and $\prod_{i=1}^{m} U_i^2 \subseteq \mathcal{V}_i$. By hypothesis, there exists $k_i \in \mathbb{N}$ such that $(f_i \times f_{m+1})^{k_i}(U_i^1 \times V_1) \cap (U_i^2 \times V_2) \neq \emptyset$. Then, for each $i \in \{1, \ldots, m\}$, there exists $(u_i, v_i) \in U_i^1 \times V_1$ such that $(f_i \times f_{m+1})^{k_i}((u_i, v_i)) \in U_i^2 \times V_2$. Consequently, for each $i \in \{1, \ldots, m\}$, $f_i^{k_i}(u_i) \in U_i^2$. Let $k = \max\{k_1, \ldots, k_m\}$. Then, by Lemma 3.2, we have that, for all $i \in \{1, \ldots, m\}$, $f_i^{k}(u_i) \in U_i^2$. Let $i_0 \in \{1, \ldots, m\}$ be such that $k = k_{i_0}$, and let $v = v_{i_0}$. Thus, $f_i^{k}(v) \in V_2$. Hence, $((u_1, \ldots, u_m), v)) \in (\prod_{i=1}^{m} U_i^1) \times V_1$ and $((\prod_{i=1}^{m} f_i) \times f_{m+1})^{k}((u_1, \ldots, u_m), v)) \in (\prod_{i=1}^{m} U_i^2) \times V_2$. Consequently, $((\prod_{i=1}^{m} f_i) \times f_{m+1})^{k}((U_1 \times V_1) \cap (U_2 \times V_2) \neq \emptyset$. Therefore, $(\prod_{i=1}^{m} f_i) \times f_{m+1}$ is transitive. \qed

Remark 4.9 Let $X$ be a topological space and let $f : X \to X$ be a function. Observe that if $X$ is +invariant over open subsets under $f$, then $f$ cannot be strongly transitive unless $X$ has the trivial topology.

Theorem 4.10 Let $X_1, \ldots, X_m$ be topological spaces and, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function. Let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, $TT_+$, Touhey, scattering, an F-system or mild mixing. If, for each $i \in \{1, \ldots, m\}$, $f_i \in \mathcal{M}$ and $X_i$ is +invariant over open subsets under $f_i$, then $\prod_{i=1}^{m} f_i \in \mathcal{M}$.

**Proof** Suppose that, for each $i \in \{1, \ldots, m\}$, $f_i$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for each $i \in \{1, \ldots, m\}$, there exist nonempty open subsets $U_i$ and $V_i$ of $X_i$ such that $\prod_{i=1}^{m} U_i \subseteq \mathcal{U}$ and $\prod_{i=1}^{m} V_i \subseteq \mathcal{V}$. By hypothesis, for each $i \in \{1, \ldots, m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(U_i) \cap V_i \neq \emptyset$. For each $i \in \{1, \ldots, m\}$, let $u_i \in U_i$ be such that $f_i^{k_i}(u_i) \in V_i$ and let $k = \max\{k_1, \ldots, k_m\}$. By Lemma 3.2, we have that, for each $i \in \{1, \ldots, m\}$, $f_i^{k}(u_i) \in V_i$. Hence, $(u_1, \ldots, u_m) \in \prod_{i=1}^{m} U_i$ and $(f_1^{k}(u_1), \ldots, f_m^{k}(u_m)) \in \prod_{i=1}^{m} V_i$. Consequently, $(\prod_{i=1}^{m} f_i)^{k}((u_1, \ldots, u_m)) \in \prod_{i=1}^{m} V_i$. It follows that $(\prod_{i=1}^{m} f_i)^{k}(U_1 \cap V_1) \neq \emptyset$. Therefore, $(\prod_{i=1}^{m} f_i)^{k}(U) \cap V \neq \emptyset$ and $\prod_{i=1}^{m} f_i$ is transitive.

Suppose that, for each $i \in \{1, \ldots, m\}$, $f_i$ is weakly mixing. Let $U_1, U_2, V_1,$ and $V_2$ be four nonempty open subsets of $\prod_{i=1}^{m} X_i$. Then, for each $(a_1, \ldots, a_m) \in \prod_{i=1}^{m} U_i$, there exist nonempty open subsets $U_i^1, U_i^2, V_i^1$ and $V_i^2$ of $X_i$, such that $\prod_{i=1}^{m} U_i^1 \subseteq U_1$, $\prod_{i=1}^{m} U_i^2 \subseteq U_2$, $\prod_{i=1}^{m} V_i^1 \subseteq V_1$ and $\prod_{i=1}^{m} V_i^2 \subseteq V_2$. Since, $f_i$ is weakly mixing, for every $i \in \{1, \ldots, m\}$, there exists $k_i \in \mathbb{N}$ such that $f_i^{k_i}(U_i^j) \cap V_i^j \neq \emptyset$, for each $j \in \{1, 2\}$. For each $i \in \{1, \ldots, m\}$, let $a_i \in U_i^1$ be such that $f_i^{k_i}(a_i) \in V_i^1$ and let $a_i' \in U_i^2$ be such that $f_i^{k_i}(a_i') \in V_i^2$. Let $k = \max\{k_1, \ldots, k_m\}$. Hence, by Lemma 3.2, for each $i \in \{1, \ldots, m\}$, $f_i^{k}(a_i) \in V_i^1$ and $f_i^{k}(a_i') \in V_i^2$. It follows that $(\prod_{i=1}^{m} f_i)^{k}((a_1, \ldots, a_m)) \in \prod_{i=1}^{m} V_i^1$ and $(\prod_{i=1}^{m} f_i)^{k}((a_1', \ldots, a_m')) \in \prod_{i=1}^{m} V_i^2$. Consequently, $(\prod_{i=1}^{m} f_i)^{k}(U_i) \cap V_i \neq \emptyset$ and $(\prod_{i=1}^{m} f_i)^{k}(U_2) \cap V_2 \neq \emptyset$. Therefore, $\prod_{i=1}^{m} f_i$ is weakly mixing.
Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is totally transitive. Let \( s \in \mathbb{N} \) and let \( \mathcal{U}, \mathcal{V} \) be two nonempty open subsets of \( \prod_{i=1}^{m} X_i \). Then, for each \( i \in \{1,\ldots,m\} \), there exist nonempty open subsets \( U_i, V_i \) of \( X_i \) such that \( \prod_{i=1}^{m} U_i \subseteq \mathcal{U} \) and \( \prod_{i=1}^{m} V_i \subseteq \mathcal{V} \). Since, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is totally transitive, for each \( i \in \{1,\ldots,m\} \), there exists \( k_i \in \mathbb{N} \) such that \( (f_i^k)^{k_i}(U_i) \cap V_i \neq \emptyset \). Hence, for all \( i \in \{1,\ldots,m\} \), \( f_i^k(U_i) \cap V_i \neq \emptyset \).

For every \( i \in \{1,\ldots,m\} \), let \( u_i \in U_i \) be such that \( f_i^k(u_i) \in V_i \). Let \( k = \max\{k_1,\ldots,k_m\} \). By Lemma 3.2, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \). Let \( k = \max\{k_1,\ldots,k_m\} \). By Lemma 3.2, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \). By Lemma 3.2, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \). By Lemma 3.2, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \).

By Remark 3.1, part (1), we have that, \( \prod_{i=1}^{m} f_i^k((u_1,\ldots,u_m)) \in \prod_{i=1}^{m} U_i \) and \( \prod_{i=1}^{m} V_i \). Consequently:

\[
\prod_{i=1}^{m} f_i^k((u_1,\ldots,u_m)) \in \left( \prod_{i=1}^{m} U_i \right) \cap \bigcap_{i=1}^{m} V_i.
\]

Hence, \( \langle \prod_{i=1}^{m} f_i \rangle^* \) is transitive. Since \( s \in \mathbb{N} \) is arbitrary, we have that \( \prod_{i=1}^{m} f_i \) is totally transitive.

Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is a chaotic function. Then, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is transitive and \( \text{Per}(f_i) \) is dense in \( X_i \). By the first part of the proof of this theorem, we have that, \( \prod_{i=1}^{m} f_i \) is transitive and by Theorem 3.15, \( \text{Per}(\prod_{i=1}^{m} f_i) \) is dense in \( \prod_{i=1}^{m} X_i \). Therefore, \( \prod_{i=1}^{m} f_i \) is chaotic.

Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is orbit-transitive. Thus, for all \( i \in \{1,\ldots,m\} \), there exists \( x_i \in X_i \) such that \( \text{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i \). Then, by Theorem 3.7, part (2), \( \text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(\langle (x_1,\ldots,x_m) \rangle, \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i \). Thence, \( \prod_{i=1}^{m} f_i \) is orbit-transitive.

Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is strictly orbit-transitive. Then, for every \( i \in \{1,\ldots,m\} \), there exists \( x_i \in X_i \) such that \( \text{cl}_{X_i}(\mathcal{O}(f_i(x_i), f_i)) = X_i \). By Theorem 3.7, part (2):

\[
\text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(\langle (f_1(x_1),\ldots,f_n(x_m)) \rangle, \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i.
\]

Consequently \( \text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(\langle (\prod_{i=1}^{m} f_i)((x_1,\ldots,x_m)), \prod_{i=1}^{m} f_i \rangle) = \prod_{i=1}^{m} X_i \). Therefore, \( \prod_{i=1}^{m} f_i \) is strictly orbit-transitive.

Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is \( \omega \)-transitive. Then, for every \( i \in \{1,\ldots,m\} \), there exists \( x_i \in X_i \) such that \( \omega(x_i, f_i) = X_i \). By Theorem 3.7, part (1), \( \omega((x_1,\ldots,x_m), \prod_{i=1}^{m} f_i) = \prod_{i=1}^{m} X_i \). Therefore, \( \prod_{i=1}^{m} f_i \) is \( \omega \)-transitive.

Suppose that, for each \( i \in \{1,\ldots,m\} \), \( f_i \) is \( TT_{+++} \). Let \( \mathcal{U} \) and \( \mathcal{V} \) be two nonempty open subsets of \( \prod_{i=1}^{m} X_i \). Then, for every \( i \in \{1,\ldots,m\} \), there exist nonempty open subsets \( U_i, V_i \) of \( X_i \) such that \( \prod_{i=1}^{m} U_i \subseteq \mathcal{U} \) and \( \prod_{i=1}^{m} V_i \subseteq \mathcal{V} \). Since, for all \( i \in \{1,\ldots,m\} \), \( f_i \) is \( TT_{+++} \), we have that, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(U_i, V_i) \) is infinite. For every \( i \in \{1,\ldots,m\} \), let \( k_i \in n_{f_i}(U_i, V_i) \). Then, for each \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(U_i) \cap V_i \neq \emptyset \). It follows that, for all \( i \in \{1,\ldots,m\} \), there exists \( u_i \in U_i \) such that \( f_i^{k_i}(u_i) \in V_i \). Let \( k = \max\{k_1,\ldots,k_m\} \). By Lemma 3.2, for every \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \). Then, \( \prod_{i=1}^{m} f_i^{k_i}(U_i) \cap V_i \neq \emptyset \). Consequently, \( \prod_{i=1}^{m} f_i^{k_i}(U_i) \cap V_i \neq \emptyset \). Therefore, \( k = n_{f_i}(U_i, V_i) \). Now, since, for each \( i \in \{1,\ldots,m\} \), \( n_{f_i}(U_i, V_i) \) is infinite, for every \( i \in \{1,\ldots,m\} \), we can take \( k_i \in n_{f_i}(U_i, V_i) \) such that \( k_i > k \). Let \( k_1 = \max\{k'_1,\ldots,k'_m\} \). By Lemma 3.2, for every \( i \in \{1,\ldots,m\} \), \( f_i^{k_i}(u_i) \in V_i \). It follows that,
\((\prod_{i=1}^{m} f_i)^{k_1}(\prod_{i=1}^{m} U_i) \cap \prod_{i=1}^{m} V_i \neq \emptyset\). Consequently, \(\prod_{i=1}^{m} f_i^{k_1}(U) \cap V \neq \emptyset\). Therefore, \(k_1 \in n\prod_{i=1}^{m} f_i(U, V)\) and \(k_1 > k\). Continuing with this process, we have that \(n\prod_{i=1}^{m} f_i(U, V)\) is an infinite set. Since \(U\) and \(V\) are arbitrary, we have that the function \(\prod_{i=1}^{m} f_i\) is \(TT_++\).

Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is Touhey. Let \(U\) and \(V\) be two nonempty open subsets of \(\prod_{i=1}^{m} X_i\). Then, for every \(i \in \{1, \ldots, m\}\), there exist two nonempty open subsets \(U_i\) and \(V_i\) of \(X_i\) such that \(\prod_{i=1}^{m} U_i \subseteq U\) and \(\prod_{i=1}^{m} V_i \subseteq V\). Since, for all \(i \in \{1, \ldots, m\}\), \(f_i\) is Touhey, for each pair of nonempty open subsets \(U_i\) and \(V_i\), there exist a periodic point \(x_i \in U_i\) and \(k_i \in \mathbb{Z}_+\) such that \(f_i^{k_i}(x_i) \in V_i\). Let \(k = \max\{k_1, \ldots, k_m\}\). Then, by Lemma 3.2, we have that for each \(i \in \{1, \ldots, m\}\), \(f_i^{k}(x_i) \in V_i\). By Theorem 3.3, part (4), we obtain that \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} U_i \subseteq U\) and \((\prod_{i=1}^{m} f_i^k(x_1, \ldots, x_m)) \in \prod_{i=1}^{m} V_i \subseteq V\). Therefore, \(\prod_{i=1}^{m} f_i\) is Touhey.

Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is an F-system. Then, for every \(i \in \{1, \ldots, m\}\), \(f_i\) is totally transitive and \(\text{Per}(f_i)\) is dense in \(X_i\). By the third paragraph of the proof of this theorem, we have that \(\prod_{i=1}^{m} f_i\) is totally transitive. Moreover, by Theorem 3.15, we know that \(\text{Per}(\prod_{i=1}^{m} f_i)\) is dense in \(\prod_{i=1}^{m} X_i\). Therefore, \(\prod_{i=1}^{m} f_i\) is an F-system.

Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is mild mixing. Let \(Y\) be a topological space, let \(g : Y \to Y\) be a transitive function. By hypothesis, for each \(i \in \{1, \ldots, m\}\), \(f_i \times g\) is transitive. Since, for each \(i \in \{1, \ldots, m\}\), \(X_i\) is +invariant over open subsets under \(f_i\), by Lemma 4.8, \(\prod_{i=1}^{m} f_i \times g\) is transitive. Therefore, \(\prod_{i=1}^{m} f_i\) is mild mixing.

Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is scattering. Let \(Y\) be a topological space and let \(g : Y \to Y\) be a minimal function. By hypothesis, for each \(i \in \{1, \ldots, m\}\), \(f_i \times g\) is transitive. Since, for each \(i \in \{1, \ldots, m\}\), \(X_i\) is +invariant over open subsets under \(f_i\), by Lemma 4.8, \(\prod_{i=1}^{m} f_i \times g\) is transitive. Therefore, \(\prod_{i=1}^{m} f_i\) is scattering.

**Proposition 4.11** Let \(X_1, \ldots, X_m\) be topological spaces and, for each \(i \in \{1, \ldots, m\}\), let \(f_i : X_i \to X_i\) be a continuous function. If for every \(i \in \{1, \ldots, m\}\), \(f_i\) is minimal and \(X_i\) is +invariant over open subsets under \(f_i\), then \(\prod_{i=1}^{m} f_i\) is minimal.

**Proof** Suppose that, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is minimal and \(X_i\) is +invariant over open subsets under \(f_i\). By hypothesis, we have that \(\prod_{i=1}^{m} f_i\) is a continuous function. Thus, it is sufficient to show that, for all \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i\), \(\text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i\). Let \((x_1, \ldots, x_m) \in \prod_{i=1}^{m} X_i\). Since, for each \(i \in \{1, \ldots, m\}\), \(f_i\) is minimal, we have that \(\text{cl}_{X_i}(\mathcal{O}(x_i, f_i)) = X_i\). Since, for every \(i \in \{1, \ldots, m\}\), \(X_i\) is +invariant over open subsets under \(f_i\), by Theorem 3.7, part (2), we have that \(\text{cl}_{\prod_{i=1}^{m} X_i}(\mathcal{O}(x_1, \ldots, x_m), \prod_{i=1}^{m} f_i)) = \prod_{i=1}^{m} X_i\). Thus, since \(\prod_{i=1}^{m} f_i\) is continuous, we have that \(\prod_{i=1}^{m} f_i\) is minimal. \(\square\)

**Corollary 4.12** Let \(X_1, \ldots, X_m\) be topological spaces and for each \(i \in \{1, \ldots, m\}\), let \(f_i : X_i \to X_i\) be a continuous function. If for each \(i \in \{1, \ldots, m\}\), \(f_i\) is totally minimal and \(X_i\) is +invariant over open subsets under \(f_i\), then \(\prod_{i=1}^{m} f_i\) is totally minimal.

**Proof** Let \(s \in \mathbb{N}\). By hypothesis, for every \(i \in \{1, \ldots, m\}\), \(f_i^s\) is minimal and continuous. Thus, by
Proposition 4.11, $\prod_{i=1}^{m} f_i^s$ is minimal. Then, by Remark 3.1, part (1), $(\prod_{i=1}^{m} f_i)^s$ is minimal. Finally, since $s \in \mathbb{N}$ is arbitrary, we have that $\prod_{i=1}^{m} f_i$ is totally minimal.

5. Dynamic properties of $n$-fold symmetric product of a product space.

Let $X_1, \ldots, X_m$ be topological spaces. In this section we analyze some topological and dynamical properties of the hyperspace $\mathcal{F}_n(\prod_{i=1}^{m} X_i)$ and their relationships with the spaces $\mathcal{F}_n(X_i)$ and $X_i$, for each $i \in \{1, \ldots, m\}$.

Lemma 5.1 Let $X_1, \ldots, X_m$ be topological spaces, let $i_0 \in \{1, \ldots, m\}$, let $n \in \mathbb{N}$, let $\{a_1, \ldots, a_r\} \subseteq \mathcal{F}_n(X_{i_0})$ with $r \leq n$, and let $U_1, \ldots, U_n$ be nonempty open subsets of $X_{i_0}$ such that $\{a_1, \ldots, a_r\} \subseteq (U_1, \ldots, U_n)$. For each $i \in \{1, \ldots, m\}\setminus\{i_0\}$ and for every $l \in \{1, \ldots, r\}$, let $a_i^l \in X_i$ and let $a_{i_0}^l = a_i$. Then, for each $j \in \{1, \ldots, n\}$, $\mathcal{F}_n(\prod_{i=1}^{m} V_i^j) = \prod_{i=1}^{m} V_i^j$.

Proof It is not difficult to see that (1) is satisfied. We show that (2) is true. Let $p \in \{1, \ldots, r\}$. Since $\{a_1, \ldots, a_r\} \subseteq (U_1, \ldots, U_n)$, there exists $j_0 \in \{1, \ldots, n\}$ such that $a_p = a_{j_0}^p \in U_{j_0}$. Thus, $b_p = (a_1^p, \ldots, a_{j_0}^p, \ldots, a_m^p) \in \prod_{i=1}^{m} V_i^{j_0} = \prod_{i=1}^{m} V_i^j$. Therefore, $b_p \subseteq \bigcup_{j=1}^{n} U_j$. Consequently, $\{b_1, \ldots, b_r\} \subseteq \bigcup_{j=1}^{n} U_j$. Now, we will prove that, for each $j \in \{1, \ldots, n\}$, $\{b_1, \ldots, b_r\} \cap U_j \neq \emptyset$. Let $k \in \{1, \ldots, n\}$. Then, $U_k = \prod_{i=1}^{m} V_i^k$. Since $\{a_1, \ldots, a_r\} \cap U_k \neq \emptyset$, there exists $i_0 \in \{1, \ldots, r\}$ such that $a_{i_0} \in U_k$. Hence, $(a_1^k, \ldots, a_{i_0}^k, \ldots, a_m^k) \in U_k$. Consequently, for each $j \in \{1, \ldots, n\}$, $\{b_1, \ldots, b_r\} \cap U_j \neq \emptyset$. Therefore, $\{b_1, \ldots, b_r\} \subseteq \prod_{i=1}^{m} U_i^j$.

Lemma 5.2 Let $X_1, \ldots, X_m$ be topological spaces, let $l, n \in \mathbb{N}$ be such that $l \leq n$, for each $i \in \{1, \ldots, m\}$, let $U_i^1, \ldots, U_i^l$ be nonempty open subsets of $X_i$, and for every $j \in \{1, \ldots, l\}$, let $(x_i^1, \ldots, x_i^j) \in \prod_{i=1}^{m} X_i$. If $\{(x_i^1, \ldots, x_i^j) : j \in \{1, \ldots, l\}\} \subseteq \prod_{i=1}^{m} U_i^j$, then, for each $i \in \{1, \ldots, m\}$, $\{x_i^1, \ldots, x_i^j\} \subseteq \prod_{i=1}^{m} U_i^j$.

Proof Let $i_0 \in \{1, \ldots, m\}$. We will show that $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \subseteq \prod_{i=1}^{m} U_i^j$. First we will prove that $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \subseteq \bigcup_{i=1}^{n} U_i^j$. Let $k \in \{1, \ldots, l\}$. By hypothesis, there exists $s \in \{1, \ldots, n\}$ such that $(x_i^1, \ldots, x_i^j) \subseteq \bigcup_{i=1}^{n} U_i^s$. Then $x_{i_0}^k \in U_i^s$. Thus, $x_{i_0}^k \subseteq \bigcup_{i=1}^{n} U_i^j$. Therefore, $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \subseteq \bigcup_{i=1}^{n} U_i^j$.

Now we will see that, for each $j \in \{1, \ldots, n\}$, $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \cap U_j^j \neq \emptyset$. Let $l \in \{1, \ldots, n\}$. By hypothesis, $\{(x_i^1, \ldots, x_i^j) : j \in \{1, \ldots, l\}\} \subseteq \prod_{i=1}^{m} U_i^j$. Then, $x_{i_0}^l \subseteq U_{i_0}^l$. Hence, $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \cap U_j^j \neq \emptyset$. Because $l \in \{1, \ldots, n\}$ is arbitrary, we have that, for every $p \in \{1, \ldots, n\}$, $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \cap U_j^j \neq \emptyset$. Therefore, $\{x_{i_0}^1, \ldots, x_{i_0}^j\} \subseteq \prod_{i=1}^{m} U_i^j$. Finally, since $i_0 \in \{1, \ldots, m\}$ is arbitrary, we have that, for all $i \in \{1, \ldots, m\}$, $\{x_i^1, \ldots, x_i^j\} \subseteq \prod_{i=1}^{m} U_i^j$. □
Lemma 5.3 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$ and let $U_1^i, \ldots, U_n^i, V_1^i, \ldots, V_n^i$ be nonempty open subsets of $X_i$. Then, for each $i \in \{1, \ldots, m\}$:

$$n_{\mathcal{F}_n}(\prod_{i=1}^m f_i) \left( \left( \prod_{i=1}^m U_1^i, \ldots, \prod_{i=1}^m U_n^i \right), \left( \prod_{i=1}^m V_1^i, \ldots, \prod_{i=1}^m V_n^i \right) \right) \subseteq n_{\mathcal{F}_n(f_i)}((U_1^i, \ldots, U_n^i), (V_1^i, \ldots, V_n^i)).$$

Proof Let $k \in n_{\mathcal{F}_n}(\prod_{i=1}^m f_i)((\prod_{i=1}^m U_1^i), (\prod_{i=1}^m V_1^i))$. Then

$$\left( \mathcal{F}_n \left( \prod_{i=1}^m f_i \right) \right)^k \left( \left( \prod_{i=1}^m U_1^i, \ldots, \prod_{i=1}^m U_n^i \right), \left( \prod_{i=1}^m V_1^i, \ldots, \prod_{i=1}^m V_n^i \right) \right) \neq \emptyset.$$

Now, let $l \leq n$ and let \{{x_1^j, \ldots, x_n^j} : j \in \{1, \ldots, l\}\} \subseteq (\prod_{i=1}^m U_1^i, \ldots, \prod_{i=1}^m U_n^i)$, such that

$$\left( \mathcal{F}_n \left( \prod_{i=1}^m f_i \right) \right)^k \left( \{x_1^j, \ldots, x_n^j} : j \in \{1, \ldots, l\}\right) \subseteq \left( \prod_{i=1}^m V_1^i, \ldots, \prod_{i=1}^m V_n^i \right).$$

By Remark 3.1, parts (1) and (2), we have that \{(f_1^k(x_1^1), \ldots, f_m^k(x_1^m)) : j \in \{1, \ldots, l\}\} \subseteq (\prod_{i=1}^m V_1^i, \ldots, \prod_{i=1}^m V_n^i).

Thus, by Lemma 5.2, for every $i \in \{1, \ldots, m\}$, \{x_1^1, \ldots, x_1^m\} \subseteq (U_1^i, \ldots, U_n^i) and \{(f_1^k(x_1^1), \ldots, f_1^k(x_1^m))\} \subseteq (V_1^i, \ldots, V_n^i).$ Hence, for all $i \in \{1, \ldots, m\}$, $(\mathcal{F}_n(f_i))^k((x_1^1, \ldots, x_1^m)) \subseteq (\mathcal{F}_n(f_i))^k((U_1^i, \ldots, U_n^i)) \cap (V_1^i, \ldots, V_n^i)$. Therefore, for every $i \in \{1, \ldots, m\}$, $k \in n_{\mathcal{F}_n(f_i)}((U_1^i, \ldots, U_n^i), (V_1^i, \ldots, V_n^i))$. 

By Corollary 3.8 and by [4, Theorem 3.14], we have the following result.

Proposition 5.4 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following hold:

1. For each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(X_i)$ is perfect if and only if $\prod_{i=1}^n X_i$ is perfect.
2. For each $i \in \{1, \ldots, m\}$, $X_i$ is perfect if and only if $\mathcal{F}_n(\prod_{i=1}^n X_i)$ is perfect.
3. For each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(X_i)$ is perfect if and only if $\mathcal{F}_n(\prod_{i=1}^n X_i)$ is perfect.

By Theorem 3.9 and [4, Theorem 3.8], we have the following result.

Proposition 5.5 Let $X_1, \ldots, X_m$ be topological spaces and let $n \in \mathbb{N}$. Then the following hold:

1. For each $i \in \{1, \ldots, m\}$, $X_i$ is pseudoregular if and only if $\mathcal{F}_n(\prod_{i=1}^n X_i)$ is pseudoregular.
2. For every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(X_i)$ is pseudoregular if and only if $\prod_{i=1}^n X_i$ is pseudoregular.

Theorem 5.6 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, and let $l, n \in \mathbb{N}$ be such that $l \leq n$. If $\mathcal{A} = \{(x_1^j, \ldots, x_n^j) : j \in \{1, \ldots, l\}\} \subseteq \mathcal{F}_n(\prod_{i=1}^n X_i)$ is a transitive point of $\mathcal{F}_n(\prod_{i=1}^n f_i)$, then, for every $i \in \{1, \ldots, m\}$, \{x_1^i, \ldots, x_n^i\} is a transitive point of $\mathcal{F}_n(f_i)$. 

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Proof Suppose that \( A \) is a transitive point of \( \mathcal{F}_n(\prod_{i=1}^m f_i) \). Let \( i_0 \in \{1, \ldots, m\} \) and let \( U \) be a nonempty open subset of \( \mathcal{F}_n(X_{i_0}) \). Hence, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n \) of \( X_{i_0} \) such that \( \langle U_1, \ldots, U_n \rangle \subseteq U \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( j \in \{1, \ldots, n\} \), let \( V_j = X_i \) and \( V_{i_0} = U_j \). Then, for all \( j \in \{1, \ldots, n\} \), let \( U_j' = \prod_{i=1}^m V_i' \). Thus, \( \langle U_1', \ldots, U_n' \rangle \) is a nonempty open subset of \( \mathcal{F}_n(\prod_{i=1}^m X_i) \). By hypothesis, \( \langle U_1', \ldots, U_n' \rangle \cap O(A, \mathcal{F}_n(\prod_{i=1}^m f_i)) \neq \emptyset \). In consequence, there exists \( k \in \mathbb{N} \) such that \( [\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(A) \in \langle U_1', \ldots, U_n' \rangle \). Then \( \{f_1^k(x_1'), \ldots, f_n^k(x_n') \} : j \in \{1, \ldots, l\} \} \in \langle U_1', \ldots, U_n' \rangle \). By Lemma 5.2, we have that \( \{f_1^k(x_1'), \ldots, f_n^k(x_n') \} \in \langle U_1', \ldots, U_n' \rangle \). Hence, \( [\mathcal{F}_n(f_{i_0})]^k\{x_1, \ldots, x_n\} \} \in \langle U_1', \ldots, U_n' \rangle \). Therefore, \( x_{i_0}^1 \ldots, x_{i_0}^m \} \) is a transitive point of \( \mathcal{F}_n(f_{i_0}) \). Because \( i_0 \in \{1, \ldots, m\} \) is arbitrary, we have that, for each \( i \in \{1, \ldots, m\} \), \( x_i^1 \ldots, x_i^m \} \) is a transitive point of \( \mathcal{F}_n(f_i) \). □

Theorem 5.7 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \rightarrow X_i \) be a function, let \( l, n \in \mathbb{N} \) be such that \( l \leq n \), and let \( A = \{x_1^1, \ldots, x_m^1 \} \} \in \mathcal{F}_n(\prod_{i=1}^m X_i) \). If \( \omega(A, \mathcal{F}_n(\prod_{i=1}^m f_i)) = \mathcal{F}_n(\prod_{i=1}^m X_i) \), then, for each \( i \in \{1, \ldots, m\} \), \( \omega(x_1^1, \ldots, x_i^m \}, \mathcal{F}_n(f_i) = \mathcal{F}_n(X_i) \).

Proof Suppose that \( \omega(A, \mathcal{F}_n(\prod_{i=1}^m f_i)) = \mathcal{F}_n(\prod_{i=1}^m X_i) \). Let \( i_0 \in \{1, \ldots, m\} \). Now we show that \( \omega(\{x_1^{i_0}, \ldots, x_m^{i_0} \} \}, \mathcal{F}_n(f_{i_0}) \} = \mathcal{F}_n(X_{i_0}) \). Let \( \{a_1, \ldots, a_r\} \} \in \mathcal{F}_n(X_{i_0}) \) with \( r \leq n \), let \( U \) be an open subset of \( \mathcal{F}_n(X_{i_0}) \) such that \( \{a_1, \ldots, a_r\} \} \in U \) and let \( k \in \mathbb{N} \). By [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n \) of \( X_{i_0} \) such that \( \{a_1, \ldots, a_r\} \} \in \langle U_1, \ldots, U_n \rangle \}. \) For each \( l \in \{1, \ldots, r\} \) and for every \( i \in \{1, \ldots, l\} \}, let \( a_i^l = X_i \) and let \( a_i^{i_0} = a_i \). Then, for all \( l \in \{1, \ldots, r\} \}, \) let \( a_i^l = (a_1^l, \ldots, a_m^l) \}. \) On the other hand, for each \( i \in \{1, \ldots, m\} \} \{i_0\} \) and for every \( j \in \{1, \ldots, n\} \), let \( V_i^j = X_i \) and \( V_{i_0}^j = U_j \}. \) Finally, for all \( j \in \{1, \ldots, n\} \), let \( U_j' = \prod_{i=1}^m V_i' \). By Lemma 5.1, part (1), \( \{a_1^1, \ldots, a_r^l \} \} \in \mathcal{F}_n(\prod_{i=1}^m X_i) \). Hence, by hypothesis, \( \{a_1^1, \ldots, a_r^l \} \} \in \omega(A, \mathcal{F}_n(\prod_{i=1}^m f_i)) \). By Lemma 5.1, part (2), \( \{a_1^1, \ldots, a_r^l \} \} \in \langle U_1', \ldots, U_n' \rangle \}. \) By Remark 3.1, parts (1) and (2), we have that \( \{f_1^k(x_1^p), \ldots, f_n^k(x_n^p) \} : p \in \{1, \ldots, l\} \} \in \langle U_1', \ldots, U_n' \rangle \}. \) By Lemma 5.2, \( \{f_1^k(x_1^p), \ldots, f_n^k(x_n^p) \} \} \in \langle U_1, \ldots, U_n \rangle \}. \) Thus, \( \mathcal{F}_n(f_{i_0})^k\{x_1^1, \ldots, x_i^m \} \} \in \langle U_1, \ldots, U_n \rangle \}. \) Then \( \{a_1, \ldots, a_r \} \} \in \omega(\{x_1^1, \ldots, x_i^m \} \}, \mathcal{F}_n(f_{i_0}) \}. \) Thus, \( \omega(\{x_1^1, \ldots, x_i^m \} \}, \mathcal{F}_n(f_{i_0}) = \mathcal{F}_n(X_{i_0}) \). □

By Theorem 3.15 and [4, Theorem 3.4], we have the following result.

Theorem 5.8 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \rightarrow X_i \) be a function, and let \( n \in \mathbb{N} \). Then the following hold:

1. For every \( i \in \{1, \ldots, m\} \), \( \text{Per}(f_i) \) is dense in \( X_i \) if and only if \( \text{Per}(\mathcal{F}_n(\prod_{i=1}^m f_i)) \) is dense in \( \mathcal{F}_n(\prod_{i=1}^m X_i) \).

2. For each \( i \in \{1, \ldots, m\} \), \( \text{Per}(\mathcal{F}_n(f_i)) \) is dense in \( \mathcal{F}_n(X_i) \) if and only if \( \text{Per}(\prod_{i=1}^m f_i) \) is dense in \( \prod_{i=1}^m X_i \).

By Proposition 3.10 and [4, Theorem 3.3], we have the following result.

Proposition 5.9 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \rightarrow X_i \) be a function, and let \( n \in \mathbb{N} \). Then the following hold:
1. For every \( i \in \{1, \ldots, m\} \), \( U_i \) is +invariant under \( f_i \) if and only if \( \prod_{i=1}^{m} U_i \) is +invariant under \( F_n(\prod_{i=1}^{m} f_i) \).

2. For each \( i \in \{1, \ldots, m\} \), \( \langle U_i \rangle \) is +invariant under \( F_n(f_i) \) if and only if \( \prod_{i=1}^{m} U_i \) is +invariant under \( \prod_{i=1}^{m} f_i \).

6. Induced functions to \( n \)-fold symmetric products of product spaces

Let \( X_1, \ldots, X_m \) be topological spaces and for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function. In this section we analyze the relationships between the functions \( F_n(\prod_{i=1}^{m} f_i) \), \( F_n(f_i) \) and \( f_i \), for every \( i \in \{1, \ldots, m\} \), when any of this is exact, mixing, transitive, weakly mixing, totally transitive, strongly transitive, chaotic, minimal, totally minimal, orbit-transitive, strictly orbit-transitive, \( \omega \)-transitive, \( TT_++ \), mild mixing, exactly Devaney chaotic, backward minimal, scattering, Touhey or an \( F \)-system.

**Theorem 6.1** Let \( X, Y \) be topological spaces, let \( f : X \to X \), \( g : Y \to Y \) be functions and let \( n \in \mathbb{N} \). If \( F_n(f) \times g \) is transitive, then \( f \times g \) is transitive.

**Proof** Suppose that \( F_n(f) \times g \) is transitive. Let \( U, V \) be two nonempty open subsets of \( X \times Y \). Then there exist nonempty open subsets \( U_1, U_2 \) of \( X \) and \( V_1, V_2 \) of \( Y \) such that \( U_1 \times V_1 \subseteq U \) and \( U_2 \times V_2 \subseteq V \). Thus, \( \langle U_1 \rangle \) and \( \langle U_2 \rangle \) are nonempty open subsets of \( F_n(X) \). By hypothesis, there exists \( k \in \mathbb{N} \) such that \( (F_n(f) \times g)^k(\langle U_1 \rangle \times V_2) \cap (\langle U_2 \rangle \times V_2) \neq \emptyset \). It follows that there exists \( \{x_1, \ldots, x_r\} \subseteq \langle U_1 \rangle \) such that \( [F_n(f) \times g]^k((x_1, \ldots, x_r), v_1) \in \langle U_2 \rangle \times V_2 \). Let \( x \in \{x_1, \ldots, x_r\} \). We have that, \( x \in U_1 \) and \( f^k(x) \in U_2 \). Consequently, for each \( x \in \{x_1, \ldots, x_r\} \), \( (x, v_1) \in U_1 \times V_1 \) and \( (f \times g)^k((x, v_1)) \in U_2 \times V_2 \). Thus, \( (f \times g)^k(U) \cap V \neq \emptyset \) and \( f \times g \) is transitive.

The proof of Proposition 6.2 is followed by [4, Theorems 3.4 and 4.10].

**Proposition 6.2** Let \( X \) be a topological space, let \( f : X \to X \) be a function, and let \( n \in \mathbb{N} \). Then \( f \) is exactly Devaney chaotic if and only if \( F_n(f) \) is exactly Devaney chaotic.

**Theorem 6.3** Let \( X \) be a topological space, let \( f : X \to X \) be a function and let \( n \in \mathbb{N} \). Let \( M \) be one of the following classes of functions: Touhey, an \( F \)-system, backward minimal, totally minimal, mild mixing or scattering. If \( F_n(f) \in M \), then \( f \in M \).

**Proof** Suppose that \( F_n(f) \) is Touhey. Let \( U, V \) be nonempty open subsets of \( X \). Hence, \( \langle U \rangle \) and \( \langle V \rangle \) are nonempty open subsets of \( F_n(X) \). Since \( F_n(f) \) is Touhey, there exist a periodic point \( \{x_1, \ldots, x_r\} \in \langle U \rangle \) and \( k \in \mathbb{Z}_+ \) such that \( [F_n(f)]^k(\{x_1, \ldots, x_r\}) \in \langle V \rangle \). Then, by [4, Theorem 3.4], for each \( i \in \{1, \ldots, r\} \), \( x_i \) is a periodic point of \( f \). Furthermore, for every \( i \in \{1, \ldots, r\} \), \( x_i \in U \) and \( f^k(x_i) \in V \). Therefore, \( f \) is Touhey.

Suppose that \( F_n(f) \) is an \( F \)-system. Then \( F_n(f) \) is totally transitive and \( \text{Per}(F_n(f)) \) is dense in \( F_n(X) \). Thus, by [4, Theorem 4.14], \( f \) is totally transitive and, by [4, Theorem 3.4], \( \text{Per}(f) \) is dense in \( X \). Therefore, \( f \) is an \( F \)-system.

Suppose that \( F_n(f) \) is backward minimal. Let \( x \in X \) and let \( U \) be a nonempty open subset of \( X \). Then \( \langle U \rangle \) is a nonempty open subset of \( F_n(X) \). Since \( F_n(f) \) is backward minimal, the set \( \{A \in F_n(X) : (F_n(f))^l(A) = \{x\}, \text{ for some } l \in \mathbb{N}\} \) is dense in \( F_n(X) \). Thus, there exist \( \{x_1, \ldots, x_r\} \in \langle U \rangle \) and
l ∈ N such that \([F_n(f)]^l([x_1, \ldots, x_r]) = \{x\}\). It follows that, for each \(i \in \{1, \ldots, r\}\), \(x_i \in U\) and \(f^l(x_i) = x\). Thus, \(\{y \in X : f^l(y) = x\} \neq \emptyset\). Therefore, the set \(\{y \in X : f^l(y) = x\}\) is dense in \(X\). Because \(x \in X\) is arbitrary, we have that \(f\) is backward minimal.

Suppose that \(F_n(f)\) is totally minimal. Let \(s \in \mathbb{N}\). By hypothesis, \((F_n(f))^s\) is minimal. Then, by Remark 3.1, part (1), \(F_n(f^s)\) is minimal. Hence, by [4, Theorem 4.18], \(f^s\) is minimal.

Suppose that \(F_n(f)\) is mild mixing. Let \(Y\) be a topological space and let \(g : Y \to Y\) be a continuous function. By hypothesis, \(F_n(f) \times g\) is transitive. Thus, by Theorem 6.1, \(f \times g\) is transitive. Therefore, \(f\) is mild mixing.

Suppose that \(F_n(f)\) is scattering. Let \(Y\) be a topological space, let \(g : Y \to Y\) be a minimal function. By hypothesis, \(F_n(f) \times g\) is transitive. By Theorem 6.1, \(f \times g\) is transitive. Therefore, \(f\) is scattering.

The converse of Theorem 6.3 is not true in general. Let us see a partial example of this in the following:

**Example 6.4** Let \(X = [0, 1]\) and let \(f : X \to X\) be a function given by:

\[
f(x) = \begin{cases} 
2x + \frac{1}{2}, & \text{if } x \in [0, \frac{1}{2}); \\
\frac{3}{2} - 2x, & \text{if } x \in \left[\frac{1}{2}, 1\right]; \\
1 - x, & \text{if } x \in \left[\frac{1}{2}, 1\right].
\end{cases}
\]

In [10, Example 4.10], it is shown that \(f\) is a chaotic function; however, the function \(F_n(f)\) is not chaotic. On the other hand, observe that \(f\) is a continuous function. Thus, by [18, Proposition 2.6], \(f\) is Touhey. If we suppose that \(F_n(f)\) is Touhey, again, by [18, Proposition 2.6], \(F_n(f)\) is a chaotic function, which is a contradiction. Therefore, \(F_n(f)\) is not Touhey.

**Theorem 6.5** Let \(X, Y\) be topological spaces, let \(f : X \to X\), \(g : Y \to Y\) be functions and let \(n \in \mathbb{N}\). If \(X\) is \(+\)-invariant over open subsets under \(f\) and \(f \times g\) is transitive, then \(F_n(f) \times g\) is transitive.

**Proof** Suppose that \(X\) is \(+\)-invariant over open subsets under \(f\) and \(f \times g\) is transitive. Let \(U\) and \(V\) be two nonempty open subsets of \(F_n(X) \times Y\). Then there exist nonempty open subsets \(U_1, U_2\) of \(F_n(X)\) and \(V_1, V_2\) of \(Y\) such that \(U_1 \times V_1 \subseteq U\) and \(U_2 \times V_2 \subseteq V\). By [10, Lemma 4.2], there exist nonempty open subsets \(U_1^1, \ldots, U_1^n\), \(U_2^1, \ldots, U_2^n\) of \(X\) such that \((U_1^1, \ldots, U_1^n) \subseteq U_1\) and \((U_2^1, \ldots, U_2^n) \subseteq U_2\). Since \(f \times g\) is transitive, for each \(i \in \{1, \ldots, n\}\), there exists \(k_i \in \mathbb{N}\) such that \((f \times g)^{k_i}(U_1^i \times V_1) \cap (U_2^i \times V_2) \neq \emptyset\). Hence, for every \(i \in \{1, \ldots, n\}\), there exists \((u_i, v_i) \in U_1^i \times V_1\) such that \((f \times g)^{k_i}(u_i, v_i) \in U_2^i \times V_2\). It follows that, for all \(i \in \{1, \ldots, n\}\), \(f^{k_i}(u_i) \in U_1^i\). Let \(k = \max\{k_1, \ldots, k_n\}\). By Lemma 3.2, for each \(i \in \{1, \ldots, n\}\), \(f^k(u_i) \in U_1^i\).

Consequently, \(\{f^k(u_1), \ldots, f^k(u_n)\} \in \langle U_1^1, \ldots, U_1^n \rangle\) which means that \(\langle F_n(f)^k(\{u_1, \ldots, u_n\}) \rangle \subseteq \langle U_1^1, \ldots, U_1^n \rangle\).

Moreover, \(\{u_1, \ldots, u_n\} \in \langle U_1^1, \ldots, U_1^n \rangle\). Suppose that \(k = k_{i_0}\), where \(i_0 \in \{1, \ldots, n\}\), and let \(v = v_{i_0}\). Then \(g^k(v) \in V_2\) and \(v \in V_1\). Finally, \(\langle F_n(f) \times g \rangle^k(\{u_1, \ldots, u_n, v\}) \in \langle U_2^1, \ldots, U_2^n \rangle \times V_2\) and \(\{\{u_1, \ldots, u_n\}, v\} \in \langle U_1^1, \ldots, U_1^n \rangle \times V_2\). Therefore, \(\langle F_n(f) \times g \rangle^k(U) \cap V \neq \emptyset\) and \(F_n(f) \times g\) is transitive.

**Theorem 6.6** Let \(X\) be a topological space, let \(f : X \to X\) be a function, and let \(n \in \mathbb{N}\). Let \(M\) be one of the following classes of function: transitive, totally transitive, chaotic, Touhey, an F-system, mild mixing or scattering. Then, if \(f \in M\) and \(X\) is \(+\)-invariant over open subsets under \(f\), then \(F_n(f) \in M\).
Proof Suppose that $f$ is transitive. Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Thence, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ of $X$ such that $(U_1, \ldots, U_n) \subseteq \mathcal{U}$ and $(V_1, \ldots, V_n) \subseteq \mathcal{V}$. Since $f$ is transitive, for each $i \in \{1, \ldots, n\}$, there exists $k_i \in \mathbb{N}$ such that $f^{k_i}(U_i) \cap V_i \neq \emptyset$. Then, for every $i \in \{1, \ldots, n\}$, there exists $u_i \in U_i$ such that $f^{k_i}(u_i) \in V_i$. Let $k = \max\{k_1, \ldots, k_n\}$.

By Lemma 3.2, for all $i \in \{1, \ldots, n\}$, $f^k(u_i) \in V_i$. It follows that, $(u_1, \ldots, u_n) \in \langle U_1, \ldots, U_n \rangle$ and $(\mathcal{F}_n(f))^k(\{u_1, \ldots, u_n\}) \in \langle V_1, \ldots, V_n \rangle$. Therefore, $(\mathcal{F}_n(f))^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$ and $\mathcal{F}_n(f)$ is transitive.

Suppose that $f$ is totally transitive. Let $s \in \mathbb{N}$ and let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Then by [10, Lemma 4.2], we have that, there exist nonempty open subsets $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ of $X$ such that, $(U_1, \ldots, U_n) \subseteq \mathcal{U}$ and $(V_1, \ldots, V_n) \subseteq \mathcal{V}$. Since $f^s$ is transitive, for each $i \in \{1, \ldots, n\}$, there exists $k_i \in \mathbb{N}$ such that $(f^s)^{k_i}(U_i) \cap V_i \neq \emptyset$. For every $i \in \{1, \ldots, n\}$, let $u_i \in U_i$ such that $(f^s)^{k_i}(u_i) \in V_i$. Let $k = \max\{k_1, \ldots, k_n\}$. Thus, by Lemma 3.2, for all $i \in \{1, \ldots, n\}$, $(f^s)^k(u_i) \in V_i$. Thus, $(u_1, \ldots, u_n) \in \langle U_1, \ldots, U_n \rangle$ and $(\{f^s\}^k(u_1), \ldots, (f^s)^k(u_n)) \in \langle V_1, \ldots, V_n \rangle$. So, $(\mathcal{F}_n(f))^s(\{u_1, \ldots, u_n\}) \in \langle V_1, \ldots, V_n \rangle$. It follows that, $(\mathcal{F}_n(f))^s(\mathcal{U}) \cap \mathcal{V} \neq \emptyset$. Consequently, $(\mathcal{F}_n(f))^s$ is transitive. Finally, because $s$ is arbitrary, we have that $\mathcal{F}_n(f)$ is totally transitive.

Suppose that $f$ is chaotic. Then $f$ is transitive and $\text{Per}(f)$ is dense in $X$. Thus, by [4, Theorem 3.4], we have that $\text{Per}(\mathcal{F}_n(f))$ is dense in $\mathcal{F}_n(X)$. Moreover, by the first part of this proof, if $f$ is transitive then $\mathcal{F}_n(f)$ is transitive. Therefore, $\mathcal{F}_n(f)$ is chaotic.

Suppose that $f$ is Touhey. Let $\mathcal{U}$, $\mathcal{V}$ be two nonempty open subsets of $\mathcal{F}_n(X)$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ of $X$ such that $(U_1, \ldots, U_n) \subseteq \mathcal{U}$ and $(V_1, \ldots, V_n) \subseteq \mathcal{V}$. Since $f$ is Touhey, for every $i \in \{1, \ldots, n\}$, there exist a periodic point $x_i \in U_i$ and $k_i \in \mathbb{Z}_+$ such that $f^{k_i}(x_i) \in V_i$. Let $k = \max\{k_1, \ldots, k_n\}$. Then, by Lemma 3.2, for each $i \in \{1, \ldots, n\}$, $f^k(x_i) \in V_i$. Consequently, $(\mathcal{F}_n(f))^k(\{x_1, \ldots, x_n\}) \in \langle V_1, \ldots, V_n \rangle$. Furthermore, $(x_1, x_2, \ldots, x_n) \in \langle U_1, \ldots, U_n \rangle$.

On the other hand, since, for all $i \in \{1, \ldots, n\}$, $x_i$ is a periodic point of $f_i$, by [4, Theorem 3.4], $(x_1, \ldots, x_n)$ is a periodic point of $\mathcal{F}_n(f)$. Therefore, $\mathcal{F}_n(f)$ is Touhey.

Suppose that $f$ is an F-system. Then $f$ is totally transitive and $\text{Per}(f)$ is dense in $X$. Thus, by the second part of this proof, we have that $\mathcal{F}_n(f)$ is totally transitive. Moreover, by [4, Theorem 3.4], $\text{Per}(\mathcal{F}_n(f))$ is dense. Therefore, $\mathcal{F}_n(f)$ is an F-system.

Suppose that $f$ is mild mixing. Let $Y$ be a topological space and let $g : Y \to Y$ be a transitive function. By hypothesis, $f \times g$ is transitive. Since $X$ is $+$ invariant over open subsets under $f$, by Theorem 6.5, $\mathcal{F}_n(f) \times g$ is transitive. Therefore, $\mathcal{F}_n(f)$ is mild mixing.

Suppose that $f$ is scattering. Let $Y$ be a topological space, let $g : Y \to Y$ be a minimal function. By hypothesis, $f \times g$ is transitive. Since, $X$ is $+$ invariant over open subsets under $f$, by Theorem 6.5, $\mathcal{F}_n(f) \times g$ is transitive. Therefore, $\mathcal{F}_n(f)$ is scattering.

Theorem 6.7 Let $X$ be a topological space, let $f : X \to X$ be a continuous function and let $n \in \mathbb{N}$. If $f$ is minimal and $X$ is $+$ invariant over open subsets under $f$, then $\mathcal{F}_n(f)$ is minimal.

Proof Suppose that $f$ is minimal and that $X$ is $+$ invariant over open subsets under $f$. Since $f$ is a continuous function, by [4, Theorem 6.1] $\mathcal{F}_n(f)$ is a continuous function. Thus, to show that $\mathcal{F}_n(f)$ is minimal, by [15, Proposition 6.2], we need to prove that, for each $A \in \mathcal{F}_n(X)$, $\overline{\text{cl} \mathcal{F}_n(X) \langle \mathcal{O}(A, \mathcal{F}_n(f)) \rangle} = \mathcal{F}_n(X)$. Let
\( \{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \). Since \( f \) is minimal, for each \( i \in \{1, \ldots, m\} \), \( \text{cl}_X(\mathcal{O}(x_i, f)) = X \). Let \( \mathcal{U} \) be a nonempty open subset of \( \mathcal{F}_n(X) \). Then, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n \) of \( X \) such that \( \langle U_1, \ldots, U_n \rangle \subseteq \mathcal{U} \). Consider the following cases:

Case (i): \( r = n \). In this case, for each \( i \in \{1, \ldots, n\} \), there exists \( k_i \in \mathbb{N} \) such that \( f^{k_i}(x_i) \in U_i \). Let \( k = \max\{k_1, \ldots, k_n\} \). Then, by Lemma 3.2, we have that, for every \( i \in \{1, \ldots, n\} \), \( f^k(x_i) \in U_i \). Thus, \( [\mathcal{F}_n(f)]^k(\{x_1, \ldots, x_r\}) \subseteq \{U_1, \ldots, U_n\} \). This implies that \( \mathcal{O}(\{x_1, \ldots, x_r\}, \mathcal{F}_n(f)) \cap \mathcal{U} \neq \emptyset \). Therefore, \( \text{cl}_{\mathcal{F}_n(X)}(\mathcal{O}(\{x_1, \ldots, x_r\}, \mathcal{F}_n(f))) = \mathcal{F}_n(X) \). Finally, since \( \{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \) is arbitrary, we have that \( \mathcal{F}_n(f) \) is minimal.

Case (ii): \( r < n \). In this case, for each \( i \in \{1, \ldots, r\} \), \( \mathcal{O}(x_i, f) \cap U_i \neq \emptyset \) and for every \( j \in \{r+1, \ldots, n\} \), \( \mathcal{O}(x_j, f) \cap U_j \neq \emptyset \). Then, for all \( i \in \{1, \ldots, r\} \), there exists \( k_i \in \mathbb{N} \) such that \( f^{k_i}(x_i) \in U_i \) and for each \( j \in \{r+1, \ldots, n\} \), there exists \( k_j \in \mathbb{N} \) such that \( f^{k_j}(x_j) \in U_j \). Let \( k = \max\{k_1, \ldots, k_n\} \). Then, by Lemma 3.2, for every \( i \in \{1, \ldots, r\} \), \( f^k(x_i) \in U_i \) and for all \( i \in \{1, \ldots, n\} \), \( f^k(x_r) \in U_i \). It follows that \( \{f^k(x_1), \ldots, f^k(x_n)\} \in \mathcal{O}(\{x_1, \ldots, x_r\}, \mathcal{F}_n(f)) \cap \mathcal{U} \neq \emptyset \). Therefore, \( \text{cl}_{\mathcal{F}_n(X)}(\mathcal{O}(\{x_1, \ldots, x_r\}, \mathcal{F}_n(f))) = \mathcal{F}_n(X) \). Because \( \{x_1, \ldots, x_r\} \in \mathcal{F}_n(X) \) is arbitrary, \( \mathcal{F}_n(f) \) is minimal. \( \square \)

**Proposition 6.8** Let \( X \) be a topological space, let \( f : X \to X \) be a continuous function, and let \( n \in \mathbb{N} \). If \( f \) is totally minimal and \( X \) is \( + \)-invariant over open subsets under \( f \), then \( \mathcal{F}_n(f)^s \) is totally minimal.

**Proof** Let \( s \in \mathbb{N} \). By hypothesis, \( f^s \) is minimal and continuous. Hence, by Theorem 6.7, \( \mathcal{F}_n(f^s) \) is minimal. Then, by Remark 3.1, part (1), \( \mathcal{F}_n(f)^s \) is minimal. Since \( s \in \mathbb{N} \) is arbitrary, we have that \( \mathcal{F}_n(f) \) is totally minimal. \( \square \)

**Theorem 6.9** Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function, and let \( n \in \mathbb{N} \). Then the following hold:

1. \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is exact if and only if, for each \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is exact.

2. \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is exact if and only if, for each \( i \in \{1, \ldots, m\} \), \( f_i \) is exact.

**Proof** Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is exact. Let \( \langle x_i \rangle \in \{1, \ldots, m\} \) and let \( \mathcal{U} \) be a nonempty open subset of \( \mathcal{F}_n(X_{i_0}) \). By [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n \) of \( X_{i_0} \) such that \( \langle U_1, \ldots, U_n \rangle \subseteq \mathcal{U} \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( j \in \{1, \ldots, n\} \), let \( U_i^j = X_i \) and \( U_i^j = U_j \). Moreover, for all \( j \in \{1, \ldots, n\} \), let \( U_j^j = \prod_{i=1}^m U_i^j \). Note that \( \langle U_1^j, \ldots, U_n^j \rangle \) is a nonempty open subset of \( \mathcal{F}_n(\prod_{i=1}^m X_i) \).

Since \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is exact, there exists \( k \in \mathbb{N} \) such that \( \mathcal{F}_n(\prod_{i=1}^m f_i)^k(\langle U_1^j, \ldots, U_n^j \rangle) = \mathcal{F}_n(\prod_{i=1}^m X_i) \). Let \( \{x_1, \ldots, x_r\} \in \mathcal{F}_n(X_{i_0}) \), with \( r \leq n \). For each \( j \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( l \in \{1, \ldots, r\} \) let \( a_j^l \subseteq X_j \) and let \( d_{i_0}^l = x_l \). Finally, for all \( l \in \{1, \ldots, r\} \), let \( x_l = (a_1^l, \ldots, a_m^l) \). By Lemma 5.1, part (1), \( \{x_1, \ldots, x_l\} \in \mathcal{F}_n(\prod_{i=1}^m X_i) \). Then \( \{x_1, \ldots, x_r\} \in \mathcal{F}_n(\prod_{i=1}^m f_i)^k(\langle U_1^j, \ldots, U_n^j \rangle) \). Thus, there exists \( \{b_1^1, \ldots, b_m^1\} \subseteq \{1, \ldots, p\} \) such that \( \mathcal{F}_n(\prod_{i=1}^m f_i)^k(\langle b_1^1, \ldots, b_m^1 \rangle) = \{x_1, \ldots, x_r\} \). Therefore, \( \mathcal{F}_n(X_{i_0}) = [\mathcal{F}_n(f_{i_0})]^k(\mathcal{U}) \) and \( \mathcal{F}_n(f_{i_0}) \) is exact.
Suppose that, for each $i \in \{1, \ldots, m\}$, $F_n(f_i)$ is exact. Then, by [4, Theorem 4.10], for every $i \in \{1, \ldots, m\}$, $f_i$ is exact. Thus, by Theorem 4.3, $\prod_{i=1}^{m} f_i$ is exact. Finally, by [4, Theorem 4.10], $F_n(\prod_{i=1}^{m} f_i)$ is exact.

Suppose that $F_n(\prod_{i=1}^{m} f_i)$ is exact. By [4, Theorem 4.10], $\prod_{i=1}^{m} f_i$ is exact. Then, by Theorem 4.3, for each $i \in \{1, \ldots, m\}$, $f_i$ is exact.

Finally, suppose that, for every $i \in \{1, \ldots, m\}$, $f_i$ is exact. By Theorem 4.3, $\prod_{i=1}^{m} f_i$ is exact. Then, by [4, Theorem 4.10], $F_n(\prod_{i=1}^{m} f_i)$ is exact.

By Theorems 6.2 and 4.5, we have the following result.

**Theorem 6.10** Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. Then the following are equivalent:

1. For each $i \in \{1, \ldots, m\}$, $f_i$ is exactly Devaney chaotic.
2. $F_n(\prod_{i=1}^{m} f_i)$ is exactly Devaney chaotic.
3. For every $i \in \{1, \ldots, m\}$, $F_n(f_i)$ is exactly Devaney chaotic.

By [4, Theorem 4.8] and Theorem 4.4, we have the following result.

**Theorem 6.11** Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function and let $n \in \mathbb{N}$. Then the following are equivalent:

1. For each $i \in \{1, \ldots, m\}$, $f_i$ is mixing.
2. $F_n(\prod_{i=1}^{m} f_i)$ is mixing.
3. For every $i \in \{1, \ldots, m\}$, $F_n(f_i)$ is mixing.

**Theorem 6.12** Let $X_1, \ldots, X_{m+1}$ be topological spaces, let $n \in \mathbb{N}$ and, for each $i \in \{1, \ldots, m+1\}$, let $f_i : X_i \to X_i$ be a function. If $F_n(\prod_{i=1}^{m} f_i) \times f_{m+1}$ is transitive, then, for each $i \in \{1, \ldots, m\}$, $F_n(f_i) \times f_{m+1}$ is transitive.

**Proof** Suppose that $F_n(\prod_{i=1}^{m} f_i) \times f_{m+1}$ is transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $U_1$, $U_2$ be two nonempty open subsets of $F_n(X_{i_0}) \times X_{m+1}$. Then there exist nonempty open subsets $U$, $V$ of $F_n(X_{i_0})$ and $F_1$, $F_2$ of $X_{m+1}$ such that $U \times F_1 \subseteq U_1$ and $V \times F_2 \subseteq U_2$. Thus, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \ldots, U_n$, $V_1, \ldots, V_n$ of $X_{i_0}$ such that $\langle U_1, \ldots, U_n \rangle \subseteq U$ and $\langle V_1, \ldots, V_n \rangle \subseteq V$. For each $i \in \{1, \ldots, m\}$, let $U_i^j = X_i$, $V_i^j = X_i$, $U_{i_0}^j = U_j$ and $V_{i_0}^j = V_j$. Finally, for all $j \in \{1, \ldots, n\}$, let $U_j^i = \prod_{i=1}^{m} U_i^j$ and let $V_j^i = \prod_{i=1}^{m} V_i^j$. It follows that, $\langle U_1', \ldots, U_n' \rangle$ and $\langle V_1', \ldots, V_n' \rangle$ are nonempty open subsets of $F_n(\prod_{i=1}^{m} X_i)$. By hypothesis, we have that, there exists $k \in \mathbb{N}$ such that $[F_n(\prod_{i=1}^{m} f_i) \times f_{m+1}]^k(\langle U_1', \ldots, U_n' \rangle \times F_1) \cap (\langle V_1', \ldots, V_n' \rangle \times F_2) \neq \emptyset$. Thus, there exists $\langle \langle (x_{1}^{1}, \ldots, x_{m}^{l}) : l \leq n \rangle, v_1 \rangle \in \langle U_1', \ldots, U_n' \rangle \times F_1$ such that $[F_n(\prod_{i=1}^{m} f_i) \times f_{m+1}]^k(\langle (x_{1}^{1}, \ldots, x_{m}^{l}) : l \leq n \rangle \times v_1) \in \langle V_1', \ldots, V_n' \rangle \times F_2$. Then, by Lemma 5.2, $\langle x_{1}^{1}, \ldots, x_{m}^{l} \rangle \in \langle U_1', \ldots, U_n' \rangle$ and $\langle f_{i_0}^{k}(x_{i_0}^{1}), \ldots, f_{i_0}^{k}(x_{i_0}^{l}) \rangle \in \langle V_1', \ldots, V_n' \rangle$. Hence, we have that $\langle (x_{1}^{1}, \ldots, x_{m}^{l}), v_1 \rangle \in \langle U_1', \ldots, U_n' \rangle \times F_1$ and $[F_n(f_{i_0}) \times f_{m+1}]^k(\langle x_{1}^{1}, \ldots, x_{m}^{l}, v_1 \rangle) \in \langle V_1', \ldots, V_n' \rangle \times F_2$. Therefore, $[F_n(f_{i_0}) \times f_{m+1}]^k(U_1) \cap (U_2) \neq \emptyset$ and hence $F_n(f_{i_0}) \times f_{m+1}$ is transitive.

\[ \square \]
**Theorem 6.13** Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$ let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, Rouche, an $F$-system, backward minimal, mild mixing, scattering or $TT_{++}$. If $F_n(\prod_{i=1}^m f_i) \in \mathcal{M}$, then, for every $i \in \{1, \ldots, m\}$, $F_n(f_i) \in \mathcal{M}$.

**Proof** Suppose that $F_n(\prod_{i=1}^m f_i)$ is transitive. Let $i_0 \in \{1, \ldots, m\}$ and let $\mathcal{U}, \mathcal{V}$ be nonempty open subsets of $F_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \ldots, U_n, V_1, \ldots, V_n$ of $X_{i_0}$ such that $\langle U_1, \ldots, U_n \rangle \subseteq \mathcal{U}$ and $\langle V_1, \ldots, V_n \rangle \subseteq \mathcal{V}$. For each $i \in \{1, \ldots, m\}\setminus\{i_0\}$ and for every $j \in \{1, \ldots, n\}$, let $U_{i,j} = X_i$, $U_{i,j} = U_j$, $V_{i,j} = X_i$ and $V_{i,j} = V_j$. Finally, for all $j \in \{1, \ldots, n\}$, let $U_j = \prod_{k=1}^m U_{i,k}$ and $V_j = \prod_{k=1}^m V_{i,k}$. Note that $\langle U'_1, \ldots, U'_n \rangle$ and $\langle V'_1, \ldots, V'_n \rangle$ are nonempty open subsets of $F_n(\prod_{i=1}^m X_i)$. By hypothesis, there exists $k \in \mathbb{N}$ such that $(F_n(\prod_{i=1}^m f_i))^k(\langle U'_1, \ldots, U'_n \rangle \cap \langle V'_1, \ldots, V'_n \rangle) \neq \emptyset$. Hence, there exists $(x^1, \ldots, x^m) : j \in \{1, \ldots, r\} \subseteq \langle U'_1, \ldots, U'_n \rangle$, with $r \leq n$ such that

$$
\left(F_n\left(\prod_{i=1}^m f_i\right)\right)^k(\langle x^1, \ldots, x^m \rangle : j \in \{1, \ldots, r\}) \subseteq \langle V'_1, \ldots, V'_n \rangle.
$$

By Remark 3.1, parts (1) and (2), $\langle f_i^k(x^1_i), \ldots, f_i^k(x^m_i) \rangle : j \in \{1, \ldots, r\} \subseteq \langle V'_1, \ldots, V'_n \rangle$. Consequently, by Lemma 5.2, $\langle f_i^k(x^1_i), \ldots, f_i^k(x^m_i) \rangle \subseteq \langle V'_1, \ldots, V'_n \rangle$. Which means that $\langle F_n(f_i)\rangle^k(\langle x^1_i, \ldots, x^m_i \rangle) \subseteq \langle V'_1, \ldots, V'_n \rangle$. On the other hand, $\langle x^1_i, \ldots, x^m_i \rangle \subseteq \langle U'_1, \ldots, U'_n \rangle$. Thus, $\langle F_n(f_i)\rangle^k(\langle U'_1, \ldots, U'_n \rangle) \subseteq \langle V'_1, \ldots, V'_n \rangle \neq \emptyset$. Therefore, $\langle F_n(f_i)\rangle$ is transitive.

Suppose that $F_n(\prod_{i=1}^m f_i)$ is weakly mixing. Let $i_0 \in \{1, \ldots, m\}$ and let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{V}_1$ and $\mathcal{V}_2$ be four nonempty open subsets of $F_n(X_{i_0})$. Then, by [10, Lemma 4.2], there exist nonempty open subsets $U_1, \ldots, U_n, U_1^2, \ldots, U_2^2, V_1, \ldots, V_n, V_1^2, \ldots, V_2^2$ of $X_{i_0}$ such that $\langle U_1, \ldots, U_n \rangle \subseteq \mathcal{U}_1$, $\langle U_1^2, \ldots, U_2^2 \rangle \subseteq \mathcal{U}_2$, $\langle V_1, \ldots, V_n \rangle \subseteq \mathcal{V}_1$ and $\langle V_1^2, \ldots, V_2^2 \rangle \subseteq \mathcal{V}_2$. For each $i \in \{1, \ldots, m\}\setminus\{i_0\}$ and for every $j \in \{1, \ldots, n\}$, let $W_{i,j} = X_i$, $T_{i,j} = X_i$, $F_{i,j} = X_i$, $L_{i,j} = X_i$, $T_{i,j} = U_j$, $T_{i,j} = U_j$, $F_{i,j} = V_j$ and $L_{i,j} = V_j$. Moreover, for all $j \in \{1, \ldots, n\}$, let, $W_j = \prod_{i=1}^m W_{i,j}$, $T_j = \prod_{i=1}^m T_{i,j}$, $F_j = \prod_{i=1}^m F_{i,j}$ and $L_j = \prod_{i=1}^m L_{i,j}$. Then, $\langle W_1, \ldots, W_n \rangle$, $\langle T_1, \ldots, T_n \rangle$, $\langle F_1, \ldots, F_n \rangle$ and $\langle L_1, \ldots, L_n \rangle$ are nonempty open subsets of $F_n(\prod_{i=1}^m X_i)$. By hypothesis, $(F_n(\prod_{i=1}^m f_i))^k(\langle W_1, \ldots, W_n \rangle) \cap \langle F_1, \ldots, F_n \rangle \neq \emptyset$ and $(F_n(\prod_{i=1}^m f_i))^k(\langle T_1, \ldots, T_n \rangle) \cap \langle L_1, \ldots, L_n \rangle \neq \emptyset$. Thus, there exist $(x^1, \ldots, x^m) : j \in \{1, \ldots, r\} \subseteq \langle W_1, \ldots, W_n \rangle$ and $(y^1, \ldots, y^m) : j \in \{1, \ldots, p\} \subseteq \langle T_1, \ldots, T_n \rangle$ such that

$$
\left(F_n\left(\prod_{i=1}^m f_i\right)\right)^k(\langle x^1, \ldots, x^m \rangle : j \in \{1, \ldots, r\}) \subseteq \langle F_1, \ldots, F_n \rangle
$$

and

$$
\left(F_n\left(\prod_{i=1}^m f_i\right)\right)^k(\langle y^1, \ldots, y^m \rangle : j \in \{1, \ldots, p\}) \subseteq \langle L_1, \ldots, L_n \rangle.
$$

Thus, by Remark 3.1, parts (1) and (2), we have that $\langle f_i^k(x^1_i), \ldots, f_i^k(x^m_i) \rangle : j \in \{1, \ldots, r\} \subseteq \langle F_1, \ldots, F_n \rangle$ and $\langle f_i^k(y^1_i), f_i^k(y^m_i) \rangle : j \in \{1, \ldots, p\} \subseteq \langle L_1, \ldots, L_n \rangle$. By Lemma 5.2, it follows that $\langle f_i^k(x^1_i), \ldots, f_i^k(x^m_i) \rangle \subseteq $
\( \langle V^1, \ldots, V^l \rangle \) and \( \{ f^1_i(y^1_i), \ldots, f^n_i(y^n_i) \} \in \langle V^2, \ldots, V^l \rangle. \) Then \( (F_n(f_i))^{k}(\{x^1_i, \ldots, x^l_i\}) \in \langle V^1, \ldots, V^l \rangle \) and \( (F_n(f_j))^{k}(\{y^1_i, \ldots, y^n_i\}) \in \langle V^2, \ldots, V^l \rangle. \) Moreover, \( \{x^1_i, \ldots, x^l_i\} \in \langle U^1, \ldots, U^n \rangle \) and \( \{y^1_i, \ldots, y^n_i\} \in \langle U^2, \ldots, U^n \rangle. \) Hence, we have that \( (F_n(f_i))^{k}(\langle U^1, \ldots, U^n \rangle) \cap \langle V^1, \ldots, V^l \rangle \neq \emptyset \) and \( (F_n(f_j))^{k}(\langle U^2, \ldots, U^n \rangle) \cap \langle V^2, \ldots, V^l \rangle \neq \emptyset. \) It follows that, for each \( i \in \{1, 2\} \), \( (F_n(f_i))^{k}(\mathcal{U}) \cap V_i \neq \emptyset. \) Finally, \( F_n(f_i) \) is weakly mixing.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is totally transitive. Let \( i_0 \in \{1, \ldots, m\} \), let \( s \in \mathbb{N} \), and let \( \mathcal{U}, \mathcal{V} \) be two nonempty open subsets of \( F_n(X_{i_0}) \). Then, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_s, V_1, \ldots, V_s \) of \( X_{i_0} \) such that \( \langle U_1, \ldots, U_s \rangle \subseteq \mathcal{U} \) and \( \langle V_1, \ldots, V_s \rangle \subseteq \mathcal{V} \). For each \( i \in \{1, \ldots, m\} \}\{i_0\} and for every \( j \in \{1, \ldots, n\} \), let \( U^j_i = X_i \), \( V^j_i = X_i \), \( U^j_{i_0} = U_j \) and \( V^j_{i_0} = V_j \). Moreover, for all \( j \in \{1, \ldots, n\} \), \( U'_j = \prod_{i=1}^m U^j_i \) and \( V'_j = \prod_{i=1}^m V^j_i \). It follows that \( \langle U'_1, \ldots, U'_s \rangle \) and \( \langle V'_1, \ldots, V'_s \rangle \) are nonempty open subsets of \( F_n(\prod_{i=1}^m f_i) \). Then, since \( (F_n(\prod_{i=1}^m f_i))^s \) is transitive, we have that, there exists \( k \in \mathbb{N} \) such that \( (\prod_{i=1}^m f_i)^s(\{x^1_i, \ldots, x^l_i\}) \cap \langle V'_1, \ldots, V'_s \rangle \neq \emptyset. \) Thus, there exists \( \{x^1_i, \ldots, x^l_i\} \in \langle U'_1, \ldots, U'_s \rangle \) such that \( (\prod_{i=1}^m f_i)^s(\{x^1_i, \ldots, x^l_i\}) \cap \langle V'_1, \ldots, V'_s \rangle \neq \emptyset. \) In consequence, \( (f^1_k(x^1_i), \ldots, f^m_k(x^l_i)) \in \langle V'_1, \ldots, V'_s \rangle. \) Then, by Lemma 5.2, \( \{f^k_i(x^1_i), \ldots, f^k_i(x^l_i)\} \in \langle V'_1, \ldots, V'_s \rangle. \) Hence, \( (F_n(f_i))^k(\{x^1_i, \ldots, x^l_i\}) \in \langle V'_1, \ldots, V'_s \rangle. \) Meanwhile, by Lemma 5.2, \( \{x^1_i, \ldots, x^l_i\} \in \langle U'_1, \ldots, U'_s \rangle \). It follows that \( (F_n(f_i))^k(\{U_1, \ldots, U_n\}) \cap \langle V'_1, \ldots, V'_s \rangle \neq \emptyset. \) Consequently, \( (F_n(f_i))^k(\mathcal{U}) \cap \mathcal{V} \neq \emptyset. \) Therefore, \( F_n(f_i) \) is transitive. Since \( s \in \mathbb{N} \) is arbitrary, we have that \( F_n(f_i) \) is totally transitive.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is strongly transitive. Let \( i_0 \in \{1, \ldots, m\} \) and let \( \mathcal{U} \) be a nonempty open subset of \( F_n(X_{i_0}) \). Hence, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n \) of \( X_{i_0} \) such that \( \langle U_1, \ldots, U_n \rangle \subseteq \mathcal{U} \). For each \( i \in \{1, \ldots, m\} \}\{i_0\} and for every \( j \in \{1, \ldots, n\} \), let \( U^j_i = X_i \) and \( U^j_{i_0} = U_j \). Moreover, for all \( j \in \{1, \ldots, n\} \), let \( U'_j = \prod_{i=1}^m U^j_i \) and \( V'_j = \prod_{i=1}^m V^j_i \). Then \( U'_j = \prod_{i=1}^m U^j_i \) and \( V'_j = \prod_{i=1}^m V^j_i \). On the other hand, by Lemma 3.1, \( \{y^1_i, \ldots, y^n_i\} \in \langle U'_1, \ldots, U'_s \rangle \). Hence, \( \{f^k_i(y^1_i), \ldots, f^k_i(y^n_i)\} \in \langle U'_1, \ldots, U'_s \rangle. \) The other hand, by Lemma 5.2, \( \{y^1_i, \ldots, y^n_i\} \in \langle U'_1, \ldots, U'_s \rangle \). Hence, \( \mathcal{U} \cap \mathcal{V} \neq \emptyset. \) Therefore, \( F_n(f_i) \) is transitive.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is chaotic. Then \( F_n(\prod_{i=1}^m f_i) \) is transitive and \( \text{Per}(F_n(\prod_{i=1}^m f_i)) \) is dense in \( F_n(\prod_{i=1}^m X_i). \) Thus, for each \( i \in \{1, \ldots, m\} \), \( F_n(f_i) \) is transitive and by Theorem 5.8, part (2), for every \( i \in \{1, \ldots, m\} \), \( \text{Per}(F_n(f_i)) \) is dense in \( F_n(X_i). \) Therefore, for all \( i \in \{1, \ldots, m\} \), \( F_n(f_i) \) is chaotic.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is orbit-transitive. Then, there exists a transitive point \( \{x^1_i, \ldots, x^l_i\} \) of \( F_n(\prod_{i=1}^m f_i) \). Thus, by Theorem 5.6, we have that, for each \( i \in \{1, \ldots, m\} \), \( \{x^1_i, \ldots, x^l_i\} \) is a transitive point of \( F_n(f_i) \). Consequently, for every \( i \in \{1, \ldots, m\} \), \( \mathcal{O}(\{x^1_i, \ldots, x^l_i\}, F_n(f_i)) \) is a dense subset in 517
\[ F_n(X_i) \]. Which implies that, for all \( i \in \{1, \ldots, m\} \), \( F_n(f_i) \) is orbit-transitive.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is strictly orbit-transitive. It follows that, there exists a transitive point \( \{(f_1(x_1^{(i)}), \ldots, f_m(x_m^{(i)})) : j \in \{1, \ldots, l\}\} \) of \( F_n(\prod_{i=1}^m f_i) \). By Theorem 5.6, for each \( i \in \{1, \ldots, m\} \), we have that \( \{f_i(x_1^{(i)}), \ldots, f_i(x_l^{(i)})\} \) is a transitive point of \( F_n(f_i) \). Thus, for every \( i \in \{1, \ldots, m\} \), \( F_n(f_i)(\{x_1^{(i)}, \ldots, x_l^{(i)}\}) \) is a transitive point of \( F_n(f_i) \). Hence, for all \( i \in \{1, \ldots, m\} \), the subset \( \mathcal{O}(F_n(f_i)(\{x_1^{(i)}, \ldots, x_l^{(i)}\}), F_n(f_i)) \) is dense in \( F_n(X_i) \). Therefore, for each \( i \in \{1, \ldots, m\} \), \( F_n(f_i) \) is strictly orbit-transitive.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is \( \omega \)-transitive. By hypothesis, there exists \( \{(x_1^{(i)}, \ldots, x_m^{(i)}): j \in \{1, \ldots, l\}\} \in F_n(\prod_{i=1}^m X_i) \) such that \( \omega\{(x_1^{(i)}, \ldots, x_m^{(i)}): j \in \{1, \ldots, l\}\} \in F_n(\prod_{i=1}^m f_i) = F_n(\prod_{i=1}^m X_i) \). Then, by Theorem 5.7, for each \( i \in \{1, \ldots, m\} \), \( \omega\{(x_1^{(i)}, \ldots, x_l^{(i)}), F_n(f_i) = F_n(X_i), \) which means that for every \( i \in \{1, \ldots, m\} \), \( F_n(f_i) \) is \( \omega \)-transitive.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is \( TT_{++} \). Let \( i_0 \in \{1, \ldots, m\} \) and let \( U, V \) be two nonempty open subsets of \( F_n(X_{i_0}) \). Then, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n, V_1, \ldots, V_n \) of \( X_{i_0} \) such that \( (U_1, \ldots, U_n) \ni \{i_0\} \) and for each \( i \in \{1, \ldots, n\} \), let \( U_i = X_i \), \( V_i = X_i \), \( U_{i_0} = U_j \) and \( V_{i_0} = V_j \). Moreover, for all \( j \in \{1, \ldots, n\} \), let \( U_j = \prod_{i=1}^m U_i^l \) and \( V_j = \prod_{i=1}^m V_i^l \). Note that \( \langle U_1', \ldots, U_n' \rangle \) and \( \langle V_1', \ldots, V_n' \rangle \) are nonempty open subsets of \( F_n(\prod_{i=1}^m X_i) \). By hypothesis, \( n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}(\langle U_1', \ldots, U_n' \rangle, \langle V_1', \ldots, V_n' \rangle) \) is infinite. On the other hand, by Lemma 5.3, we have that
\[
 n_{\mathcal{F}_n(\prod_{i=1}^m f_i)}(\langle U_1', \ldots, U_n' \rangle, \langle V_1', \ldots, V_n' \rangle) \subseteq n_{\mathcal{F}_n(f_{i_0})}(\langle U_1, \ldots, U_n \rangle, \langle V_1, \ldots, V_n \rangle).
\]
Consequently, \( n_{\mathcal{F}_n(f_{i_0})}(\langle U_1, \ldots, U_n \rangle, \langle V_1, \ldots, V_n \rangle) \) is infinite. Therefore, \( F_n(f_{i_0}) \) is \( TT_{++} \).

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is Touhy. Let \( i_0 \in \{1, \ldots, m\} \) and let \( U, V \) be two nonempty open subsets of \( F_n(X_{i_0}) \). Thence, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_n, V_1, \ldots, V_n \) of \( X_{i_0} \) such that \( (U_1, \ldots, U_n) \subseteq \mathcal{U} \) and \( (V_1, \ldots, V_n) \subseteq \mathcal{V} \). For each \( i \in \{1, \ldots, m\} \) and for each \( j \in \{1, \ldots, n\} \), let \( U_i = X_i \), \( V_i = X_i \), \( U_{i_0} = U_j \) and \( V_{i_0} = V_j \). Finally, for all \( j \in \{1, \ldots, n\} \), let \( U_j = \prod_{i=1}^m U_i^l \) and \( V_j = \prod_{i=1}^m V_i^l \). It follows that \( \langle U_1', \ldots, U_n' \rangle \) and \( \langle V_1', \ldots, V_n' \rangle \) are nonempty open subsets of \( F_n(\prod_{i=1}^m X_i) \). Since \( F_n(\prod_{i=1}^m X_i) \) is Touhy, there exist a periodic point \( \{(x_1^{(i)}, \ldots, x_m^{(i)}): r \leq n \in \{1, \ldots, l\}\} \in \langle U_1', \ldots, U_n' \rangle \) and \( k \in \mathbb{Z}_+ \) such that \( [\mathcal{F}_n(\prod_{i=1}^m f_i)]^k(\{(x_1^{(i)}, \ldots, x_m^{(i)}): r \leq n \in \{1, \ldots, l\}\}) \subseteq \langle V_1', \ldots, V_n' \rangle \). By Remark 3.1, part (2), \( \{(f_1^k(x_1^{(i)}), \ldots, f_m^k(x_m^{(i)})): r \leq n \in \{1, \ldots, l\}\} \in \langle V_1', \ldots, V_n' \rangle \). Then, by Lemma 5.2, \( \{f_1^k(x_1), \ldots, f_m^k(x_m)\} \subseteq \mathcal{V} \). Thus, \( \mathcal{F}_n(f_{i_0})^k(\{(x_1^{(i)}, \ldots, x_m^{(i)}): r \leq n \in \{1, \ldots, l\}\}) \subseteq \mathcal{V} \). On the other hand, since \( \{(x_1^{(i)}, \ldots, x_m^{(i)}): r \leq n \in \{1, \ldots, l\}\} \in \langle U_1', \ldots, U_n' \rangle \), by Lemma 5.2, \( \{x_1^{(i)}, \ldots, x_m^{(i)}\} \subseteq \langle U_1', \ldots, U_n' \rangle \). Moreover, since \( \{(x_1^{(i)}, \ldots, x_m^{(i)}): r \leq n \in \{1, \ldots, l\}\} \) is a periodic point of \( F_n(\prod_{i=1}^m f_i) \), by [4, Theorem 3.4], for each \( l \in \{1, \ldots, r\} \), \( (x_1^{(i)}, \ldots, x_m^{(i)}) \) is a periodic point of \( \prod_{i=1}^m f_i \). Then, by Theorem 3.3, part (4), for each \( l \in \{1, \ldots, r\} \), \( x_1^{(i)} \) is a periodic point of \( f_{i_0} \). Thus, by [4, Theorem 3.4], \( \{x_1^{(i)}, \ldots, x_m^{(i)}\} \) is a periodic point of \( F_n(f_{i_0}) \). Therefore, \( F_n(f_{i_0}) \) is Touhy.

Suppose that \( F_n(\prod_{i=1}^m f_i) \) is an F-system. Then, \( F_n(\prod_{i=1}^m f_i) \) is totally transitive and the subset \( \mathcal{P}(\mathcal{F}_n(\prod_{i=1}^m f_i)) \) is dense in \( F_n(\prod_{i=1}^m X_i) \). By [4, Theorem 3.4], \( \mathcal{P}(\prod_{i=1}^m f_i) \) is dense in \( \prod_{i=1}^m X_i \). In consequence, by Theorem 3.15, for each \( i \in \{1, \ldots, m\} \), \( \mathcal{P}(f_i) \) is dense in \( X_i \). Again, by [4, Theorem 3.4], for every \( i \in \{1, \ldots, m\} \), \( \mathcal{P}(\mathcal{F}_n(f_i)) \) is dense in \( F_n(X_i) \). On the other hand, by the third paragraph of this
proof, we have that, for all \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is totally transitive. Therefore, for each \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is an F-system.

Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is backward minimal. Let \( i_0 \in \{1, \ldots, m\} \) and let \( \{x_1, \ldots, x_r\} \subseteq \mathcal{F}_n(X_{i_0}) \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( j \in \{1, \ldots, r\} \), let \( y_i^j \in X_i \) and let \( y_{i_0}^j = x_j \). Hence, \( \{(y_i^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\} \subseteq \mathcal{F}_n(\prod_{i=1}^m X_i) \). Since \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is backward minimal, the set \( \{A \subseteq \mathcal{F}_n(\prod_{i=1}^m X_i) : \mathcal{F}_n(\prod_{i=1}^m f_i)^I(A) = \{(y_1^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\}, \text{ for some } l \in \mathbb{N}\} \), is dense in \( \mathcal{F}_n(\prod_{i=1}^m X_i) \). Let \( \mathcal{U} \) be a nonempty open subset of \( \mathcal{F}_n(X_{i_0}) \). Then, by [10, Lemma 4.2], there exist nonempty open subsets \( U_1, \ldots, U_m \) of \( X_{i_0} \) such that \( \langle U_1, \ldots, U_m \rangle \subseteq \mathcal{U} \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( j \in \{1, \ldots, n\} \), let \( U_i^j = X_i \) and \( U_{i_0}^j = U_j \). Finally, for all \( j \in \{1, \ldots, n\} \), let \( U'_j = \prod_{i=1}^m U_i^j \). Thus, \( \langle U'_1, \ldots, U'_n \rangle \) is a nonempty open subset of \( \mathcal{F}_n(\prod_{i=1}^m X_i) \). By hypothesis, there exist \( \{(z_1^p, \ldots, z_m^p) : p \geq n \text{ and } j \in \{1, \ldots, p\}\} \subseteq \langle U'_1, \ldots, U'_m \rangle \) and \( l \in \mathbb{N} \) such that \( \mathcal{F}_n(\prod_{i=1}^m f_i)^I(\{(z_1^p, \ldots, z_m^p) : p \geq n \text{ and } j \in \{1, \ldots, p\}\}) = \{(y_1^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\} \). Meanwhile, by Lemma 5.2, \( \{(z_1^p, \ldots, z_m^p) \subseteq \langle U_1, \ldots, U_m \rangle \). Moreover, by Remark 3.1, parts (1) and (2), \( \{(f_1^j(z_1^p), \ldots, f_m^j(z_1^p)) : p \geq n \text{ and } j \in \{1, \ldots, p\}\} = \{(y_1^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\} \). It follows that \( \{(f_1^j(z_1^p), \ldots, f_m^j(z_1^p) = \langle y_1^j, \ldots, y_m^j \rangle \}. Consequently, \( \mathcal{F}_n(f_{i_0})^I(\{(z_1^p, \ldots, z_m^p) = \{(y_1^j, \ldots, y_m^j) \}. Therefore, the \( \{A \subseteq \mathcal{F}_n(X_{i_0}) : \mathcal{F}_n(f_{i_0})^I(A) = \{x_1, \ldots, x_r\}, \text{ for some } l \in \mathbb{N}\} \), is dense in \( \mathcal{F}_n(X_{i_0}) \) and \( \mathcal{F}_n(f_{i_0}) \) is backward minimal.

Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is mild mixing. Let \( i_0 \in \{1, \ldots, m\} \), let \( Y \) be a topological space and let \( g : Y \to Y \) be a transitive function. By hypothesis, \( \mathcal{F}_n(\prod_{i=1}^m f_i) \times g \) is transitive. Thus, by Theorem 6.12, \( \mathcal{F}_n(f_{i_0}) \times g \) is transitive. Therefore, \( \mathcal{F}_n(f_{i_0}) \) is mild mixing.

Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is scattering. Let \( i_0 \in \{1, \ldots, m\} \), let \( Y \) be a topological space and let \( g : Y \to Y \) be a minimal function. By hypothesis, \( \mathcal{F}_n(\prod_{i=1}^m f_i) \times g \) is transitive. Thus, by Theorem 6.12, \( \mathcal{F}_n(f_{i_0}) \times g \) is transitive. Therefore, \( \mathcal{F}_n(f_{i_0}) \) is scattering.

\[ \square \]

\textbf{Theorem 6.14} Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a continuous function, and let \( n \in \mathbb{N} \). If \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is minimal, then, for every \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is minimal.

\textbf{Proof} Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is minimal. Let \( i_0 \in \{1, \ldots, m\} \). By hypothesis, \( f_{i_0} \) is continuous. Hence, \( \mathcal{F}_n(f_{i_0}) \) is continuous. Thus, by [15, Proposition 6.2], it is sufficient to prove that for each \( A \subseteq \mathcal{F}_n(X_{i_0}) \), \( \mathcal{F}_n(X_{i_0}) \) is open \( A, \mathcal{F}_n(f_{i_0}) \) with \( r \leq n \). For each \( i \in \{1, \ldots, m\} \setminus \{i_0\} \) and for every \( j \in \{1, \ldots, r\} \), let \( y_i^j \in X_i \) and \( y_{i_0}^j = x_j \). Thus, \( \{(y_1^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\} \subseteq \mathcal{F}_n(\prod_{i=1}^m X_i) \). Since \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is minimal. We have that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is \( \bigcup \{(y_1^j, \ldots, y_m^j) : j \in \{1, \ldots, r\}\} \subseteq \mathcal{F}_n(\prod_{i=1}^m f_i) \). Thus, by Theorem 5.6, for all \( i \in \{1, \ldots, m\} \) we have that, \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is a minimal function. Consequently, \( \mathcal{F}_n(f_{i_0}) \) is a minimal function.

\[ \mathcal{F}_n(f_{i_0}) \] is arbitrary, \( \mathcal{F}_n(f_{i_0}) \) is minimal.

\[ \square \]

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Theorem 6.15 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a continuous function, and let \( n \in \mathbb{N} \). If \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is totally minimal, then, for every \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is totally minimal.

Proof Suppose that \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is totally minimal. Let \( s \in \mathbb{N} \). By hypothesis, \( [\mathcal{F}_n(\prod_{i=1}^m f_i)]^s \) is minimal. Then, by Remark 3.1, part (2), \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is minimal. Then, by Theorem 6.14, for each \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i^s) \) is minimal. Again, by Remark 3.1, part (2), for every \( i \in \{1, \ldots, m\} \), \( [\mathcal{F}_n(f_i)]^s \) is minimal. Since \( s \in \mathbb{N} \) is arbitrary, we have that, for all \( i \in \{1, \ldots, m\} \), \( \mathcal{F}_n(f_i) \) is totally minimal. \( \square \)

By [4, Theorems 4.11, 4.12, 4.14, 4.15, 4.19, 5.1, 5.3, 5.6, 5.9], Theorem 6.3, and Theorem 6.13, we have the following result.

Theorem 6.16 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function, let \( n \in \mathbb{N} \), and let \( \mathcal{M} \) be one of the following classes of functions: transitive, weakly mixing, totally transitive, strongly transitive, chaotic, orbit-transitive, strictly orbit-transitive, \( \omega \)-transitive, \( TT_{\alpha\beta} \), Touhey, an \( F \)-system, backward minimal, mild mixing or scattering. If \( \mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M} \), then, for every \( i \in \{1, \ldots, m\} \), \( f_i \in \mathcal{M} \).

The converse of Theorem 6.16 is not true in general. Let us see a partly example of this in the following:

Example 6.17 Let \( f : [0, 2] \to [0, 2] \) be a function given by:

\[
f(x) = \begin{cases} 
2x + 1, & 0 \leq x \leq \frac{1}{2}, \\
-2x + 3, & \frac{1}{2} \leq x \leq 1, \\
-x + 2, & 1 \leq x \leq 2.
\end{cases}
\]

In [8, Example 1], it is shown that \( f \) is transitive; however, \( f \times f : [0, 2] \times [0, 2] \to [0, 2] \times [0, 2] \) is not transitive. If we suppose that \( \mathcal{F}_n(f \times f) \) is transitive, by [4, Theorem 4.11], we have that \( f \times f \) is transitive. Which is a contradiction. Therefore, \( \mathcal{F}_n(f \times f) \) is not transitive.

By Theorems 6.3, 6.14, 6.15, and [4, Theorem 4.18], we have the following result.

Theorem 6.18 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a continuous function, and let \( n \in \mathbb{N} \). Then the following hold:

1. If \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is minimal, then, for every \( i \in \{1, \ldots, m\} \), \( f_i \) is minimal.

2. If \( \mathcal{F}_n(\prod_{i=1}^m f_i) \) is totally minimal, then, for all \( i \in \{1, \ldots, m\} \), \( f_i \) is totally minimal.

By Theorems 3.14, 4.10, 6.6, and [4, Theorems 5.2, 5.4, 5.7], we obtain the following result.

Theorem 6.19 Let \( X_1, \ldots, X_m \) be topological spaces, for each \( i \in \{1, \ldots, m\} \), let \( f_i : X_i \to X_i \) be a function, let \( n \in \mathbb{N} \), and let \( \mathcal{M} \) be one of the following classes of functions: transitive, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, \( \omega \)-transitive, Touhey, an \( F \)-system, mild mixing or scattering. If for every \( i \in \{1, \ldots, m\} \), \( X_i \) is + invariant over open subsets under \( f_i \) and \( f_i \in \mathcal{M} \), then \( \mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M} \).

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Corollary 6.20 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, let $n \in \mathbb{N}$, and let $\mathcal{M}$ be one of the following classes of functions: transitive, totally transitive, chaotic, orbit-transitive, strictly orbit-transitive, $\omega$-transitive, Touhey, an $F$-system, mild mixing, scattering or $TT_{++}$. If for every $i \in \{1, \ldots, m\}$, $X_i$ is $+\text{invariant}$ over open subsets under $f_i$ and $\mathcal{F}_n(f_i) \in \mathcal{M}$, then $\mathcal{F}_n(\prod_{i=1}^m f_i) \in \mathcal{M}$.

Theorem 6.21 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function, and let $n \in \mathbb{N}$. If for every $i \in \{1, \ldots, m\}$, $f_i$ is weakly mixing and continuous and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $f_i$ is weakly mixing and continuous and that $X_i$ is $+\text{invariant}$ over open subsets under $f_i$. Then, by Theorem 4.10, $\prod_{i=1}^m f_i$ is weakly mixing. Even more, $\prod_{i=1}^m f_i$ is continuous. Thus, by [4, Theorem 4.13], we have that $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.

Corollary 6.22 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a function such that $\prod_{i=1}^m f_i$ is continuous, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is weakly mixing and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is weakly mixing, and that $X_i$ is $+\text{invariant}$ over open subsets under $f_i$ and $\prod_{i=1}^m f_i$ is continuous. Then, by [4, Theorem 4.12], for each $i \in \{1, \ldots, m\}$, $f_i$ is weakly mixing. Even more, for each $i \in \{1, \ldots, m\}$, $f_i$ is continuous. Thus, by Theorem 6.21, $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is weakly mixing.

Theorem 6.23 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $f_i$ is minimal and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal.

Proof Suppose that, for each $i \in \{1, \ldots, m\}$, $f_i$ is minimal and that $X_i$ is $+\text{invariant}$ over open subsets under $f_i$. Then, by Proposition 4.11, $\prod_{i=1}^m f_i$ is minimal. Even more, $\prod_{i=1}^m f_i$ is continuous and by Theorem 3.14, $\prod_{i=1}^m f_i$ is $+\text{invariant}$ over open subsets under $\prod_{i=1}^m f_i$. Thus, by Theorem 6.7, $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal.

As a consequence of Theorem 6.23 and [4, Theorem 4.18], we have the following result.

Corollary 6.24 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $\mathcal{F}_n(f_i)$ is minimal and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is minimal.

As a consequence of Corollary 4.12, Theorem 3.14, and Proposition 6.8, we obtain the following.

Corollary 6.25 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function, and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $f_i$ is totally minimal and $X_i$ is $+\text{invariant}$ over open subsets under $f_i$, then $\mathcal{F}_n(\prod_{i=1}^m f_i)$ is totally minimal.

As a consequence of Theorem 6.3 and Corollary 6.25, we have:
Corollary 6.26 Let $X_1, \ldots, X_m$ be topological spaces, for each $i \in \{1, \ldots, m\}$, let $f_i : X_i \to X_i$ be a continuous function and let $n \in \mathbb{N}$. If, for every $i \in \{1, \ldots, m\}$, $F_n(f_i)$ is totally minimal and $X_i$ is $+$-invariant over open subsets under $f_i$, then $F_n(\prod_{i=1}^{m} f_i)$ is totally minimal.

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References


