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Global existence and blow-up of solutions of the time-fractional space-involution reaction-diffusion equation

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Abstract: A time-fractional space-nonlocal reaction-diffusion equation in a bounded domain is considered. First, the existence of a unique local mild solution is proved. Applying Poincaré inequality it is obtained the existence and boundedness of global classical solution for small initial data. Under some conditions on the initial data, we show that solutions may experience blow-up in a finite time.

Key words: Caputo derivative, reaction-diffusion equation, involution, global existence, blow-up

1. Introduction

The purpose of this paper is to study Cauchy problem for the time-fractional space-nonlocal reaction-diffusion equation

$$\partial^{\alpha}_{+0,t}u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(1-x,t) = u(x,t)(u(x,t)-1),\; x \in (0,1), t > 0, \tag{1.1}$$

supplemented with boundary conditions

$$u(0,t) = 0,\; u(1,t) = 0,\; t \geq 0, \tag{1.2}$$

and initial condition

$$u(x,0) = u_0(x),\; x \in [0,1], \tag{1.3}$$

where $\varepsilon \in \mathbb{R}$, $\partial^{\alpha}_{+0,t}$ is the Caputo fractional derivative of order $\alpha \in (0,1]$ (see. Def. 1.3).

When $0 < \alpha < 1$ and $\varepsilon = 0$, equation (1.1) is the time-fractional reaction-diffusion equation. When $\alpha = 1, \varepsilon = 0$, it represents the classical reaction-diffusion equation. Let us mention that with the change of variable $v := 1-u$, (1.1) is transformed to the Fisher equation, if $\alpha = 1, \varepsilon = 0$.

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [7, 14]. Furthermore, for the equations containing transformation of the spatial variable in the diffusion term, we can cite Cabada and Tojo [8], where an example that describes a concrete situation in physics is given. Note that,

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the direct and inverse problems for diffusion and fractional diffusion equations with involutions were studied in [4, 5, 12, 13].

Our paper is motivated by the recent paper [1] in which the authors considered the questions of global solutions and blowing-up solutions to the equation (1.1), when $\varepsilon = 0$. Our problem (1.1)-(1.3) is a simple generalization of results in [1]. We will prove the existence of globally bounded solutions, as well as blowing-up solutions, according to the condition imposed on the initial data. Note that, similar studies for time-fractional reaction-diffusion equations were considered in [3, 9].

Thus, let us briefly summarise the results of this paper:

- **Existence of local mild solution** Suppose that $|\varepsilon| < 1$ and $u_0 \in C([0,1])$, then there exists a unique local mild solution $u \in C([0,1], C(0, T_{max}))$ of problem (1.1)-(1.3) with the alternative:
  - either $T_{max} = +\infty$;
  - or $T_{max} < +\infty$ and $\lim\limits_{t \to T_{max}} \|u(t)\|_{L^\infty([0,1])} = +\infty$.
- **Existence of global classical solution** Let $|\varepsilon| < 1$ and $u_0(x) \in C([0,1])$ satisfy the estimates $0 \leq u_0(x) \leq 1$. Then problem (1.1)-(1.3) admits a global classical solution

$$u \in C^{2,1}((0,1) \times \mathbb{R}_+) \cap C([0,1] \times \mathbb{R}_+),$$

  that satisfies

$$0 \leq u(x,t) \leq 1 \quad \text{for} \quad (x,t) \in [0,1] \times \mathbb{R}_+.$$

- **Large time behavior of global solutions** Assume that $|\varepsilon| < 1$, $0 \leq u_0 \leq 1$ and $u_0 \in C([0,1])$. Then the global classical solution $0 \leq u \leq 1$ of nonlocal reaction-diffusion problem (1.1)-(1.3) satisfies the following estimate

$$\|u(t,\cdot)\|^2_{L^2([0,1])} \leq \frac{\|u_0\|^2_{L^2([0,1])}}{1 + \frac{(1-\varepsilon)\pi^2}{\Gamma(1+\alpha)} t^\alpha}, \quad t \geq 0. \tag{1.4}$$

- **Blow-up of solution** Let $|\varepsilon| < 1$. If $1 + (1-\varepsilon)\pi^2 \leq \sqrt{2}\int\limits_0^1 u_0(x) \sin \pi x dx = F_0$, then the classical solution of problem (1.1)-(1.3) blows-up in a finite time

$$\left(\frac{\Gamma(\alpha+1)}{4(F_0 - 1/2 - (1-\varepsilon)\pi^2)}\right)^{\frac{1}{\alpha}} \leq T^* \leq \left(\frac{\Gamma(\alpha+1)}{F_0 - 1 - (1-\varepsilon)\pi^2}\right)^{\frac{1}{\alpha}}.$$

1.1. Preliminaries

Let us give basic definitions of fractional differentiation and integration of the Riemann–Liouville and Caputo types.

Definition 1.1 *[11] Let f be an integrable real-valued function on the interval $[a,b]$, $-\infty < a < b < +\infty$. The following integral*

$$I^\alpha_{a+}[f](t) = (f * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_a^t (t-s)^{\alpha-1} f(s) ds$$

is called the Riemann–Liouville integral operator of the fractional order $\alpha > 0$. *Here* $K_\alpha = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, Γ *denotes the Euler gamma function.*

Definition 1.2 *[11] Let* $f \in L^1([a,b])$ *and* $f * K_{1-\alpha} \in W^{1,1}([a,b])$, *where* $W^{1,1}([a,b])$ *is the Sobolev space defined as*

$$W^{1,1}([a,b]) = \left\{ f \in L^1([a,b]) : \frac{d}{dt} f \in L^1([a,b]) \right\}.$$

The Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ *is defined as*

$$D^\alpha_{a+}[f](t) = \frac{d}{dt} I^{1-\alpha}_{a+}[f](t).$$

Definition 1.3 *[11] Let* $f \in L^1([a,b])$ *and* $f * K_{1-\alpha} \in W^{1,1}([a,b])$. *For* $0 < \alpha < 1$, *the fractional derivative*

$$\mathcal{D}^\alpha_{a+}[f](t) = D^\alpha_{a+}[f(t) - f(a)]$$

is the differential operator of the fractional order α $(0 < \alpha < 1)$ *in the Caputo sense.*

If $f \in C^1([a,b])$, *then the Caputo fractional derivative is defined as*

$$\mathcal{D}^\alpha_{a+}[f](t) = I^{1-\alpha}_{a+} f'(t).$$

Proposition 1.4 *[2] Let* $v \in C^1([0,T])$. *Then*

$$2v(t)\partial^\alpha_{+0} v(t) \geq \partial^\alpha_{+0} v^2(t).$$

1.2. Finite time blow-up of solutions of a fractional differential equation

We consider the fractional differential equation

$$\begin{aligned} &\partial^\alpha_{0+} y(t) = y^2(t), \ t > 0, \ 0 < \alpha < 1, \\ &y(0) = y_0 \in \mathbb{R}. \end{aligned} \tag{1.5}$$

The blow-up of solutions to (1.5) is assured by the following.

Proposition 1.5 *[10] If* $y_0 > 0$, *then the solution of problem* (1.5) *blows-up in a finite time*

$$\left(\frac{\Gamma(\alpha+1)}{4(y_0 + 1/2)} \right)^{\frac{1}{\alpha}} \leq T^* \leq \left(\frac{\Gamma(\alpha+1)}{y_0} \right)^{\frac{1}{\alpha}},$$

that is $\lim_{t \to T^*} u(t) = +\infty$.

1.3. Poincaré inequality for the differential operator with involution

We consider the following eigenvalue problem

$$\begin{aligned} &-e''(x) + \varepsilon e''(1-x) = \lambda e(x), \ x \in (0,1), \\ &e(0) = 0, e(1) = 0, \end{aligned} \tag{1.6}$$

where $\varepsilon \in \mathbb{R}$.

Proposition 1.6 *[13] Let $|\varepsilon| < 1$. Then, the eigenvalue problem (1.6) is a selfadjoint in $L^2([0,1])$ and it has the eigenvalues*

$$\lambda_k = (1 + (-1)^k \varepsilon)\pi^2 k^2,\ k \in \mathbb{N},$$

and corresponding eigenfunctions

$$e_k(x) = \sqrt{2}\sin \pi k x,\ k \in \mathbb{N}$$

which form a complete orthonormal basis in $L^2([0,1])$.

Below we give the Poincaré inequality for the eigenvalue problem (1.6).

Proposition 1.7 *Let $|\varepsilon| < 1$. Then the following inequality is true*

$$\int_0^1 |e'(x)|^2 dx \geq \pi^2 \int_0^1 e^2(x)dx \geq 0. \tag{1.7}$$

Proof Multiplying scalarly in $L^2([0,1])$ equation (1.6) by $e(x)$ and integrating by part, we obtain

$$\begin{aligned} -\lambda \int_0^1 e^2(x)dx &= \int_0^1 e''(x)e(x)dx - \varepsilon \int_0^1 e''(1-x)e(x)dx \\ &= -\int_0^1 e'(x)e'(x)dx + \varepsilon \int_0^1 e'(1-x)e'(x)dx \\ &\leq -\int_0^1 e'(x)e'(x)dx + |\varepsilon| \int_0^1 e'(1-x)e'(x)dx \\ &\leq -\int_0^1 |e'(x)|^2 dx + |\varepsilon| \left(\int_0^1 |e'(x)|^2 dx\right)^{1/2} \left(\int_0^1 |e'(1-x)|^2 dx\right)^{1/2} \\ &= -(1-|\varepsilon|)\int_0^1 |e'(x)|^2 dx, \end{aligned}$$

thanks to Cauchy-Schwarz inequality, that is

$$\lambda \int_0^1 e^2(x)dx \leq (1-|\varepsilon|)\int_0^1 |e'(x)|^2 dx.$$

Since $\lambda \geq \lambda_1 = (1-\varepsilon)\pi^2$, we have

$$\int_0^1 |e'(x)|^2 dx \geq \frac{(1-\varepsilon)}{(1-|\varepsilon|)}\pi^2 \int_0^1 e^2(x)dx \geq \pi^2 \int_0^1 e^2(x)dx.$$

The proof is complete. □

2. Existence of local mild solutions

By $\mathcal{L}$ we denote L^2 realization of the operator (1.6), is given by the standard operator calculus for selfadjoint operators.

Definition 2.1 *Let* $u_0 \in C([0,1])$ *and* $T_{max} > 0$. *We say* $u \in C([0,1], C(0,T_{max}))$ *is a mild solution of* (1.1)-(1.3) *if* u *satisfies the following integral equation*

$$u(t) = E_{\alpha,1}(-t^{\alpha}\mathcal{L})u_0 + \int\limits_0^t s^{\alpha-1}E_{\alpha,\alpha}(-s^{\alpha}\mathcal{L})f(u(t-s))ds,\ t \in (0,T_{max}), \tag{2.1}$$

where $f(u(s)) = u(s)(u(s)-1)$ *and* $E_{\alpha,\beta}(z)$ *is the Mittag-Leffler function (see, e.g. [11]):*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

Theorem 2.2 *Suppose that* $u_o \in C([0,1])$, *then there exists a unique local mild solution* $u \in C([0,1], C(0,T_{max}))$ *of problem* (1.1)-(1.3) *with the alternative:*

- *either* $T_{max} = +\infty$;
- *or* $T_{max} < +\infty$ *and* $\lim\limits_{t\to T_{max}} \|u(t)\|_{L^\infty([0,1])} = +\infty$.

Proof The following properties [6]

$$0 < E_{\alpha,1}(-z) \le 1,\ 0 < E_{\alpha,\alpha}(-z) \le \frac{1}{\Gamma(\alpha)},\ z \ge 0,\ 0 < \alpha \le 1,$$

implies that

$$\|E_{\alpha,1}(-t^{\alpha}\mathcal{L})u_0\|_{L^\infty([0,1])} \le \|u_0\|_{L^\infty([0,1])} \tag{2.2}$$

and

$$\|E_{\alpha,\alpha}(-t^{\alpha}\mathcal{L})u_0\|_{L^\infty([0,1])} \le \frac{1}{\Gamma(\alpha)}\|u_0\|_{L^\infty([0,1])}. \tag{2.3}$$

The proof is based on Banach fixed point theorem. Let us define the following Banach space

$$\mathcal{B} = \left\{u \in C([0,\tau] : C([0,1])) : \sup_{t\in[0,\tau]}\|u(t)\|_{L^\infty([0,1])} \le 2\|u_0\|_{L^\infty([0,1])},\right\}$$

where τ will be determined later. We consider the equation

$$\mathcal{I}u(t) \equiv E_{\alpha,1}(-t^{\alpha}\mathcal{L})u_0 + \int\limits_0^t s^{\alpha-1}E_{\alpha,\alpha}(-s^{\alpha}\mathcal{L})f(u(t-s))ds = u(t),\ t \in [0,\tau].$$

Note that $f(u(t-s)) = u(t-s)(u(t-s)-1)$ is locally Lipschitzian function.

Firstly, we need to show that $\mathcal{I}:\mathcal{B}\to\mathcal{B}$. Let there exist τ such that the following inequality holds

$$\|f(u(t))\|_{L^\infty([0,1])}\leq M\|u(t)\|_{L^\infty([0,1])},\ M>0. \tag{2.4}$$

If $u\in\mathcal{B}$, then by (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}\|\mathcal{I}u(t)\|_{L^\infty([0,\tau],L^\infty([0,1]))}&\leq \|E_{\alpha,1}(-t^\alpha\mathcal{L})u_0\|_{L^\infty([0,\tau],L^\infty([0,1]))}\\&+\frac{\tau^\alpha}{\alpha}\|E_{\alpha,\alpha}(-s^\alpha\mathcal{L})f(u)\|_{L^\infty([0,\tau];L^\infty([0,1]))}\\&\leq\|u_0\|_{L^\infty([0,1])}+\frac{2M\tau^\alpha}{\Gamma(\alpha+1)}\|u_0\|_{L^\infty([0,1])}.\end{aligned}$$

Now, choosing τ small enough such that $\tau^\alpha\leq\frac{\Gamma(\alpha+1)}{2M}$, we conclude that

$$\|\mathcal{I}u(t)\|_{L^\infty([0,\tau];L^\infty([0,1]))}\leq 2\|u_0\|_{L^\infty([0,1])},$$

and then $\mathcal{I}u(t)\in\mathcal{B}$.

Next, we show that $\mathcal{I}$ is a contraction map. Letting $u,v\in\mathcal{B}$, we have

$$\begin{aligned}\|\mathcal{I}u(t)-\mathcal{I}v(t)\|_{L^\infty([0,1])}&\leq\frac{t^\alpha}{\Gamma(\alpha+1)}\sup_{0\leq s\leq t}\|f(u(t-s))-f(v(t-s))\|_{L^\infty([0,1])}\\&\leq\frac{t^\alpha M}{\Gamma(\alpha+1)}\sup_{0\leq s\leq t}\|u(t-s)-v(t-s)\|_{L^\infty([0,1])}\end{aligned}$$

thanks to the locally Lipschitz property of function f. Consequently

$$\|\mathcal{I}u-\mathcal{I}v\|_{L^\infty([0,\tau],L^\infty([0,1]))}\leq\frac{\tau^\alpha M}{\Gamma(\alpha+1)}\|u-v\|_{L^\infty([0,\tau],L^\infty([0,1]))}$$

Chosen τ so that $\frac{\tau^\alpha M}{\Gamma(\alpha+1)}<1$, we conclude that $\mathcal{I}$ the contraction map on $\mathcal{B}$. So, by the Banach fixed point theorem, problem (2.1) admits a unique mild solution $u\in\mathcal{B}$. □

3. Existence of global solutions

Theorem 3.1 *Let* $|\varepsilon|<1$ *and* $u_0(x)\in C([0,1])$ *satisfy the estimates* $0\leq u_0(x)\leq 1$. *Then problem* (1.1)-(1.3) *admits a global classical solution*

$$u\in C^{2,1}((0,1)\times\mathbb{R}_+)\cap C([0,1]\times\mathbb{R}_+),$$

that satisfies

$$0\leq u(x,t)\leq 1 \ \ for\ \ (x,t)\in[0,1]\times\mathbb{R}_+.$$

Proof Firstly, we show that $u\geq 0$. Multiplying scalarly in $L^2([0,1])$ equation (1.1) by $\tilde{u}:=\min(u,0)$, we obtain

$$\begin{aligned}\int_0^1\partial^\alpha_{+0,t}\tilde{u}(x,t)\cdot\tilde{u}(x,t)dx-\int_0^1\tilde{u}_{xx}(x,t)\cdot\tilde{u}(x,t)dx+\varepsilon\int_0^1\tilde{u}_{xx}(1-x,t)\cdot\tilde{u}(x,t)dx\\=\int_0^1\tilde{u}^2(x,t)\left(\tilde{u}(x,t)-1\right)dx.\end{aligned}$$

Integrating by part and using Poincaré's inequality (1.7) imply

$$-\int_0^1 \tilde{u}_{xx}(x,t)\cdot\tilde{u}(x,t)dx+\varepsilon\int_0^1 \tilde{u}_{xx}(1-x,t)\cdot\tilde{u}(x,t)dx\leq 0$$

for $|\varepsilon|<1$. Then, using Proposition 1.4, we have

$$\partial^{\alpha}_{+0,t}\int_0^1 \tilde{u}^2(x,t)\,dx\lesssim\int_0^1 \tilde{u}^2(x,t)\,dx. \tag{3.1}$$

By denoting $\int_0^1 \tilde{u}^2(x,t)\,dx=E(t)$ in (3.1), we obtain

$$\begin{cases}\partial^{\alpha}_{+0}E(t)\lesssim E(t),\\ E(0)=0,\end{cases}$$

which implies $\int_0^1 \tilde{u}^2(x,t)\,dx=0$. Consequently $u\geq 0$.

Now we show that $u\leq 1$. Multiplying scalarly in $L^2([0,1])$ equation (1.1) by $\hat{u}:=\min(1-u,0)$, we get

$$\int_0^1 \partial^{\alpha}_{+0,t}\hat{u}(x,t)\cdot\hat{u}(x,t)dx-\int_0^1 \hat{u}_{xx}(x,t)\cdot\hat{u}(x,t)dx+\varepsilon\int_0^1 \hat{u}_{xx}(1-x,t)\cdot\hat{u}(x,t)dx$$
$$=\int_0^1 \hat{u}^2(x,t)\,(\hat{u}(x,t)-1)\,dx.$$

As the above calculations, for the function $\hat{u}:=\min(1-u,0)$ we have

$$\partial^{\alpha}_{+0,t}\int_0^1 \hat{u}^2(x,t)\,dx\lesssim\int_0^1 \hat{u}^2(x,t)\,dx.$$

Hence $\int_0^1 \hat{u}^2(x,t)\,dx=0$, which implies $u\leq 1$. The result follows as $0\leq u\leq 1$. □

3.1. Large time behavior of global solutions

Theorem 3.2 *Assume that $0\leq u_0\leq 1$ and $u_0\in C([0,1])$. Then the global classical solution $0\leq u\leq 1$ of nonlocal reaction-diffusion problem (1.1)-(1.3) satisfies the following estimate*

$$\|u(t,\cdot)\|^2_{L^2([0,1])}\leq\frac{\|u_0\|^2_{L^2([0,1])}}{1+\frac{(1-\varepsilon)\pi^2}{\Gamma(1+\alpha)}t^{\alpha}},\ \ t\geq 0. \tag{3.2}$$

Proof As $0 \le u \le 1$, the right hand side of (1.1) satisfies $-u+u^2<0$, so u satisfies

$$\begin{aligned}&\partial^{\alpha}_{+0,t}u(x,t)-u_{xx}(x,t)+\varepsilon u_{xx}(1-x,t)\le 0, x\in(0,1),\ t>0,\\&u(0,t)=0,\ u(1,t)=0,\ t\ge 0,\\&u(x,0)=u_0(x),\ \ x\in[0,1],\end{aligned}$$

By multiplying scalarly in $L^2([0,1])$ equation (1.1) by u and using Poincaré's inequality (1.7), we obtain

$$\begin{aligned}&\partial^{\alpha}E(t)+(1-\varepsilon)\pi^2E(t)\le 0,\ t>0,\\&E(0)=E_0=\|u_0\|^2_{L^2([0,1])}\ge 0.\end{aligned}$$

where $E(t)=\int\limits_0^1 u^2(x,t)dx$. Let $\bar{E}(t)$ be the solution of problem

$$\partial^{\alpha}\bar{E}(t)+(1-\varepsilon)\pi^2\bar{E}(t)=0,\ t>0,\ \ \bar{E}(0)=E_0\ge 0,$$

which has the unique solution

$$E(t)\le E_0E_{\alpha}(-(1-\varepsilon)\pi^2t^{\alpha}),\ t\ge 0,$$

where $E_{\alpha,1}(z)$ is the Mittag-Leffler function. Since $E(t)\le\bar{E}(t)$, then, using the following estimate for the Mittag-Leffler function (see [6])

$$E_{\alpha}(-z)\le\frac{1}{1+\frac{1}{\Gamma(1+\alpha)}z},\ z\ge 0,\ 0<\alpha\le 1,$$

we have

$$E(t)\le\frac{E_0}{1+\frac{(1-\varepsilon)\pi^2}{\Gamma(1+\alpha)}t^{\alpha}},\ t\ge 0.$$

The proof is complete. □

4. Blow-up of solutions

In [13] it was proved that the first eigenvalue and the first eigenfunction of problem (1.6), respectively, have the form $\lambda_1=(1-\varepsilon)\pi^2$ and $e_1(x)=\sqrt{2}\sin\pi x$, where $\int\limits_0^1 e_1(x)dx=1$.

Theorem 4.1 *If* $1+(1-\varepsilon)\pi^2\le\int\limits_0^1 u_0(x)e_1(x)dx=F_0$, *then the classical solution of problem* (1.1)-(1.3) *blows-up in a finite time*

$$\left(\frac{\Gamma(\alpha+1)}{4(F_0-1/2-(1-\varepsilon)\pi^2)}\right)^{\frac{1}{\alpha}}\le T^*\le\left(\frac{\Gamma(\alpha+1)}{F_0-1-(1-\varepsilon)\pi^2}\right)^{\frac{1}{\alpha}}.$$

Proof Multiplying equation (1.1) by $e_1(x)$ and integrating over $[0,1]$, leads to

$$\begin{aligned}&\partial^\alpha \int_0^1 u(x,t)e_1(x)dx + \int_0^1 u_{xx}(x,t)e_1(x)dx - \varepsilon \int_0^1 u_{xx}(1-x,t)e_1(x)dx \\ &= \int_0^1 u(x,t)\left(u(x,t)-1\right)e_1(x)dx.\end{aligned} \tag{4.1}$$

Let us set $F(t) = \int_0^1 u(x,t)e_1(x)dx$.

Since

$$\begin{aligned}\int_0^1 (u_{xx}(x,t) - \varepsilon u_{xx}(1-x,t))e_1(x)dx &= \int_0^1 u(x,t)(e_1''(x) - \varepsilon e_1''(1-x))dx \\ &= -\lambda_1 \int_0^1 u(x,t)e_1(x)dx,\end{aligned}$$

for $u(0,t) = u(1,t) = 0$, $e_1(0) = e_1(1) = 0$, and

$$F^2(t) \le \int_0^1 u^2(x,t)e_1(x)dx$$

via Hölder's inequality, we have for (4.1) that

$$\partial^\alpha F(t) + (1+\lambda_1)F(t) \ge F^2(t). \tag{4.2}$$

Let $\tilde{F}(t) = F(t) - (1+\lambda_1)$, then from (4.2) we get

$$\partial^\alpha \tilde{F}(t) \ge \tilde{F}(t)\left(\tilde{F}(t) + 1 + \lambda_1\right) \ge \tilde{F}^2(t). \tag{4.3}$$

Since $0 \le \tilde{F}_0 = \tilde{F}(0)$, from the results in [10] (see Proposition 1.5) the solution of inequality (4.3) blows-up in a finite time. □

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References

[1] Ahmad B,Alhothuali MS,Alsulami HH,Kirane M,Timoshin S. On a time fractional reaction diffusion equation. Applied Mathematics and Computation 2015; 257: 199-204.

[2] Alsaedi A,Ahmad B,Kirane M. A survey of useful inequalities in fractional calculus. Fractional Calculus and Applied Analysis 2017; 20 (3): 574-594.

[3] Alsaedi A,Kirane M,Torebek BT. Global existence and blow-up for space and time nonlocal reaction-diffusion equation. Quaestiones Mathematicae 2019. (to appear).

[4] Al-Salti N,Kirane M,Torebek BT, On a class of inverse problems for a heat equation with involution perturbation. Hacettepe Journal of Mathematics and Statistics 2019; 48 (3): 669-681.

[5] Ashyralyev A,Sarsenbi AM. Well-posedness of a parabolic equation with involution. Numerical Functional Analysis and Optimization 2017; 38 (10): 1295-1304.

[6] Boudabsa L,Simon T,Vallois P. Fractional extreme distributions. ArXiv 2019: 1-46. arXiv: 1908.00584v1.

[7] Cabada A,Tojo FAF. General Results for Differential Equations with Involutions. In Differential Equations with Involutions. Paris, France: Atlantis Press, 2015, pp. 17-23.

[8] Cabada A,Tojo FAF. Equations with involutions. In: Workshop on Differential Equations University of Santiago de Compostela; Santiago de Compostela, Spain; 2014.

[9] Cao J,Song G, Wang J,Shi Q,Sun S. Blow-up and global solutions for a class of time fractional nonlinear reaction-diffusion equation with weakly spatial source. Applied Mathematics Letters 2019; 91: 201-206.

[10] Hnaien D,Kellil F,Lassoued R. Blowing-up solutions and global solutions to a fractional differential equation. Fractional Differential Calculus 2014; 4 (1): 45-53.

[11] Kilbas AA, Srivastava HM,Trujillo JJ. Theory and Applications of Fractional Differential Equations, Mathematics Studies Vol. 204. Amsterdam, Netherlands: Elsevier North-Holland Publishing Company, 2006.

[12] Kirane M,Sadybekov MA,Sarsenbi AA. On an inverse problem of reconstructing a subdiffusion process from nonlocal data. Mathematical Methods in the Applied Sciences 2019; 42 (6): 2043-2052.

[13] Torebek BT,Tapdigoglu R. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. Mathematical Methods in the Applied Sciences 2017; 40 (18): 6468-6479.

[14] Wiener J. Generalized Solutions of Functional-Differential Equations. New Jersey, USA: World Scientific Publishing, 1993.