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DANIYAL İSRAFİLZADE

ELİFE GÜRSEL

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Faber–Laurent series in variable Smirnov classes

Daniyal M. ISRAFILOV^{1,2,*}, Elife GÜRSEL¹

¹Balıkesir University, Department of Mathematics, Balıkesir Turkey

²Institute of Mathematics and Mechanics of ANAS, Baku, Azerbaijan

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Abstract: In this work, the maximal convergence properties of partial sums of Faber–Laurent series in the variable exponent Smirnov classes of analytic functions defined on a doubly connected domain of the complex plane are investigated.

Key words: Faber–Laurent series, variable spaces, maximal convergence

1. Introduction and main results

Let K be a bounded continuum of the complex plane \mathbb{C} with the complementary $\mathbb{C} \setminus K$, consisting of two simple connected domains G and B . We assume that B is bounded and G is unbounded component of this complementary. Without loss of generality, we assume that $0 \in B$. Moreover, let $D := \{w \in \mathbb{C} : |w| < 1\}$, $D^- := \{w \in \mathbb{C} : |w| > 1\}$, and $\mathbb{T} := \partial D$.

We denote by $w = \varphi_1(z)$ the conformal mapping of G onto D^- normalized by the conditions

$$\varphi_1(\infty) = \infty, \quad \lim_{z \rightarrow \infty} \frac{\varphi_1(z)}{z} > 0$$

and by ψ_1 the inverse mapping of φ_1 .

We also denote by $w = \varphi_2(z)$ the conformal mapping of B onto D^- normalized by the conditions

$$\varphi_2(0) = \infty, \quad \lim_{z \rightarrow 0} z\varphi_2(z) > 0$$

and by ψ_2 the inverse mapping of φ_2 .

Since the Laurent expansion of φ_1 in some neighborhood of the infinity has the form

$$\varphi_1(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \dots,$$

we have

$$[\varphi_1(z)]^n = \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k + \sum_{k<0} \gamma_{n,k} z^k.$$

*Correspondence: mdaniyal@balikesir.edu.tr

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The polynomial

$$\Phi_n^1(z) := \gamma^n z^n + \sum_{k=0}^{n-1} \gamma_{n,k} z^k$$

is called the Faber polynomial of order n for continuum K . Let $E_n^1(z) := -\sum_{k<0} \gamma_{n,k} z^k$.

The function φ_2 has an expansion in some neighborhood of the origin:

$$\varphi_2(z) = \frac{\beta}{z} + \beta_0 + \beta_1 z + \dots + \beta_k z^k \dots$$

Raising this function to the power n , we obtain

$$[\varphi_2(z)]^n = \Phi_n^2(1/z) - E_n^2(z), \quad z \in B \tag{1.1}$$

where $\Phi_n^2(1/z)$ denotes the polynomial of negative powers of z and the term $E_n^2(z)$ contains nonnegative powers of z ; hence, this is an analytic function in the domain B .

Note that the polynomials Φ_n^1 and Φ_n^2 can be found also as the Taylor coefficients of the series representations

$$\frac{\psi_1'(t)}{\psi_1(t) - z} = \sum_{k=0}^{\infty} \Phi_k^1(z) \frac{1}{t^{k+1}}, \quad |t| > R_1 > 1, z \in K \tag{1.2}$$

and

$$\frac{\psi_2'(t)}{\psi_2(t) - z} = -\sum_{k=0}^{\infty} \Phi_k^2(1/z) \frac{1}{t^{k+1}}, \quad |t| > R_2 > 1, z \in K \tag{1.3}$$

respectively.

Let $p(\cdot) : \Gamma \rightarrow \mathbb{R}^+ := [0, \infty)$ be a Lebesgue measurable function defined on the Jordan rectifiable curve $\Gamma \subset \mathbb{C}$, such that

$$1 < p_- := \operatorname{ess\,inf}_{z \in \Gamma} p(z) \leq \operatorname{ess\,sup}_{z \in \Gamma} p(z) := p_+ < \infty. \tag{1.4}$$

Definition 1.1 We say that $p(\cdot) \in P_0(\Gamma)$, if $p(\cdot)$ satisfies the conditions (1.4) and for some constant $c_0 > 0$ the inequality

$$|p(z_1) - p(z_2)| \leq \frac{c_0}{\log(|\Gamma|/|z_1 - z_2|)}, \quad \forall z_1, z_2 \in \Gamma$$

holds, where $|\Gamma|$ is the Lebesgue measure of Γ .

For a given exponent $p(\cdot)$ we define the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ as the set of Lebesgue measurable functions f defined on Γ such that $\int_{\Gamma} |f(z)|^{p(z)} |dz| < \infty$. Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf \left\{ \lambda \geq 0 : \int_{\Gamma} |f(z) / \lambda|^{p(z)} |dz| \leq 1 \right\} < \infty,$$

it becomes a Banach space, which in the case of $[0, 2\pi]$ is the variable exponent Lebesgue space $L^{p(\cdot)}([0, 2\pi])$, investigated in [3, 5, 19].

Let us take the level lines

$$\Gamma_{R_1} := \{z : |\varphi_1(z)| = R_1 > 1\},$$

$$L_{R_2} := \{z : |\varphi_2(z)| = R_2 > 1\}$$

and let $G_{R_1} := \text{int}\Gamma_{R_1}$, $G_{R_1}^- := \text{ext}\Gamma_{R_1}$, $B_{R_2} := \text{int}L_{R_2}$, $B_{R_2}^- := \text{ext}L_{R_2}$. Moreover, let G_{R_1, R_2} be a doubly-connected domain bounded by the curves Γ_{R_1} and L_{R_2} .

Let $E^p(G_{R_1, R_2})$, $p \geq 1$, be a classical Smirnov class of analytic functions in the doubly-connected domain G_{R_1, R_2} . We mention [6] that $f \in E^p(G_{R_1, R_2})$, iff f is analytic in G_{R_1, R_2} and there exists a sequence $(\Delta_\nu)_{\nu=1}^\infty$, $\Delta_\nu \subset G_{R_1, R_2}$ of domains Δ_ν whose boundaries $(\Gamma_\nu)_{\nu=1}^\infty$ consist of two rectifiable Jordan curves, such that the domain Δ_ν contains each compact subset G^* of G_{R_1, R_2} for every $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and

$$\limsup_{\nu \rightarrow \infty} \int_{\Gamma_\nu} |f(z)|^p |dz| < \infty.$$

Definition 1.2 Let $p_1(\cdot)$ and $p_2(\cdot)$ be the Lebesgue measurable functions defined on Γ_{R_1} and L_{R_2} , respectively. The set

$$E^{p_1(\cdot), p_2(\cdot)}(G_{R_1, R_2}) := \left\{ f \in E^1(G_{R_1, R_2}) : f \in L^{p_1(\cdot)}(\Gamma_{R_1}) \cap L^{p_2(\cdot)}(L_{R_2}) \right\}$$

is called the variable exponent Smirnov class of analytic functions in G_{R_1, R_2} .

Since $L^{p(\cdot)}(\mathbb{T})$ is noninvariant with respect to the usual shift operator, we consider the mean value operator: $\sigma_h : f \rightarrow \sigma_h f := \frac{1}{h} \int_0^h f(we^{it}) dt$, $w \in \mathbb{T}$, $0 < h < \pi$, which is bounded [4] in $L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$. Using this operator we define the modulus of smoothness as following.

Definition 1.3 Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in P_0(\mathbb{T})$. The function $\Omega(f, \cdot)_{p(\cdot), \mathbb{T}} : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\Omega(f, \cdot)_{p(\cdot), \mathbb{T}} := \sup_{0 < h \leq \delta} \|f - \sigma_h f\|_{L^{p(\cdot)}(\mathbb{T})}$$

is called the modulus of smoothness of f in $L^{p(\cdot)}(\mathbb{T})$.

If $f(z) \in E^1(G_{R_1, R_2})$ is an analytic function in the doubly-connected domain G_{R_1, R_2} , then it has the integral representation [20, pp. 256]:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{L_{R_2}} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in K.$$

Let

$$\begin{aligned} f_{R_1}^+(z) & : = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \frac{f(\zeta)}{\zeta - z} d\zeta \\ & = \frac{1}{2\pi i} \int_{|w|=R_1} \frac{\psi_1'(w)}{\psi_1(w) - z} f(\psi_1(w)) dw, \end{aligned} \tag{1.5}$$

$$\begin{aligned}
 f_{R_2}^-(z) & : = \frac{-1}{2\pi i} \int_{L_{R_2}} \frac{f(\zeta)}{\zeta - z} d\zeta \\
 & = \frac{-1}{2\pi i} \int_{|w|=R_2} \frac{\psi_2'(w)}{\psi_2(w) - z} f(\psi_2(w)) dw.
 \end{aligned}
 \tag{1.6}$$

Combining (1.5) and (1.6) with the series representations (1.2) and (1.3), respectively we can write

$$f_{R_1}^+(z) \sim \sum_{k=0}^{\infty} a_k (f_{R_1}^+) \Phi_k^1(z), \quad z \in G_{R_1},
 \tag{1.7}$$

$$f_{R_2}^-(z) \sim \sum_{k=0}^{\infty} b_k (f_{R_2}^-) \Phi_k^2(1/z), \quad z \in B_{R_2}^-,
 \tag{1.8}$$

where

$$\begin{aligned}
 a_k (f_{R_1}^+) & = \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \frac{f_{R_1}^+(z) \varphi_1'(z)}{[\varphi_1(z)]^{k+1}} dz \\
 & = \frac{1}{2\pi i} \int_{|w|=R_1} \frac{f_{R_1}^+(\psi_1(w))}{w^{k+1}} dw,
 \end{aligned}
 \tag{1.9}$$

$$\begin{aligned}
 b_k (f_{R_2}^-) & = \frac{1}{2\pi i} \int_{L_{R_2}} \frac{f_{R_2}^-(z) \varphi_2'(z)}{[\varphi_2(z)]^{k+1}} dz \\
 & = \frac{1}{2\pi i} \int_{|w|=R_2} \frac{f_{R_2}^-(\psi_2(w))}{w^{k+1}} dw.
 \end{aligned}
 \tag{1.10}$$

Let us introduce the value

$$\begin{aligned}
 R_n(z, f) & : = f(z) - \left\{ \sum_{k=0}^n a_k (f_{R_1}^+) \Phi_k^1(z) \right. \\
 & \quad \left. + \sum_{k=1}^n b_k (f_{R_2}^-) \Phi_k^2(1/z) \right\}
 \end{aligned}
 \tag{1.11}$$

and the best approximation numbers

$$E_n(f, G_{R_1})_{p_1(\cdot)} := \inf \|f - p_n\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \quad \text{for } f \in E^{p_1(\cdot)}(G_{R_1}),
 \tag{1.12}$$

$$E_n(f, B_{R_2}^-)_{p_2(\cdot)} := \inf \|f - q_n\|_{L^{p_2(\cdot)}(L_{R_2})} \quad \text{for } f \in E^{p_2(\cdot)}(B_{R_2}^-),
 \tag{1.13}$$

where inf is taken over the polynomials $p_n(z)$ and $q_n(1/z)$, respectively.

Since Γ_{R_1} and L_{R_2} are analytic curves, the following lemmas are true[14]:

Lemma 1.4 *If $f \in L^{p_1(\cdot)}(\Gamma_{R_1})$, $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, then $f_{R_1}^+(z) \in E^{p_1(\cdot)}(G_{R_1})$.*

Lemma 1.5 *If $f \in L^{p_2(\cdot)}(L_{R_2})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$, then $f_{R_2}^-(z) \in E^{p_2(\cdot)}(B_{R_2}^-)$.*

By $c(\cdot)$, $c_1(\cdot)$, $c_2(\cdot), \dots$, we denote the constants depending in general of parameters given in the brackets.

Our main results are as follows:

Theorem 1.6 *Let $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$. If $f \in E^{p_1(\cdot), p_2(\cdot)}(G_{R_1, R_2})$, $R_1, R_2 > 1$, then there is a constant $c(p_1, p_2) > 0$ such that for $\forall z \in K$*

$$|R_n(z, f)| \leq c(p_1, p_2) \sqrt{n \ln n} \left[\frac{E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}}{R_1^{n+1}(R_1 - 1)} + \frac{E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - 1)} \right].$$

Theorem 1.6 is new also for the simple connected domains. When G is a simple connected domain, we have

Corollary 1.7 *Let $p(\cdot) \in \mathcal{P}_0(\Gamma_R)$. If $f \in E^{p(\cdot)}(G_R)$, $R > 1$, then there is a constant $c(p) > 0$ such that*

$$|R_n(z, f)| \leq c(p) \sqrt{n \ln n} \frac{E_n(f_R^+, G_R)_{p(\cdot)}}{R_1^{n+1}(R_1 - 1)}, \quad z \in K,$$

where

$$R_n(z, f) := f(z) - \sum_{k=0}^n a_k(f_R^+) \Phi_k^1(z).$$

We denote by $w = \varphi_1(z)$ the conformal mapping of $G_{R_1}^-$ onto D^- and by φ_1^{-1} the inverse mapping of φ_1 . Moreover, we denote by $w = \varphi_2(z)$ the conformal mapping of B_{R_2} onto D^- and by φ_2^{-1} the inverse mapping of φ_2 . It is easy to see that

$$\begin{aligned} \varphi_1(z) &= \frac{\varphi_1(z)}{R_1} \quad \text{and} \quad \varphi_1^{-1}(w) = \psi_1(R_1 w), \\ \varphi_2(z) &= \frac{\varphi_2(z)}{R_2} \quad \text{and} \quad \varphi_2^{-1}(w) = \psi_2(R_2 w). \end{aligned}$$

Let

$$\begin{aligned} f_1(w) &:= f(\varphi_1^{-1}(w)) \quad \text{and} \quad p_{1,1}(w) := p_1(\varphi_1^{-1}(w)), \\ f_2(w) &:= f(\varphi_2^{-1}(w)) \quad \text{and} \quad p_{2,2}(w) := p_2(\varphi_2^{-1}(w)). \end{aligned}$$

The following theorem gives a qualitative estimation for the error of maximal convergence:

Theorem 1.8 *Let $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$. If $f \in E^{p_1(\cdot), p_2(\cdot)}(G_{R_1, R_2})$, $R_1, R_2 > 1$, then there exists a constant $c(p_1, p_2) > 0$ such that*

$$|R_n(z, f)| \leq c(p_1, p_2) \sqrt{n \ln n} \left[\frac{\Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}}}{R_1^{n+1}(R_1 - 1)} + \frac{\Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}}{R_2^{n+1}(R_2 - 1)} \right].$$

When $p(\cdot) = const > 1$, different versions of Theorems 1.6 and 1.8 in the classical Smirnov, Smirnov–Orlicz classes of analytic functions defined on the simple connected domains can be found in the monograph [20, chapter X] and also in [8–12]. In the case of variable exponent $p(\cdot)$ similar problems were investigated in [1, 2, 7, 13–17, 22].

2. Auxiliary results

For fixed $R_1, R_2 > 1$ the level curves Γ_{R_1} and L_{R_2} are analytic curves and hence by [21] there exist the positive constants $c_i > 0, i = 1, 2, \dots, 8$, such that

$$\begin{aligned} 0 < c_1 \leq |\varphi'_1(z)| \leq c_2 < \infty, \quad z \in \Gamma_{R_1}, \\ 0 < c_3 \leq |\psi'_1(w)| \leq c_4 < \infty, \quad |w| = R_1 \\ 0 < c_5 \leq |\varphi'_2(z)| \leq c_6 < \infty, \quad z \in L_{R_2}, \\ 0 < c_7 \leq |\psi'_2(w)| \leq c_8 < \infty, \quad |w| = R_2 \end{aligned} \tag{2.1}$$

$$0 < c_7 \leq |\psi'_2(w)| \leq c_8 < \infty, \quad |w| = R_2 \tag{2.2}$$

The proof of the following Lemma goes by a similar way to the proof of Lemma 1 in [14].

Lemma 2.1 *The following equivalences are true*

$$\begin{aligned} p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1}) \Leftrightarrow p_{1,1}(\cdot) \in \mathcal{P}_0(\mathbb{T}), \quad p_2(\cdot) \in \mathcal{P}_0(L_{R_2}) \Leftrightarrow p_{2,2}(\cdot) \in \mathcal{P}_0(\mathbb{T}), \\ f \in L^{p_1(\cdot)}(\Gamma_{R_1}) \Leftrightarrow f_1 \in L^{p_{1,1}(\cdot)}(\mathbb{T}), \quad f \in L^{p_2(\cdot)}(L_{R_2}) \Leftrightarrow f_2 \in L^{p_{2,2}(\cdot)}(\mathbb{T}). \end{aligned}$$

We mention that if Γ is a rectifiable Jordan curve and $f \in L^1(\Gamma)$, then the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\Gamma \setminus \Gamma(z, \varepsilon)} \frac{f(\zeta)}{\zeta - z} d\zeta := \frac{1}{2\pi i} (P.V) \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\Gamma(z, \varepsilon) := \{\zeta \in \Gamma : |\zeta - z| < \varepsilon\}$, existing for almost all $z \in \Gamma$ is called the Cauchy singular integral of f at $z \in \Gamma$.

For a given $f \in L^1(\Gamma)$ we associate the singular integral $S_\Gamma(f)$ taking the value $S_\Gamma(f)(z)$ a.e. on Γ . The linear operator S_Γ defined in such way is called the *Cauchy singular operator*. By [18] it is a bounded linear operator from $L^{p(\cdot)}(\Gamma)$ to $L^{p(\cdot)}(\Gamma)$.

If $f \in L^{p(\cdot)}(\Gamma), p(\cdot) \in \mathcal{P}_0(\Gamma)$, then the functions

$$\begin{aligned} f^+(z) &:= \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in int\Gamma, \\ f^-(z) &:= \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in ext\Gamma \end{aligned}$$

are analytic in $int\Gamma$ and $ext\Gamma$, respectively. According to Privalov’s theorem they have nontangential limits a.e. on Γ and the relations

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z) \quad \text{and} \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z) \tag{2.3}$$

are valid *a.e.* on Γ . Hence,

$$f(z) = f^+(z) - f^-(z)$$

holds *a.e.* on Γ .

Lemma 2.2 *If $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, then $\Omega(S_{\mathbb{T}}(f), \cdot)_{p(\cdot)} \leq c(p)\Omega(f, \cdot)_{p(\cdot)}$.*

Proof Let $\delta \in (0, \pi)$, $h < \delta$ and $w \in \mathbb{T}$. By Fubini's theorem

$$\begin{aligned} \sigma_h(S_{\mathbb{T}}(f)(w)) &= (1/h) \int_0^h S_{\mathbb{T}}(f)(we^{it}) dt \\ &= (1/h) \int_0^h \frac{1}{2\pi i} (P.V.) \left(\int_{\mathbb{T}} \frac{f(\tau e^{it}) d\tau}{\tau - w} \right) dt \\ &= \frac{1}{2\pi i} (P.V.) \int_{\mathbb{T}} \frac{(1/h) \int_0^h f(\tau e^{it}) dt}{\tau - w} d\tau \\ &= \frac{1}{2\pi i} (P.V.) \int_{\mathbb{T}} \frac{\sigma_h(f)(\tau)}{\tau - w} d\tau \\ &= S_{\mathbb{T}}(\sigma_h(f)(w)) \end{aligned}$$

and hence using the boundedness of singular operator $S_{\mathbb{T}}$ in $L^{p(\cdot)}(\mathbb{T})$, we have that

$$\begin{aligned} \|S_{\mathbb{T}}(f) - \sigma_h(S_{\mathbb{T}}(f))\|_{L^{p(\cdot)}(\mathbb{T})} &= \|S_{\mathbb{T}}(f - \sigma_h(f))\|_{L^{p(\cdot)}(\mathbb{T})} \\ &\leq c(p) \|f - \sigma_h(f)\|_{L^{p(\cdot)}(\mathbb{T})}, \end{aligned}$$

which implies the desired relation $\Omega(S_{\mathbb{T}}(f), \cdot)_{p(\cdot)} \leq c(p)\Omega(f, \cdot)_{p(\cdot)}$. □

In [14] were proved some direct theorems of approximation theory in the variable exponent Smirnov classes of analytic functions which in our terms can be formulated as follows:

Theorem 2.3 *If $f \in E^{p_1(\cdot)}(G_{R_1})$, $p_1(\cdot) \in \mathcal{P}_0(\Gamma_{R_1})$, then for $\forall n \in \mathbb{N}$ there is an algebraic polynomial $p_n(z, f)$ such that for some constant $c(p_1) > 0$ the inequality*

$$\|f - p_n(\cdot, f)\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \leq c(p_1)\Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}}$$

holds.

Theorem 2.4 *If $f \in E^{p_2(\cdot)}(B_{R_2}^-)$, $p_2(\cdot) \in \mathcal{P}_0(L_{R_2})$, then for $\forall n \in \mathbb{N}$ there is an algebraic polynomial $p_n(1/z, f)$ such that for some constant $c(p_2) > 0$ the inequality*

$$\|f - p_n(\cdot, f)\|_{L^{p_2(\cdot)}(L_{R_2})} \leq c(p_2)\Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}$$

holds.

The following theorem also will be used.

Theorem 2.5 *If $r_i > 1$ and $|w| \geq r > 1$, $i = 1, 2$, then the inequality*

$$\frac{1}{2\pi} \int_{|t|=r_i} \left| \frac{\psi'_i(t)}{\psi_i(t) - \psi_i(w)} - \frac{1}{t-w} \right| |dt| \leq \sqrt{\frac{r_i^2}{r_i^4 - 1} \ln \frac{r^2}{r^2 - 1}} \tag{2.4}$$

holds.

Note that Theorem 2.5 in the case of $i = 1$ was proved in [20, pp. 174], and in the case of $i = 2$ the proof goes by a similar way.

Using (1.7) and (1.8) in (1.11), for $z \in K$ we obtain

$$\begin{aligned} |R_n(z, f)| &= \left| f(z) - \sum_{k=0}^n a_k (f_{R_1}^+) \Phi_k^1(z) - \sum_{k=1}^n b_k (f_{R_2}^-) \Phi_k^2(1/z) \right| \\ &= \left| f_{R_1}^+(z) + f_{R_2}^-(z) - \sum_{k=0}^n a_k (f_{R_1}^+) \Phi_k^1(z) - \sum_{k=1}^n b_k (f_{R_2}^-) \Phi_k^2(1/z) \right| \\ &\leq |f_{R_1}^+(z) - \sum_{k=0}^n a_k (f_{R_1}^+) \Phi_k^1(z)| + |f_{R_2}^-(z) - \sum_{k=1}^n b_k (f_{R_2}^-) \Phi_k^2(1/z)| \\ &= \left| \sum_{k=n+1}^{\infty} a_k (f_{R_1}^+) \Phi_k^1(z) \right| + \left| \sum_{k=n+1}^{\infty} b_k (f_{R_2}^-) \Phi_k^2(1/z) \right| \\ &= : |R_n^1(z, f_{R_1}^+)| + |R_n^2(z, f_{R_2}^-)|. \end{aligned} \tag{2.5}$$

It is clear that

$$\begin{aligned} R_n^1(z, f_{R_1}^+) &= \sum_{k=n+1}^{\infty} a_k (f_{R_1}^+) \Phi_k^1(z) \\ &= \frac{1}{2\pi i} \int_{|t|=R_1} f_{R_1}^+(\psi(t)) \left[\sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} \right] dt \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} R_n^2(z, f_{R_2}^-) &= \sum_{k=n+1}^{\infty} b_k (f_{R_2}^-) \Phi_k^2(1/z) \\ &= \frac{1}{2\pi i} \int_{|t|=R_2} f_{R_2}^-(\psi_2(t)) \sum_{k=n+1}^{\infty} \frac{\Phi_k^2(1/z)}{t^{k+1}} dt. \end{aligned} \tag{2.7}$$

If p_n is a polynomial of degree at most n , then

$$R_n^1(z, f_{R_1}^+) = \frac{1}{2\pi i} \int_{|t|=R_1} [f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))] \sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} dt. \tag{2.8}$$

Since

$$\Phi_k^1(z) = [\varphi_1(z)]^k + E_k^1(z), \quad z \in K, \tag{2.9}$$

we have

$$\sum_{k=n+1}^{\infty} \frac{\Phi_k^1(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty} \frac{[\varphi_1(z)]^k}{t^{k+1}} + \sum_{k=n+1}^{\infty} \frac{E_k^1(z)}{t^{k+1}}. \tag{2.10}$$

Hence, from (2.8), taking into account (2.10), for $z = \psi_1(w)$ we get

$$\begin{aligned} |R_n^1(z, f_{R_1}^+)| &\leq \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt| \\ &+ \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} E_k^1(\psi_1(w)) \frac{1}{t^{k+1}} \right| |dt|. \end{aligned} \tag{2.11}$$

Similarly, using (1.1) we obtain

$$\begin{aligned} |R_n^2(z, f_{R_2}^-)| &\leq \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| dt \\ &+ \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}} \right| dt. \end{aligned} \tag{2.12}$$

We shall also use the relations [20, pp. 63]:

$$E_k^i(\psi_i(w)) = \frac{1}{2\pi i} \int_{|\tau|=r_i} \tau^k F_i(\tau, w) d\tau, \quad |w| \geq r_i > 1, \quad i = 1, 2 \tag{2.13}$$

where

$$F_i(\tau, w) := \frac{\psi_i'(\tau)}{\psi_i(\tau) - \psi_i(w)} - \frac{1}{\tau - w}, \quad |\tau| > 1, \quad |w| > 1.$$

3. Proof of main results

Proof [Proof of Theorem 1.6] Let $z \in \Gamma_{r_1}$, $1 < r_1 < R_1$ and p_n be the best approximating polynomial of degree at most n to $f_{R_1}^+ \in E^{p_1(\cdot)}(G_{R_1})$. Denoting

$$\begin{aligned} I_1 &:= \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt|, \\ I_2 &:= \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} E_k^1(\psi_1(w)) \frac{1}{t^{k+1}} \right| |dt| \end{aligned}$$

by virtue of (2.11), we see that

$$|R_n^1(z, f_{R_1}^+)| \leq I_1 + I_2. \tag{3.1}$$

Using relations (2.1), (1.12) and applying Hölder’s inequality [3, pp. 27], we have

$$\begin{aligned}
 I_1 &= \frac{1}{2\pi} \int_{\Gamma_{R_1}} |f_{R_1}^+(\zeta) - p_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}} \right| |\varphi_1'(\zeta)| |d\zeta| \\
 &\leq c_1 \int_{\Gamma_{R_1}} |f_{R_1}^+(\zeta) - p_n(\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}} \right| |d\zeta| \\
 &\leq c_2(p_1) \|f_{R_1}^+(\zeta) - p_n(\zeta)\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \left\| \sum_{k=n+1}^{\infty} \frac{[\varphi_1(z)]^k}{[\varphi_1(\zeta)]^{k+1}} \right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})} \\
 &\leq c_3(p_1) E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)} \left\| \frac{|\varphi_1(z)|^{n+1} |\varphi_1(\cdot)|^{-n-1}}{|\varphi_1(\cdot)| - |\varphi_1(z)|} \right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})} \\
 &\leq \frac{c_4(p_1) r_1^{n+1}}{R_1^{n+1} (R_1 - r_1)} E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}. \tag{3.2}
 \end{aligned}$$

Now, we estimate the integral I_2 . By (2.13) we have

$$\begin{aligned}
 I_2 &= \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k^1(\psi_1(w))}{t^{k+1}} \right| |dt| \\
 &= \frac{1}{2\pi} \int_{|t|=R_1} \left\{ |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \left| \sum_{k=n+1}^{\infty} \left[\frac{1}{2\pi i} \int_{|\tau|=r_1} \tau^k F_1(\tau, w) d\tau \right] \frac{1}{t^{k+1}} \right| \right\} |dt| \\
 &\leq \frac{1}{2\pi} \int_{|t|=R_1} \left\{ |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \frac{1}{2\pi} \int_{|\tau|=r_1} \left| \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} \right| |F_1(\tau, w)| |d\tau| \right\} |dt| \\
 &= \frac{1}{2\pi} \int_{|t|=R_1} \left\{ |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \frac{1}{2\pi} \int_{|\tau|=r_1} \left| \frac{\tau^{n+1}}{t^{n+1} (t - \tau)} \right| |F_1(\tau, w)| |d\tau| \right\} |dt|.
 \end{aligned}$$

By Fubini’s theorem

$$I_2 \leq \frac{r_1^{n+1}}{2\pi R_1^{n+1}} \int_{|\tau|=r_1} \left\{ |F_1(\tau, w)| \frac{1}{2\pi} \int_{|t|=R_1} |f_{R_1}^+(\psi_1(t)) - p_n(\psi_1(t))| \frac{|dt|}{|t - \tau|} \right\} |d\tau|$$

and then changing the variables and using Hölder’s inequality we obtain

$$\begin{aligned}
 I_2 &\leq \frac{r_1^{n+1}}{2\pi R_1^{n+1}} \int_{|\tau|=r_1} \left\{ |F_1(\tau, w)| \frac{1}{2\pi} \int_{\Gamma_{R_1}} |f_{R_1}^+(\zeta) - p_n(\zeta)| \frac{|\varphi_1'(\zeta)| |d\zeta|}{|\varphi_1(\zeta) - \varphi_1(z)|} \right\} |d\tau| \\
 &\leq \frac{c_5(p_1) r_1^{n+1}}{R_1^{n+1}} \int_{|\tau|=r_1} |F_1(\tau, w)| \left\{ \|f_{R_1}^+(\zeta) - p_n(\zeta)\|_{L^{p_1(\cdot)}(\Gamma_{R_1})} \left\| \frac{\varphi_1'(\zeta)}{\varphi_1(\zeta) - \varphi_1(z)} \right\|_{L^{q_1(\cdot)}(\Gamma_{R_1})} \right\} |d\tau| \\
 &\leq \frac{c_6(p_1) r_1^{n+1}}{R_1^{n+1} (R_1 - r_1)} \int_{|\tau|=r_1} |F_1(\tau, w)| E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)} |d\tau|.
 \end{aligned}$$

From this, by using (2.4), we have that

$$I_2 \leq \frac{c_7(p_1)r_1^{n+1}}{R_1^{n+1}(R_1-r_1)} E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)} \sqrt{\frac{r_1^2}{r_1^4-1} \ln \frac{r_1^2}{r_1^2-1}}. \tag{3.3}$$

Now, the inequalities (3.1)–(3.3) imply that

$$|R_n^1(z, f_{R_1}^+)| \leq \frac{c_8(p_1)r_1^{n+1} E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}}{R_1^{n+1}(R_1-r_1)} \sqrt{\frac{r_1^2}{r_1^4-1} \ln \frac{r_1^2}{r_1^2-1}}.$$

Consequently, setting $z \in K$ and $r_1 = 1 + \frac{1}{n}$ in this estimate, we obtain the inequality

$$|R_n^1(z, f_{R_1}^+)| \leq \frac{c_9(p_1) E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}}{R_1^{n+1}(R_1-1)} \sqrt{n \ln n} \tag{3.4}$$

with $c_9(p_1) > 0$. Now, let $z \in \Gamma_{r_2}$, $1 < r_2 < R_2$. Denoting

$$I_1^* := \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| dt$$

$$I_2^* := \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}} \right| dt$$

by virtue of (2.12), we see that

$$|R_n^2(z, f_{R_2}^-)| \leq I_1^* + I_2^*. \tag{3.5}$$

Using (2.2) and Hölder’s inequality, we have

$$\begin{aligned} I_1^* &= \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{w^k}{t^{k+1}} \right| |dt| \\ &= \frac{1}{2\pi} \int_{L_{R_2}} |f_{R_2}^-(\zeta) - p_n(1/\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\zeta)]^{k+1}} \right| |\varphi_2'(\zeta)| |d\zeta| \\ &\leq c_{10} \int_{L_{R_2}} |f_{R_2}^-(\zeta) - p_n(1/\zeta)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\zeta)]^{k+1}} \right| |d\zeta| \\ &\leq c_{11}(p_2) \|f_{R_2}^-(\cdot) - p_n(1/\cdot)\|_{L^{p_2(\cdot)}(L_{R_2})} \left\| \sum_{k=n+1}^{\infty} \frac{[\varphi_2(z)]^k}{[\varphi_2(\cdot)]^{k+1}} \right\|_{L^{q_2(\cdot)}(L_{R_2})} \\ &\leq c_{12}(p_2) \|f_{R_2}^-(\cdot) - p_n(1/\cdot)\|_{L^{p_2(\cdot)}(L_{R_2})} \left\| \frac{|\varphi_2(z)|^{n+1}}{|\varphi_2(\cdot)|^{n+1} (|\varphi_2(\cdot)| - |\varphi_2(z)|)} \right\|_{L^{q_2(\cdot)}(L_{R_2})} \\ &\leq c_{13}(p_2) E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)} \frac{r_2^{n+1}}{R_2^{n+1}(R_2-r_2)}. \end{aligned} \tag{3.6}$$

Now, we estimate the integral I_2^* . By (2.13) we have

$$\begin{aligned} I_2^* &= \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \left| \sum_{k=n+1}^{\infty} \frac{E_k^2(\psi_2(w))}{t^{k+1}} \right| |dt| \\ &= \frac{1}{2\pi} \int_{|t|=R_2} \left\{ |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \sum_{k=n+1}^{\infty} \frac{1}{2\pi i} \int_{|\tau|=r_2} \frac{\tau^k}{t^{k+1}} F_2(\tau, w) d\tau \right\} |dt| \\ &\leq \frac{1}{2\pi} \int_{|t|=R_2} \left\{ |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \frac{1}{2\pi} \int_{|\tau|=r_2} \left| \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} \right| |F_2(\tau, w)| |d\tau| \right\} |dt| \\ &= \frac{1}{2\pi} \int_{|t|=R_2} \left\{ |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \frac{1}{2\pi} \int_{|\tau|=r_2} \left| \frac{\tau^{n+1}}{t^{n+1}(t-\tau)} \right| |F_2(\tau, w)| |d\tau| \right\} |dt|. \end{aligned}$$

By Fubini's theorem

$$I_2^* \leq \frac{1}{2\pi} \int_{|\tau|=r_2} \left\{ |\tau|^{n+1} |F_2(\tau, w)| \frac{1}{2\pi} \int_{|t|=R_2} |f_{R_2}^-(\psi_2(t)) - p_n(1/\psi_2(t))| \frac{|t|^{-n-1}}{|t-\tau|} |dt| \right\} |d\tau|$$

and hence changing the variables and using Hölder's inequality, we obtain

$$\begin{aligned} I_2^* &\leq \frac{r_2^{n+1}}{2\pi R_2^{n+1}} \int_{|\tau|=r_2} \left\{ |F_2(\tau, w)| \frac{1}{2\pi} \int_{L_{R_2}} |f_{R_2}^-(\zeta) - p_n(1/\zeta)| \frac{|\varphi_2'(\zeta)|}{|\varphi_2(\zeta) - \varphi_2(z)|} |d\zeta| \right\} |d\tau| \\ &\leq \frac{c_{14}(p_2)r_2^{n+1}}{R_2^{n+1}} \int_{|\tau|=r_2} \left\{ |F_2(\tau, w)| \|f_{R_2}^-(\cdot) - p_n(1/\cdot)\|_{L^{p_2(\cdot)}(L_{R_2})} \left\| \frac{|\varphi_2'(\cdot)|}{|\varphi_2(\cdot) - \varphi_2(z)|} \right\|_{L^{q_2(\cdot)}(L_{R_2})} \right\} |d\tau|. \end{aligned}$$

From this, using Theorem 2.5, in the case of $i = 2$ we have

$$I_2^* \leq \frac{c_{15}(p_2)r_2^{n+1}}{R_2^{n+1}(R_2 - r_2)} E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)} \sqrt{\frac{r_2^2}{r_2^4 - 1} \ln \frac{r_2^2}{r_2^2 - 1}},$$

which, in combination with (3.5) and (3.6), implies that

$$|R_n^2(z, f_{R_2}^-)| \leq \frac{c_{16}(p_2)r_2^{n+1} E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - r_2)} \sqrt{\frac{r_1^2}{r_1^4 - 1} \ln \frac{r_1^2}{r_1^2 - 1}}.$$

Setting in this estimate $r_2 = 1 + \frac{1}{n}$, we obtain for $z \in K$ the inequality

$$|R_n^2(z, f_{R_2}^-)| \leq \frac{c_{17}(p_2) E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - 1)} \sqrt{n \ln n} \tag{3.7}$$

with $c_{17}(p_2) > 0$. Finally, combining the relations (3.4), (3.7), and (2.5) we have

$$|R_n(z, f)| \leq c(p_1, p_2) \sqrt{n \ln n} \left[\frac{E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}}{R_1^{n+1}(R_1 - 1)} + \frac{E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - 1)} \right].$$

□

Now we can proof Theorem 1.8.

Proof [Proof of Theorem 1.8] From Theorem 1.6, 2.3 and 2.4 we have

$$\begin{aligned} |R_n(z, f)| &\leq c(p_1, p_2) \sqrt{n \ln n} \left[\frac{E_n(f_{R_1}^+, G_{R_1})_{p_1(\cdot)}}{R_1^{n+1}(R_1 - 1)} + \frac{E_n(f_{R_2}^-, B_{R_2}^-)_{p_2(\cdot)}}{R_2^{n+1}(R_2 - 1)} \right] \\ &\leq c_1(p_1, p_2) \sqrt{n \ln n} \left[\frac{\Omega(f_{R_1}^+ \circ \varphi_1^{-1}, 1/n)_{p_{1,1}(\cdot), \mathbb{T}}}{R_1^{n+1}(R_1 - 1)} + \frac{\Omega(f_{R_2}^- \circ \varphi_2^{-1}, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}}{R_2^{n+1}(R_2 - 1)} \right]. \end{aligned} \tag{3.8}$$

Using the subadditivity property of modulus and (2.3), and also Lemma (2.2), we obtain that

$$\begin{aligned} \Omega(f_{R_1}^+ \circ \varphi_1^{-1}, 1/n)_{p_{1,1}(\cdot), \mathbb{T}} &= \Omega(f_1/2 + S_{\mathbb{T}}(f_1), 1/n)_{p_{1,1}(\cdot), \mathbb{T}} \\ &\leq c_{18}(p_{1,1}) \left[\Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}} + \Omega(S_{\mathbb{T}}(f_1), 1/n)_{p_{1,1}(\cdot), \mathbb{T}} \right] \\ &\leq c_{19}(p_{1,1}) \Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}} \end{aligned}$$

and

$$\begin{aligned} \Omega(f_{R_2}^- \circ \varphi_2^{-1}, 1/n)_{p_{2,2}(\cdot), \mathbb{T}} &= \Omega(f_2/2 - S_{\mathbb{T}}(f_2), 1/n)_{p_{2,2}(\cdot), \mathbb{T}} \\ &\leq c_{20}(p_{2,2}) \left[\Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}} + \Omega(S_{\mathbb{T}}(f_2), 1/n)_{p_{2,2}(\cdot), \mathbb{T}} \right] \\ &\leq c_{21}(p_{2,2}) \Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}. \end{aligned}$$

Now, these inequalities and (3.8) imply that

$$|R_n(z, f)| \leq c(p_{1,1}, p_{2,2}) \sqrt{n \ln n} \left[\frac{\Omega(f_1, 1/n)_{p_{1,1}(\cdot), \mathbb{T}}}{R_1^{n+1}(R_1 - 1)} + \frac{\Omega(f_2, 1/n)_{p_{2,2}(\cdot), \mathbb{T}}}{R_2^{n+1}(R_2 - 1)} \right].$$

□

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