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# Oscillation criteria for higher-order neutral type difference equations

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**Abstract:** In this paper, oscillation criteria are obtained for higher-order neutral-type nonlinear delay difference equations of the form

$$\Delta(r_n(\Delta^{k-1}(y_n + p_n y_{\tau_n})) + q_n f(y_{\sigma_n}) = 0,\ n \geq n_0, \tag{0.1}$$

where $r_n, p_n, q_n \in [n_0, \infty)$, $r_n > 0$, $q_n > 0$; $0 \leq p_n \leq p_0 < \infty$; $\lim_{n\to\infty} \tau_n = \infty$, $\lim_{n\to\infty} \sigma_n = \infty$; $\sigma_n \leq n$, $\sigma_n$ is nondecreasing; $\Delta\tau_n \geq \tau_0 > 0$; $\tau_\sigma = \sigma_\tau$; $\frac{f(u)}{u} \geq m > 0$ for $u \neq 0$. Moreover, we provide some examples to illustrate our main results.

**Key words:** Oscillation, oscillatory, difference equations

## 1. Introduction

In recent years, the methods used to construct the solutions and wide applications of the theory of difference equations have been intensively studied ([l-4,12] and the references therein). Especially, neutral difference equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines. Recently, much researches have been carried out on the oscillatory and asymptotic behaviour of solutions of higher-order neutral type delay and advanced difference equations ([5,6,8–12,14–18]). In these studies, the authors considered the case of $n - \tau \leq n$ or $\tau_n \leq n$ for higher order neutral type linear delay difference equations of the form

$$\Delta^m(y_n + p_n y_{\tau_n}) + q_n y_{\sigma_n} = 0,\ m \geq 2,$$

and they obtained many results on this assumption. Only in [7], the authors considered the differential equation

$$(r(t)(x(t) + p(t)x(\tau(t)))^{(n-1)})' + q(t)x(\sigma(t)) = 0$$

which is the continuity analogue of linear form of the equation (0.1) where $\tau(t) \geq t$ and $\sigma(t) < t$.

In this paper, we investigate the asymptotic and oscillation behavior of solutions of a certain higher-order neutral-type nonlinear difference equation

$$\Delta\left(r_n \Delta^{k-1}(y_n + p_n y_{\tau_n})\right) + q_n f(y_{\sigma_n}) = 0 \tag{1.1}$$

under the cases $\tau_n \geq n$ and $\tau_n \leq n$ where $n \in \mathbb{N}$, $k \in \mathbb{N}_2 = \{2, 3, ...\}$ and the following conditions hold:

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$(H_1)$ $r_n > 0$, $q_n > 0$; $0 \leq p_n \leq p_0 < \infty$ on $\mathbb{N}_{n_0} = \{n_0, n_0+1, ...\}$.

$(H_2)$ $\tau_n$ and $\sigma_n$ are defined on $\mathbb{N}$ and $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} \sigma_n = \infty$; $\sigma_n \leq n$, $\sigma_n$ is nondecreasing.

$(H_3)$ $\tau_{\sigma_n} = \sigma_{\tau_n}$.

$(H_4)$ $\frac{f(u)}{u} \geq m > 0$ for $u \neq 0$.

For our further references, we assume that

$$R_n = \sum_{s=n_0}^{n-1} \frac{1}{r_s} \to \infty \text{ as } n \to \infty,$$

$z_n = y_n + p_n y_{\tau_n}$, $a = \min\{|\sigma_n|, |\tau_n| : n \in \mathbb{N}\}$ and $I_0 = [a, n_0]$. A function $y_n$ is called the solution of Eq. (1.1) with $y_n = \varphi_n$ when $n \in I_0$, or it satisfies Eq. (1.1) for $n \geq n_0$, or $z_n$ and $r_n \Delta^{k-1} z_n$ are defined on $\mathbb{N}_{n_0}$. We consider only those solutions $y_n$ of Eq. (1.1) which satisfy $sup\{|y_n| : n \geq n_0 \in \mathbb{N}\} > 0$ for all $n \geq n_0$. Then, the solution $y_n$ of Eq. (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and otherwise it is oscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

## 2. Main results

Let $y_n$ be a positive solution of Eq.(1.1). Then, we say the sequence $z_n = y_n + p_n y_{\tau_n}$ is of degree $l$ if

$$\Delta^i z_n > 0, \text{ for } 0 \leq i \leq l, \tag{2.1}$$

$$(-1)^{i-1} \Delta^i z_n > 0, \text{ for } l < i \leq k-1, \tag{2.2}$$

$$\Delta(r_n \Delta^{k-1} z_n) < 0 \tag{2.3}$$

eventually.

For our incoming references, we will denote by $N_l$ the set of sequences $z_n$ of degree $l$.

**Lemma 2.1** *If $y_n$ is a positive solution of Eq.(1.1), then the set $N$ of all corresponding sequences $z_n = y_n + p_n y_{\tau_n}$ has the following structure:*

$$N = N_0 \cup N_2 \cup ... \cup N_{k-1} \textit{ if } k \textit{ is odd}$$

*and*

$$N = N_1 \cup N_3 \cup ... \cup N_{k-1} \textit{ if } k \textit{ is even.}$$

**Proof** Since $y_n$ is a positive solution of Eq. (1.1),

$$\Delta(r_n \Delta^{k-1} z_n) = -q_n f(y_{\tau_n}) < 0.$$

Thus, $r_n \Delta^{k-1} z_n$ is decreasing; moreover, all differences $\Delta^i z_n$, $0 \leq i \leq k-1$, are the fixed signs eventually. Hence, our assertion is a consequence of the well-known Knesser's lemma and its proof can be seen in [2]. Therefore, we omit it in here. □

For our following references, we denote

$$Q_n = \inf\{q_n, q_{\tau_n}\}, \ \ n \geq n_1$$

where $n_1$ is large enough. Then, we define

$$\begin{aligned} I_n^1 &= R_n - R_{n_1};\ I_n^i = \sum_{s=n_1}^{n} I_s^{i-1},\ 2 \leq i \leq k-1 \\ J_n^2 &= \sum_{u=n}^{\infty} \frac{1}{r_u} \sum_{s=u}^{\infty} Q_s;\ J_n^i = \sum_{s=n}^{\infty} J_s^{i-1},\ 3 \leq i \leq k. \end{aligned}$$

where we write $I_i(n)$ and $J_i(n)$ are indicated by $I_n^i$ and $J_n^i$, respectively. Hence, we can define the functions $Q_n^i = Q_i(n)$ that will be used as the coefficients of the supply difference equations. In here, we find

$$Q_n^{k-1} = Q_n I_{\sigma_n}^{k-1} \tag{2.4}$$

and for $k \geq 4$, we get

$$Q_n^i = \frac{(\sigma_n - n_1)^i}{i!} J_n^{k-1-i},\ 1 \leq i \leq k-3. \tag{2.5}$$

**Theorem 2.2** *Let $\tau_n \geq n$. Assume that*

$$J_{n_0}^k = \infty \tag{2.6}$$

*and the first order difference equations*

$$\Delta x_n + \frac{1}{1+p_0} Q_n^i x_{\sigma_n} = 0 \tag{$E_i$}$$

*are oscillatory for $i = 2, 4, ..., k-1$ if $k$ is odd, and for $i = 1, 3, ..., k-1$ if $k$ is even. Then, either Eq. (1.1) is oscillatory for even $k$ or every nonoscillatory solution $y_n$ of Eq.(1.1) satisfies $\lim_{n\to\infty} y_n = 0$ for odd $k$.*

**Proof** Without losing generality, assume that $y_n$ is a positive solution of Eq. (1.1). Then the corresponding sequence $z_n$ satisfies

$$z_{\sigma_n} = y_{\sigma_n} + p_{\sigma_n} y_{\tau_{\sigma_n}} \leq y_{\sigma_n} + p_0 y_{\sigma_{\tau_n}} \tag{2.7}$$

in view of hypotheses $(H_1)$ and $(H_3)$.

On the other hand, Eq. (1.1) can be written as

$$\Delta(r_n \Delta^{k-1} z_n) + q_n f(y_{\sigma_n}) = 0 \tag{2.8}$$

and also we have

$$0 = p_0 \Delta(r_{\tau_n} \Delta^{k-1} z_{\tau_n}) + p_0 q_{\tau_n} f(y_{\sigma_{\tau_n}}) \geq p_0 \Delta(r_{\tau_n} \Delta^{k-1} z_{\tau_n}) + p_0 q_{\tau_n} m y_{\sigma_{\tau_n}} \tag{2.9}$$

from using $(H_3)$ and $(H_4)$. Combining $(2.8)$ and $(2.9)$, and considering (2.7), we are led to

$$\begin{aligned} 0 &\geq \Delta(r_n \Delta^{k-1} z_n) + p_0 \Delta(r_{\tau_n} \Delta^{k-1} z_{\tau_n}) + q_n f(y_{\sigma_n}) + p_0 m q_{\tau_n} y_{\sigma_{\tau_n}} \\ &\geq \Delta(r_n \Delta^{k-1} z_n) + p_0 \Delta(r_{\tau_n} \Delta^{k-1} z_{\tau_n}) + q_n m y_{\sigma_{\tau_n}} + p_0 m q_{\tau_n} y_{\sigma_{\tau_n}} \\ &\geq \Delta(r_n \Delta^{k-1} z_n) + p_0 \Delta(r_{\tau_n} \Delta^{k-1} z_{\tau_n}) + m Q_n z_{\sigma_n} \end{aligned} \tag{2.10}$$

In addition, it follows from Lemma 2.1 that $z_n$ is of degree $l$, where $l \in \{0, 2, ..., k-1\}$ if $k$ is odd, and $l \in \{1, 3, ..., k-1\}$ if $k$ is even. Firstly, assume that $l = k-1$ i.e.

$$z_n > 0,\ \Delta z_n > 0, \ldots,\ r_n \Delta^{k-1} z_n > 0,\ \Delta(r_n \Delta^{k-1} z_n) < 0.$$

Then, since $w_n = r_n \Delta^{k-1} z_n > 0$ is decreasing, we are led to

$$\begin{aligned} \Delta^{k-2} z_n &\geq \Delta^{k-2} z_n - \Delta^{k-2} z_{n_1} \\ &= \sum_{s=n_1}^{n-1} \frac{1}{r_s} \left[ r_s \Delta^{k-1} z_s \right] \\ &\geq w_n \sum_{s=n_1}^{n-1} \frac{1}{r_s} \\ &\geq w_n \left[ R_n - R_{n_1} \right]. \end{aligned}$$

Repeating $k-2$ times sum from $n_1$ to $n-1$, we obtain $z_n \geq w_n I_n^{k-1}$. That is,

$$z_{\sigma_n} \geq w_{\sigma_n} I_{\sigma_n}^{k-1}. \tag{2.11}$$

Combining (2.11) together with (2.10), we see that $w_n$ is a positive solution of the inequalities

$$\Delta(w_n + p_0 w_{\tau_n}) + m Q_n^{k-1} w_{\sigma_n} \leq 0. \tag{2.12}$$

Let us denote $x_n = w_n + p_0 w_{\tau_n}$. Since $w_n$ is decreasing and $\tau_n \geq n$, one can see that $x_n \leq w_n + p_0 w_n = w_n(1 + p_0)$, that is,

$$w_{\sigma_n} \geq \frac{1}{1 + p_0} x_{\sigma_n},$$

which together with (2.12) gives that $x_n$ is a positive solution of the inequalities

$$\Delta x_n + \frac{m}{1 + p_0} Q_n^{k-1} x_{\sigma_n} \leq 0.$$

Finally, by a well-known result (see [13, p. 186, Corollary 7.6.1]), we conclude that the corresponding equation $(E_{k-1})$ has also an eventually positive solution. This contradicts to oscillation of $(E_{k-1})$. Therefore, $l \neq k-1$.

Now, assume that $l \geq 1$. That is,

$$z_n > 0,\ \Delta z_n > 0,\ ...,\ \Delta^l z_n > 0,\ \Delta^{l+1} z_n < 0,\ \Delta^{l+2} z_n > 0,\ ...,\ \Delta(r_n \Delta^{k-1} z_n) < 0.$$

Since $\Delta^l z_n$ is decreasing, we can verify that

$$\Delta^{l-1} z_n \geq \Delta^{l-1} z_n - \Delta^{l-1} z_{n_1} = \sum_{s=n_1}^{n-1} \Delta^l z_s \geq (n - n_1) \Delta^l z_n.$$

Repeating this procedure $l-1$ times, one leads to $z_n \geq \frac{(n-n_1)^l}{l!} \Delta^l z_n$, which implies that

$$z_{\sigma_n} \geq \Delta^l z_{\sigma_n} \frac{(\sigma_n - n_1)^l}{l!}. \tag{2.13}$$

Furthermore, summing (2.10) from $n$ to $\infty$, we obtain that

$$\Delta \sum_{s=n}^{\infty} r_s \Delta^{k-1} z_s + p_0 \Delta \sum_{s=n}^{\infty} (r_{\tau_s} \Delta^{k-1} z_{\tau_s}) \leq -m \sum_{s=n}^{\infty} Q_s z_{\sigma_s}$$

and

$$-\sum_{s=n+1}^{\infty} r_s \Delta^{k-1} z_s + \sum_{s=n}^{\infty} r_s \Delta^{k-1} z_s - p_0 \sum_{s=n+1}^{\infty} r_{\tau_s} \Delta^{k-1} z_{\tau_s} + p_0 \sum_{s=n}^{\infty} r_{\tau_s} \Delta^{k-1} z_{\tau_s} \geq m \sum_{s=n}^{\infty} Q_s z_{\sigma_s}.$$

Therefore, we obtain that

$$r_n \Delta^{k-1} z_n + p_0 r_{\tau_n} \Delta^{k-1} z_{\tau_n} \geq m \sum_{s=n}^{\infty} Q_s z_{\sigma_s}. \tag{2.14}$$

Since $r_n \Delta^{k-1} z_n$ is decreasing, $z_{\sigma_n}$ is increasing for $l \geq 1$ and $\tau_n \geq n$, we have

$$r_n \Delta^{k-1} z_n (1 + p_0) \geq m \sum_{s=n}^{\infty} Q_s z_{\sigma_s} \geq m z_{\sigma_n} \sum_{s=n}^{\infty} Q_s.$$

If we multiply this inequality by $1/r_n$ and then getting sum from $n$ to $\infty$, we are led to

$$\Delta \sum_{u=n}^{\infty} \Delta^{k-2} z_u \geq \frac{m}{1+p_0} z_{\sigma_n} \sum_{u=n}^{\infty} \frac{1}{r_u} \sum_{s=u}^{\infty} Q_s$$

and we get

$$-\Delta^{k-2} z_n \geq \frac{m}{1+p_0} z_{\sigma_n} J_n^2.$$

If $k \geq 4$, then repeating $k-3-l$ times sum from $n$ to $\infty$, we obtain that

$$-\Delta^{l+1} z_n \geq \frac{m}{1+p_0} z_{\sigma_n} J_n^{k-1-l}. \tag{2.15}$$

Combining (2.15) together with (2.13), we get that $x_n = \Delta^l z_n > 0$ is a positive solution of

$$\Delta x_n + \frac{m}{1+p_0} Q_n^l x_{\sigma_n} \leq 0.$$

Consequently, by a well-known result (see [13, p. 186, Corollary 7.6.1]), we see that the corresponding equation $(E_l)$ has also a positive solution. This contradicts our assumption. So that for $k$ is even, we have eliminated all possible cases for $z_n$ and we conclude that Eq.(1.1) is oscillatory. If $k$ is odd, then there is the only one case remaining $l = 0$. That is,

$$z_n > 0,\, \Delta z_n < 0,\, \Delta^2 z_n > 0, ..., r_n \Delta^{k-1} z_n > 0, \Delta(r_n \Delta^{k-1} z_n) < 0 \tag{2.16}$$

which implies that there exists $\lim_{n\to\infty} z_n = c \geq 0$. If we take $c > 0$, then summing (2.10) from $n$ to $\infty$, we get

$$r_n \Delta^{k-1} z_n + p_0 r_{\tau_n} \Delta^{k-1} z_{\tau_n} \geq \sum_{s=n}^{\infty} m Q_s z_{\sigma_s} \geq mc \sum_{s=n}^{\infty} Q_s. \tag{2.17}$$

Noting that $r_n\Delta^{k-1}z_n$ is decreasing and $\tau_n \geq n$, then it follows from (2.17) that

$$r_n\Delta^{k-1}z_n \geq \frac{mc}{1+p_0}\sum_{s=n}^{\infty} Q_s.$$

Multiplying this inequality by $1/r_n$ and then summing from $n$ to $\infty$, we have

$$-\Delta^{k-2}z_n \geq \frac{mc}{1+p_0}J_n^2.$$

Repeating $k-3$ times sum from $n$ to $\infty$, we obtain that

$$-\Delta z_n \geq \frac{mc}{1+p_0}J_n^{k-1}.$$

Now, summing from $n_1$ to $n-1$ and considering (2.6), we have

$$z_{n_1} \geq \frac{mc}{1+p_0}\sum_{s=n_1}^{n} J_s^{k-1} \to \infty \text{ as } n \to \infty$$

and this is a contradiction. Thus, $\lim\limits_{n\to\infty} z_n = c = 0$. Hence, it follows from $0 \leq y_n \leq z_n$ that $\lim\limits_{n\to\infty} y_n = 0$. □

**Example 2.3** *Consider the difference equation for* $k=3$

$$\Delta\left(r_n\Delta^2(y_n + p_n y_{\tau_n})\right) + q_n f(y_{\sigma_n}) = 0 \tag{2.18}$$

*where* $r_n = 1$, $p_n = \frac{1}{2}$, $q_n = 3^2 2^{2n-15}$, $\tau_n = n+2$, $\sigma_n = n-3$, *and* $f(u) = u^3$. *Since* $R_n = \sum_{s=n_0}^{n-1}\frac{1}{r_s} = \sum_{s=n_0}^{n-1} 1 \to \infty$ *as* $n \to \infty$, $Q_n = \inf\{q_n, q_{\tau_n}\} = \inf\{3^2 2^{2n-15}, 3^2 2^{2n-11}\} = 3^2 2^{2n-11}$, $n \geq n_1$, *and* $J_n^2 = \sum_{u=n}^{\infty}\frac{1}{r_u}\sum_{s=u}^{\infty} Q_s = \sum_{u=n}^{\infty}\sum_{s=u}^{\infty} 3^2 2^{2s-15} = \infty$. *We easily see that all conditions of the Theorem 2.2 are satisfied. Therefore, all solutions of equation (2.18) are nonoscillatory. One of the such solutions is* $y_n = (\frac{1}{2})^n$ *that tends to zero as* $n \to \infty$.

**Example 2.4** *Consider the difference equation for* $k=2$

$$\Delta^2(y_n + \frac{1}{2}y_{n+2}) + \frac{3^4}{2^{14}}4^n y_{n-3}^3 = 0 \tag{2.19}$$

*where* $r_n = 1$, $p_n = \frac{1}{2}$, $q_n = 3^4 2^{2n-14}$, $\tau_n = n+2$, $\sigma_n = n-3$, *and* $f(u) = u^3$. *Since* $R_n = \sum_{s=n_0}^{n-1}\frac{1}{r_s} = \sum_{s=n_0}^{n-1} 1 \to \infty$ *as* $n \to \infty$, $Q_n = \inf\{q_n, q_{\tau_n}\} = \inf\{3^4 2^{2n-14}, 3^4 2^{2n-10}\} = 3^4 2^{2n-14}$, $n \geq n_1$, *and* $J_n^2 = \sum_{u=n}^{\infty}\frac{1}{r_u}\sum_{s=u}^{\infty} Q_s = \sum_{u=n}^{\infty}\sum_{s=u}^{\infty} 3^4 2^{2s-14} = \infty$, *it is easily seen that all conditions of the Theorem 2.2 are satisfied. Therefore, for* $k=2$, *all solutions of equation (2.19) are oscillatory. One of such solutions is* $y_n = (-\frac{1}{2})^n$.

**Corollary 2.5** *Let* $\tau_n \geq n$ *and* (2.6) *hold. Assume that*

$$\lim_{n\to\infty}\inf\sum_{s=\sigma_n}^{n} Q_s^i > \frac{1+p_0}{e} \tag{2.20}$$

*for* $i = 2, 4, ..., k-1$ *if* $k$ *is odd and for* $i = 1, 3, ..., k-1$ *if* $k$ *is even. Then either Eq. (1.1) is oscillatory for even* $k$ *or every nonoscillatory solution* $y_n$ *of Eq. (1.1) satisfies* $\lim_{n\to\infty} y_n = 0$ *for odd* $k$.

**Proof** According to [2, p. 423, Theorem 6.20.5], we see that (2.20) guarantees that $(E_i)$ are oscillatory and our assertion follows from Theorem 2.2. □

Now, consider the case of when $\tau_n$ is delay argument, that is $\tau_n \le n$. We use the notation $\tau_n^{-1}$ for the inverse function of $\tau_n$. For our further references, let us denote

$$J_n^{*2} = \sum_{u=n}^{\infty} \frac{1}{r_{\tau_u}} \sum_{s=u}^{\infty} Q_s; \; J_n^{*i} = \sum_{s=n}^{\infty} J_s^{*i-1}, \, 3 \le i \le k.$$

We put

$$Q_n^{*k-1} = Q_n I_{\sigma_n}^{k-1} \tag{2.21}$$

and if $k \ge 4$, we set

$$Q_n^{*i} = \frac{(\tau_{\sigma_n}^{-1} - n_1)^i}{i!} J_{\tau_n^{-1}}^{*k-1-i}, \, 1 \le i \le k-3. \tag{2.22}$$

**Theorem 2.6** *Let* $\sigma_n \le \tau_n \le n$. *Assume that*

$$J_n^{*k} = \infty \tag{2.23}$$

*and the first order difference equations*

$$\Delta y_n + \frac{1}{1+p_0} Q_n^{*i} y_{\tau_{\sigma_n}^{-1}} = 0 \tag{$E_i^*$}$$

*are oscillatory for* $i = 2, 4, ..., k-1$ *if* $k$ *is odd and for* $i = 1, 3, ..., k-1$ *if* $k$ *is even. Then either Eq. (1.1) is oscillatory for* $k$ *even or every nonoscillatory solution* $y_n$ *of Eq. (1.1) satisfies* $\lim_{n\to\infty} y_n = 0$ *for* $k$ *odd.*

**Proof** Assume that $y_n$ is a positive solution of Eq. (1.1). Then using the same arguments as in the proof of Theorem 2.2, we verify that the corresponding function $z_n$ is of degree $l$, where $l \in \{0, 2, ..., k-1\}$ if $k$ is odd and $l \in \{1, 3, ..., k-1\}$ if $k$ is even. If we assume that $l = k-1$, then $w_n = r_n \Delta^{k-1} z_n > 0$ satisfies (2.12). Let us denote $v_n = w_n + p_0 w_{\tau_n}$. Since $w_n$ is decreasing and $\tau_n \le n$, it can be easily seen that $v_n \le w_{\tau_n}(1+p_0)$, and then

$$w_{\sigma_n} \ge \frac{1}{1+p_0} v_{\tau_{\sigma_n}^{-1}}$$

which together with (2.12) yields that $v_n$ is a positive solution of

$$\Delta v_n + \frac{1}{1+p_0} Q_n^{k-1} v_{\tau_{\sigma_n}^{-1}} \le 0.$$

Therefore, the equation $\left(E_{n-1}^*\right)$ has also a positive solution which contradicts our assumption. Therefore, we get that $l \ne k-1$. Now, assume that $l \ge 1$. Then $z_n$ satisfies (2.13) and (2.14). Considering $r_n \Delta^{k-1} z_n$ is decreasing and $\tau_n \le n$, it follows from (2.14) that

$$r_{\tau_n} \Delta^{k-1} z_{\tau_n} (1+p_0) \ge \sum_{s=n}^{\infty} Q_s z_{\sigma_s} \ge z_{\sigma_n} \sum_{s=n}^{\infty} Q_s.$$

If we multiply by $1/r_{\tau_n}$ and then summing from $n$ to $\infty$, we get

$$-\Delta^{k-2} z_{\tau_n} \geq \frac{1}{1+p_0} z_{\sigma_n} \sum_{u=n}^{\infty} \frac{1}{r_{\tau_u}} \sum_{s=u}^{\infty} Q_s = \frac{1}{1+p_0} z_{\sigma_n} J_n^{*2}.$$

For $k \geq 4$, repeating $k-3-l$ times sum from $n$ to $\infty$, we provide

$$-\Delta^{l+1} z_{\tau_n} \geq \frac{1}{1+p_0} z_{\sigma_n} J_n^{*k-1-l}. \tag{2.24}$$

Combining (2.22) together with (2.13), we get $v_n = \Delta^l z_n > 0$ which satisfies

$$\Delta v_{\tau_n} + \frac{1}{1+p_0} \frac{(\sigma_n - n_1)^l}{l!} J_n^{*k-1-l} v_{\sigma_n} \leq 0. \tag{2.25}$$

Since $\tau_{\sigma_n} = \sigma_{\tau_n}$ implies $\tau_{\tau_n^{-1}} = \tau_{\tau_n}^{-1} = n$, then $v_n$ is a positive solution of

$$\Delta v_n + \frac{1}{1+p_0} Q_n^{*l} v_{\tau_{\sigma_n}^{-1}} \leq 0$$

by (2.23). This means that $(E_l^*)$ has also a positive solution, but it is impossible. So that for $k$ is even, we have eliminated all possible cases for $z_n$ and we conclude that Eq. (1.1) is oscillatory. If $k$ is odd, then we have only one case $l = 0$. Hence, (2.16) holds and so $\lim_{n\to\infty} z_n = c \geq 0$. If $c > 0$, then a sum of (2.10) from $n$ to $\infty$ leads to (2.16). Since $r_n \Delta^{k-1} z_n$ is decreasing and $\tau_n \leq n$, then

$$r_{\tau_n} \Delta^{k-1} z_{\tau_n} \geq \frac{c}{1+p_0} \sum_{s=n}^{\infty} Q_s$$

by (2.17). Thus, we get

$$z_{\tau_{n_1}} \geq \frac{c}{1+p_0} \sum_{s=n_1}^{n} J_s^{*k-1} \to \infty \text{ as } n \to \infty$$

which contradicts (2.21); thus, $\lim_{n\to\infty} z_n = 0$. Finally, since $0 \leq v_n \leq z_n$, we conclude that $\lim_{n\to\infty} v_n = 0$. □

**Example 2.7** *Consider the difference equation for* $k = 2$

$$\Delta \left( r_n \Delta (y_n + p_n y_{\tau_n}) \right) + q_n f(y_{\sigma_n}) = 0 \tag{2.26}$$

*where* $r_n = (\frac{1}{4})^n$, $p_n = \frac{1}{2}$, $q_n = \frac{3^4}{2^{13}}$, $\tau_n = n-2$, $\sigma_n = n-3$, *and* $f(u) = u^3$. *Then it can be easily seen that all conditions of the Theorem 2.6 are satisfied. Therefore, all solutions of equation (2.26) are oscillatory for* $k = 2$. *One of the such solutions is* $y_n = (-\frac{1}{2})^n$.

**Example 2.8** *Consider the difference equation for* $k = 3$

$$\Delta \left( r_n \Delta^2 (y_n + p_n y_{\tau_n}) \right) + q_n f(y_{\sigma_n}) = 0 \tag{2.27}$$

*where* $r_n = (\frac{1}{4})^n$, $p_n = \frac{1}{2}$, $q_n = \frac{21}{2^{14}}$, $\tau_n = n-2$, $\sigma_n = n-3$, *and* $f(u) = u^3$, *then it is easily seen that all conditions of the Theorem 2.6 are satisfied. Therefore, all solutions of equation (2.27) are nonoscillatory. One of such solutions is* $y_n = (\frac{1}{2})^n$ *that tends to zero as* $n \to \infty$.

**Remark 2.9** *The oscillation of solutions for (2.18), (2.19), (2.26), and (2.27) cannot be determined by using the techniques considered in the literature. Indeed, it can be easily figured out that Eq. (1.1) reduces to equations in [5,6,9] and similar results contained there when* $r(n) = 1$*. However, these similar results cannot be found under the conditions* $r(n) = 1$ *and* $\tau_n \geq n$*. Moreover, for* $r(n) \neq 1$*, our results are completely different from those in [10,11,15,17,18] by the condition* $\tau_n \geq n$*.*

## 3. Conclusion

In this paper we have introduced new oscillation theorems for the investigation of the oscillation of Eq. (1.1). For this, we have considered both cases of $\tau_n \geq n$ and $\tau_n \leq n$ in a certain higher order neutral type nonlinear difference equation of the form

$$\Delta\left(r_n \Delta^{k-1}(y_n + p_n y_{\tau_n})\right) + q_n f(y_{\sigma_n}) = 0.$$

We have also provided some examples to illustrate our main results.

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