Turkish Journal of Mathematics

Volume 44 | Number 3 Article 7

1-1-2020

# Some uncertainty Inequalities related to the multivariate Laguerre function

LOTFI KAMOUN

RIM SELMI

**Turkish Journal of Mathematics**

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 712 – 728
© TÜBİTAK
doi:10.3906/mat-1908-27

# Some uncertainty Inequalities related to the multivariate Laguerre function

**Lotfi KAMOUN**[1,2]*; **Rim SELMI**[2]
[1]Laboratory LR11ES11, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunis, Tunisia
[2]Department of Mathematics, Faculty of Science, University of Monastir, Monastir, Tunisia

**Received:** 08.08.2019 • **Accepted/Published Online:** 11.03.2020 • **Final Version:** 08.05.2020

**Abstract:** In this paper, an analogous of the Heisenberg's inequality is established and three inequalities that constitute local uncertainty principle for the generalized Fourier–Laguerre transform in several variables are developed.

**Key words:** Generalized Fourier–Laguerre transform, local uncertainty principle, the Heisenberg inequality

## 1. Introduction

It is well known that the uncertainty principle asserts that a nonzero function and its Fourier transform cannot both be sharply localized. In quantum mechanics, this principle says that an observer cannot simultaneously and precisely determine the values of position and momentum of quantum particule. A mathematical formulation of this physical idea, usually called Heisenberg's inequality, was developed by Heisenberg [5] in 1927, for $f \in L^2(\mathbb{R}^n)$, as follows:

$$\left(\int_{\mathbb{R}^n} x_j^2 \left|f(x)\right|^2 dx\right)\left(\int_{\mathbb{R}^n} \xi_j^2 \left|\widehat{f}(\xi)\right|^2 d\xi\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}^n} \left|f(x)\right|^2 dx\right)^2, \quad 1 \leq j \leq n,$$

where $\widehat{f}$ is the Fourier–Plancherel transform given for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)\, e^{-i\langle \xi, x\rangle} dx.$$

Other formulations of this principle have been given in several works [1, 11–13]. Recently, many works have been dedicated to generalize this principle by considering generalized Fourier transforms like Dunkl transform [14, 17], Hankel transform [15], Fourier–Laguerre transform [10] (in case of Laguerre function of one variable).

In 1978, Faris [2] obtained various inequalities that generalize and improve the Heisenberg uncertainty principle. These results were called local uncertainty principles and they say that not only must the transform of a concentrated function be spread out, but that it cannot be too localized at any point. In 1987, Price [9] developed a family of inequalities in their sharpest forms which more directly displays the principle of local uncertainty as follows:

*Correspondence: kamoun.lotfi@planet.tn
2010 *AMS Mathematics Subject Classification:* 42B10, 42A38, 44A35, 35K08

- Let $\alpha > \frac{n}{2}$, there exists a constant $K_{\alpha,n}$ which verifies the following inequality for all measurable set $E$ of $\mathbb{R}^n$ such that $0 < m(E) < +\infty$ and $f \in L^2(\mathbb{R}^n)$:

$$\int_E \left|\widehat{f}(\xi)\right|^2 d\xi < K_{\alpha,n} m(E) \|f\|_2^{2-\frac{n}{\alpha}} \||t|^\alpha f\|_2^{\frac{n}{\alpha}} ,$$

where

$$K_{\alpha,n} = \frac{\Gamma\left(\frac{n}{2\alpha}\right)\Gamma\left(1-\frac{n}{2\alpha}\right)\left(\frac{2\alpha}{n}-1\right)^{\frac{n}{2\alpha}}}{\alpha 2^n \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)\left(1-\frac{n}{2\alpha}\right)} .$$

- Let $0 < \alpha < \frac{n}{2}$, there exists a constant $C_{\alpha,n}$ such that for all measurable set $E$ of $\mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$ we have:

$$\int_E \left|\widehat{f}(\xi)\right|^2 d\xi < C_{\alpha,n} (m(E))^{\frac{\alpha}{n}} \||x|^\alpha f\|_2^2 ,$$

where

$$C_{\alpha,n} = \left(\frac{n}{2^{n+1}\alpha^2 \pi^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)}\right)^{\frac{2\alpha}{n}} \left(1-\frac{2\alpha}{n}\right)^{\frac{2\alpha}{n}-1} .$$

The aim of this paper is to establish Heisenberg inequality and local principle inequalities related to the multivariate Laguerre function. At the end of this paper, we prove that the Heisenberg inequality can be deduced from the inequalities of local uncertainty principle that we will establish. Notice that, in framework of Laguerre function of one variable, analogous results are obtained by Rahmouni in [10].

The outline of the content of this paper is as follows:

In the second section, we give some results concerning the generalized Fourier–Laguerre transform $\mathfrak{F}$, (in case of Laguerre function of several variables), which can be useful later. In the third section, we establish the Heisenberg's inequality : if $a$, $b > 0$ then there exists $C := C(\alpha, n, a, b) > 0$ such that for all $f \in L^2_\alpha(\mathbb{K})$ we have

$$\||(x,t)|^a_{\mathbb{K}} f\|_{2,\eta_\alpha}^{\frac{2b}{a+2b}} \left\||(m,\lambda)|^b_{\widehat{\mathbb{K}}} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{a}{a+2b}} \geq C \|f\|_{2,\eta_\alpha} .$$

In the fourth section, building on the ideas of Faris [2] and Price [8, 9], we develop a family of inequalities in their sharpest forms which constitute the local uncertainty principle for the generalized Fourier–Laguerre transform as follows :

- Let $\beta$ be a real number such that $0 < \beta < |\alpha| + n + 1$, there is a constant $K := K(\alpha, \beta, n)$ such that for all $f \in L^2_\alpha(\mathbb{K})$ and every measurable set $E \subset \mathbb{N}^n \times \mathbb{R}$ with $0 < \gamma_\alpha(E) < \infty$,

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} \leq K(\gamma_\alpha(E))^{\frac{\beta}{2(|\alpha|+n+1)}} \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|_{2,\eta_\alpha} .$$

- Let $\beta$ be a real number such that $\beta > |\alpha| + n + 1$, there is a constant $M := M(\alpha, \beta, n)$ such that for every $f \in L^2_\alpha(\mathbb{K})$ and every measurable set $E \subset \mathbb{N}^n \times \mathbb{R}$ with $0 < \gamma_\alpha(E) < \infty$, we have

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} \leq M(\gamma_\alpha(E))^{\frac{1}{2}} \|f\|_{2,\eta_\alpha}^{1-\frac{|\alpha|+n+1}{\beta}} \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|_{2,\eta_\alpha}^{\frac{|\alpha|+n+1}{\beta}} .$$

- Let $\beta = |\alpha| + n + 1$, then there is a constant $\Omega := \Omega(\alpha, n)$ such that for all nonzero function $f \in L^2_\alpha(\mathbb{K})$ and measurable set $E \subset \mathbb{N}^n \times \mathbb{R}$ such that $0 < \gamma_\alpha(E) < \infty$, we have

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} \leq \Omega(\gamma_\alpha(E))^{\frac{1}{2(|\alpha|+n+1)}} \|f\|_{2,\eta_\alpha}^{1-\frac{1}{\beta}} \left\| |(x,t)|_{\mathbb{K}}^{\beta} f \right\|_{2,\eta_\alpha}^{\frac{1}{\beta}} .$$

## 2. Preliminaries

In this section, we collect some results which constitute harmonic analysis associated with the multivariate Laguerre function. For more details we refer the reader to [7].

Let $\mathbb{K} = [0, +\infty)^n \times \mathbb{R}$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in (0, +\infty)^n$. For $(m, \lambda) \in \widehat{\mathbb{K}} = \mathbb{N}^n \times \mathbb{R}$, we define the function $\Psi^\alpha_{m,\lambda}$ on $\mathbb{K}$ by

$$\Psi^\alpha_{m,\lambda}(x,t) = e^{i\lambda t}\mathfrak{L}^\alpha_m\left(|\lambda| x^2\right) ,$$

where $x^2 = (x_1^2, \cdots, x_n^2)$ for $x = (x_1, \cdots, x_n) \in (0, +\infty)^n$ and $\mathfrak{L}^\alpha_m$ is the Laguerre function with several variables of degree $|m|$ and order $\alpha$ defined on $[0, +\infty)^n$ by

$$\mathfrak{L}^\alpha_m(x) = \prod_{k=1}^{n} \mathcal{L}^{\alpha_k}_{m_k}(x_k) ,$$

$\mathcal{L}^{\alpha_k}_{m_k}$ being the Laguerre function on $\mathbb{R}_+$ of degree $m_k$ and order $\alpha_k$ which is

$$\mathcal{L}^{\alpha_k}_{m_k}(r) = e^{-\frac{r}{2}} \frac{L^{\alpha_k}_{m_k}(r)}{L^{\alpha_k}_{m_k}(0)} , \qquad r \geq 0 ,$$

$L^{\alpha_k}_{m_k}$ is the Laguerre polynomial of degree $m_k$ and order $\alpha_k$.

Notice that for $k \in \mathbb{N}$ and $\beta > 0$ the Laguerre polnomial $L^\beta_k$ is defined in terms of the generating function by

$$\sum_{k=0}^{+\infty} t^k L^\beta_k(r) = \frac{1}{(1-t)^{\beta+1}} e^{-\frac{rt}{1-t}} . \tag{2.1}$$

We denote by $\mathbb{L}^\alpha_m$ the multivariate Laguerre polynomial of degree $|m|$ and order $\alpha$ defined by

$$\mathbb{L}^\alpha_m(x) = \prod_{k=1}^{n} L^{\alpha_k}_{m_k}(x_k) ; \, x \in [0, +\infty)^n .$$

The generalized Fourier–Laguerre transform $\mathfrak{F}$ is defined on $\widehat{\mathbb{K}} = \mathbb{N}^n \times \mathbb{R}$ by :

$$\mathfrak{F}(f)(m,\lambda) = \int_{\mathbb{K}} f(x,t)\, \Psi^\alpha_{m,-\lambda}(x,t)\, d\eta_\alpha(x,t) , \; f \in L^1_\alpha(\mathbb{K}) ,$$

where $d\eta_\alpha$ is the positive measure defined on $\mathbb{K}$ by :

$$d\eta_\alpha(x,t) = \left(\prod_{k=1}^{n} \frac{x^{2\alpha_k+1}}{\pi\Gamma(\alpha_k+1)}\right) dxdt .$$

We denote $L^p_\alpha(\mathbb{K})$, $1 \le p < +\infty$, the space of measurable functions on $\mathbb{K}$ such that

$$\|f\|_{p,\eta_\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^p \, d\eta_\alpha(x,t)\right)^{\frac{1}{p}} < +\infty$$

and $L^p_\alpha\left(\widehat{\mathbb{K}}\right)$, $1 \le p < +\infty$, the space of measurable functions on $\widehat{\mathbb{K}}$ such that

$$\|g\|_{p,\gamma_\alpha} = \left(\int_{\widehat{\mathbb{K}}} |g(m,\lambda)|^p \, d\gamma_\alpha(m,\lambda)\right)^{\frac{1}{p}} < +\infty \ ,$$

$d\gamma_\alpha$ being the positive measure defined on $\widehat{\mathbb{K}}$ by :

$$\int_{\widehat{\mathbb{K}}} g(m,\lambda) \, d\gamma_\alpha(m,\lambda) = (2\pi)^{n-1} \sum_{m\in\mathbb{N}^n} \mathbb{L}^\alpha_m(0) \int_{\mathbb{R}} g(m,\lambda) \, |\lambda|^{|\alpha|+n} \, d\lambda \ .$$

The generalized Fourier–Laguerre transform $\mathfrak{F}$ satisfies the Plancherel formula

$$\|\mathfrak{F}(f)\|^2_{2,\gamma_\alpha} = \|f\|^2_{2,\eta_\alpha} \ . \tag{2.2}$$

A generalized convolution product $*$ is defined on $\mathbb{K}$ and verifies the inequality

$$\|f * g\|_{2,\eta_\alpha} \le \|f\|_{2,\eta_\alpha} \|g\|_{1,\eta_\alpha} \ , \tag{2.3}$$

where $f \in L^2_\alpha(\mathbb{K})$ and $g \in L^1_\alpha(\mathbb{K})$ (see [7]).

**Notation 2.1** *We denote by :*

1. $\delta_\varrho$ *the dilation on* $\mathbb{K}$ *defined by :*

$$\delta_\varrho(x,t) = \left(\varrho x, \varrho^2 t\right) \ , \quad \varrho > 0 \ .$$

2. $\delta'_r$ *the dilation on* $\widehat{\mathbb{K}}$ *defined by :*

$$\delta'_r(m,\lambda) = \left(m, r^2\lambda\right) \ , \quad r > 0.$$

3. $|.|_{\mathbb{K}}$ *the homogenous norm on* $\mathbb{K}$ *related to the family of dilations* $(\delta_\varrho)_{\varrho>0}$

$$|(x,t)|_{\mathbb{K}} = \left(\|x\|^4 + 4t^2\right)^{\frac{1}{4}} \ .$$

4. $\Lambda$ *the operator defined on* $\widehat{\mathbb{K}}$ *by :*

$$\Lambda = \Gamma_1^2 - 4\left(\Gamma_2 + \frac{\partial}{\partial\lambda}\right)^2 \ ,$$

*where* $\Gamma_1$ *and* $\Gamma_2$ *are defined on [7,* Page *5].*

*5.* $|.|_{\widehat{\mathbb{K}}}$ *the quasinorm on* $\widehat{\mathbb{K}}$ *defined by :*

$$|(m,\lambda)|_{\widehat{\mathbb{K}}} = 4\,|\lambda|\left(|m| + \frac{|\alpha|+n}{2}\right) .$$

*6.* $\mathcal{B}_{\mathbb{K},r}$ *the ball in* $\mathbb{K}$ *centered at* $(0_{\mathbb{R}^n},0)$ *of radius* $r$

$$\mathcal{B}_{\mathbb{K},r} = \{(x,t)\in\mathbb{K};\ |(x,t)|_{\mathbb{K}} < r\} ,$$

$\mathcal{B}^{c}_{\mathbb{K},r}$ *its complementary in* $\mathbb{K}$ *and* $\chi_{\mathcal{B}_{\mathbb{K},r}}$, $\chi_{\mathcal{B}^{c}_{\mathbb{K},r}}$ *their characteristic functions.*

*Similarly,* $\mathcal{B}_{\widehat{\mathbb{K}},r}$ *is the ball in* $\widehat{\mathbb{K}}$ *centered at* $(0_{\mathbb{N}^n,0})$ *of radius* $r$

$$\mathcal{B}_{\widehat{\mathbb{K}},r} = \left\{(m,\lambda)\in\widehat{\mathbb{K}};\ |(m,\lambda)|_{\widehat{\mathbb{K}}} < r\right\} .$$

*7.* $f_\varrho$ *the dilated of the function* $f$ *defined on* $\mathbb{K}$ *by :*

$$f_\varrho(x,t) = \varrho^{-2(|\alpha|+n+1)} f\left(\delta_{\frac{1}{\varrho}}(x,t)\right) ,$$

*preserving the* $L^1_\alpha(\mathbb{K})$ *norm of* $f$ *with respect to the measure* $d\eta_\alpha$.

**Lemma 2.2** (1) *Let* $\Sigma = \{(x,t)\in\mathbb{K} \mid |(x,t)|_{\mathbb{K}} = 1\}$ *be the unit sphere in* $\mathbb{K}$. *We denote by* $\omega_{\alpha,n}$ *the surface area of* $\Sigma$ *and* $\Omega_{\alpha,n}$ *the volume of the ball* $\mathcal{B}_{\mathbb{K},1}$. *Then,*

$$\omega_{\alpha,n} = \frac{\Gamma\left(\frac{|\alpha|+n}{2}\right)}{2^n \pi^{n-\frac{1}{2}}\Gamma(|\alpha|+n)\,\Gamma\left(\frac{|\alpha|+n+1}{2}\right)}$$

*and*

$$\Omega_{\alpha,n} = \frac{\omega_{\alpha,n}}{2(|\alpha|+n+1)} .$$

(2) *The measure of* $\mathcal{B}_{\widehat{\mathbb{K}},r}$ *with respect to the Plancherel measure* $d\gamma_\alpha$ *is finite and we have*

$$\gamma_\alpha\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right) = r^{|\alpha|+n+1}\,\Xi_{\alpha,n} , \tag{2.4}$$

*where*

$$\Xi_{\alpha,n} = \frac{\pi^{n-1}2^n}{|\alpha|+n+1}\sum_{m\in\mathbb{N}^n}\mathbb{L}^\alpha_m(0)\left(\frac{1}{4|m|+2|\alpha|+2n}\right)^{|\alpha|+n+1} .$$

**Proof**

(1) We begin by remembering that the polar coordinates in $\mathbb{R}^n$ are given by the following smooth diffeomorphism $\varphi$, from the open set

$$\left\{(r,\theta_1,\ldots,\theta_{n-1})\ ;\quad r\in(0,+\infty)\ ;\quad (\theta_i)_{1\le i\le n-2}\in(0,\pi)^{n-2}\ \text{ and }\ \theta_{n-1}\in(-\pi,\pi)\right\}$$

into $\mathbb{R}^n$ deprived of a half closed hyperplane,

$$\varphi\left(r,\theta_1,\ldots,\theta_{n-1}\right) = r\left(\left(\cos\theta_k\prod_{j=1}^{k-1}\sin\theta_j\right)_{1\le k\le n-1},\prod_{j=1}^{n-1}\sin\theta_j\right).$$

The Jacobian of $\varphi$ is given by :

$$r^{n-1}\prod_{k=1}^{n-2}\left(\sin\theta_k\right)^{n-k-1}.$$

By a similar reasoning to [4, Lemma 1], we consider the following smooth diffeomorphism

$$\begin{array}{rcl} G:(0,+\infty)^n\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right) & \longrightarrow & (0,+\infty)^n\times\mathbb{R} \\ (s,\theta) & \longmapsto & \left(\sqrt{\cos(\theta)}\,s\,,\ \frac{\|s\|^2}{2}\sin(\theta)\right) \end{array}$$

of which the Jacobian is equal to $\frac{\|s\|^2}{2}\left(\cos\theta\right)^{\frac{n}{2}-1}$. Using firstly $G$ and then $\varphi$, we can deduce that if $f$ is an integrable function on $\mathbb{K}$, then

$$\int_{\mathbb{K}} f\left(x,t\right)d\eta_\alpha\left(x,t\right) = \int_0^{+\infty}\int_\Sigma r^{2|\alpha|+2n+1} f\left(\delta_r\left(\xi\right)\right)d\xi dr\ , \tag{2.5}$$

where

$$\xi = \left(\sqrt{\cos\theta}\ \varphi\left(1,\theta_1,\ldots,\theta_{n-1}\right),\frac{\sin\theta}{2}\right)\in\Sigma\,,$$

for $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $\left(\theta_i\right)_{1\le i\le n-1}\in\left(0,\frac{\pi}{2}\right)^{n-1}$.
The surface area of $\Sigma$ is given by :

$$\begin{aligned} \omega_{\alpha,n} &= \int_\Sigma d\xi \\ &= \frac{1}{2\pi^n}\left(\prod_{k=1}^{n}\frac{1}{\Gamma\left(\alpha_k+1\right)}\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\cos\theta\right)^{|\alpha|+n-1}d\theta\right) \\ &\times\prod_{k=1}^{n-1}\int_0^{\frac{\pi}{2}}\left(\cos\theta_k\right)^{2\alpha_k+1}\left(\sin\theta_k\right)^{2\left|\alpha\right|_k+2n-2k-1}d\theta_k\ , \end{aligned}$$

where $\left|\alpha\right|_k = \sum_{j=k+1}^{n}\alpha_j$. We remind that the beta function is defined for $\Re\left(a\right)>0$ and $\Re\left(b\right)>0$ by :

$$B\left(a,b\right) = \int_0^1 s^{a-1}\left(1-s\right)^{b-1}ds = \frac{\Gamma\left(a\right)\Gamma\left(b\right)}{\Gamma\left(a+b\right)}\ .$$

By straightforward calculus using a suitable change of variable and the beta function, we find the advertised value of $\omega_{\alpha,n}$. The volume $\Omega_{\alpha,n}$ is simply deduced from relation (2.5) :

$$\Omega_{\alpha,n}=\int_{\mathcal{B}_{\mathbb{K},1}}d\eta_{\alpha}\left(x,t\right)=\int_{0}^{1}\int_{\Sigma}r^{2|\alpha|+2n+1}dr\;d\xi=\frac{\omega_{\alpha,n}}{2\left(|\alpha|+n+1\right)}\;.$$

(2) We notice that

$$(m,\lambda)\in\mathcal{B}_{\widehat{\mathbb{K}},r}\Longleftrightarrow|\lambda|<\frac{r}{4\left(|m|+\frac{|\alpha|+n}{2}\right)}\;.$$

Then, the equality (2.4) is simply deduced since

$$\gamma_{\alpha}\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right)\quad=\quad2^{n}\pi^{n-1}\sum_{m\in\mathbb{N}^{n}}\mathbb{L}_{m}^{\alpha}\left(0\right)\int_{0}^{\frac{r}{4\left(|m|+\frac{|\alpha|+n}{2}\right)}}\lambda^{|\alpha|+n}d\lambda\;.$$

Furthermore, we have

$$\frac{\mathbb{L}_{m}^{\alpha}\left(0\right)}{\left(4\left|m\right|+2\left|\alpha\right|+2n\right)^{|\alpha|+n+1}}\leq\frac{\mathbb{L}_{m}^{\alpha}\left(0\right)}{|m|^{|\alpha|+n+1}}$$

and we know that $L_{m_k}^{\alpha_k}\left(0\right)\underset{\infty}{\sim}\dfrac{m_k^{\alpha_k}}{\Gamma\left(\alpha_k+1\right)}$. Then, the family $\displaystyle\sum_{m\in\mathbb{N}^{n}}\frac{\mathbb{L}_{m}^{\alpha}\left(0\right)}{|m|^{|\alpha|+n+1}}$ is summable if and only if the family $\displaystyle\sum_{m\in\mathbb{N}^{n}}\frac{m_1^{\alpha_1}\dots m_n^{\alpha_n}}{|m|^{|\alpha|+n+1}}$ is also summable. We deduce that $\gamma_{\alpha}\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right)$ is finite since from [16, chapter XIV], $\displaystyle\sum_{m\in\mathbb{N}^{n}}\frac{1}{|m|^{n+1}}$ is summable.

□

## 3. Heisenberg's inequality for the generalized Fourier-Laguerre transform

In this section, we use the heat kernel associated with the differential operator

$$L=-\sum_{k=1}^{n}\left(\frac{\partial^{2}}{\partial x_{k}^{2}}+\frac{2\alpha_{k}+1}{x_{k}}\frac{\partial}{\partial x_{k}}+x_{k}^{2}\frac{\partial^{2}}{\partial t^{2}}\right)$$

to establish the Heisenberg's inequality.

Notice that the operators $\Lambda$ and $L$ satisfy these properties which are proved in [7]

$$\begin{aligned}L\Psi_{m,\lambda}^{\alpha}&=&|(m,\lambda)|_{\widehat{\mathbb{K}}}\,\Psi_{m,\lambda}^{\alpha}\;,\\ \Lambda\Psi_{m,\lambda}^{\alpha}&=&|(x,t)|_{\mathbb{K}}^{4}\,\Psi_{(m,\lambda)}^{\alpha}\;,\\ \mathfrak{F}\left(Lf\right)(m,\lambda)&=&-\left|(m,\lambda)\right|_{\widehat{\mathbb{K}}}\mathfrak{F}\left(f\right)(m,\lambda)\;.\end{aligned}\qquad(3.1)$$

We define $L^{b}$ for $b\in\mathbb{R}$, as in [18, Page 117] . Then, by (3.1) we get

$$\mathfrak{F}\left(L^{b}f\right)(m,\lambda)=|(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}\left(f\right)(m,\lambda)\;.\qquad(3.2)$$

**Proposition 3.1** *Let* $s > 0$*. Define the heat kernel* $h_s$ *associated with the operator* $L$ *by, for each* $(x,t) \in \mathbb{K}$*,*

$$h_s(x,t) = \int_{\widehat{\mathbb{K}}} e^{-s|(m,\lambda)|_{\widehat{\mathbb{K}}}} \Psi^{\alpha}_{m,\lambda}(x,t)\, d\gamma_\alpha(m,\lambda).$$

*Then*

$$h_s(x,t) = (2\pi)^{n-1} \int_{\mathbb{R}} \left( \frac{|\lambda|}{2\sinh(2|\lambda| s)} \right)^{|\alpha|+n} e^{-\frac{|\lambda| \|x\|^2}{2} \coth(2|\lambda|s)} e^{i\lambda t} d\lambda.$$

**Proof** For $s > 0$ and $(x,t) \in \mathbb{K}$, we have

$$\begin{aligned} h_s(x,t) &= (2\pi)^{n-1} \sum_{m\in\mathbb{N}^n} \mathbb{L}^{\alpha}_m(0) \int_{\mathbb{R}} e^{-s|(m,\lambda)|_{\widehat{\mathbb{K}}}} \Psi^{\alpha}_{m,\lambda}(x,t)\, |\lambda|^{|\alpha|+n} d\lambda \\ &= (2\pi)^{n-1} \int_{\mathbb{R}} h^{\lambda}_s(x)\, e^{i\lambda t} d\lambda, \end{aligned}$$

where

$$\begin{aligned} h^{\lambda}_s(x) &= \sum_{m\in\mathbb{N}^n} \left( e^{-s|(m,\lambda)|_{\widehat{\mathbb{K}}}} e^{-\frac{|\lambda|\|x\|^2}{2}} \prod_{k=1}^{n} L^{\alpha_k}_{m_k}\left(|\lambda| x_k^2\right) |\lambda|^{|\alpha|+n} \right) \\ &= e^{-\frac{|\lambda|\|x\|^2}{2}} e^{-2|\lambda|(|\alpha|+n)s} |\lambda|^{|\alpha|+n} \prod_{k=1}^{n} \left( \sum_{m_k\in\mathbb{N}} e^{-4|\lambda| m_k s} L^{\alpha_k}_{m_k}\left(|\lambda| x_k^2\right) \right). \end{aligned}$$

We use relation (2.1) to obtain the desired equality. □

By a straightforward calculation, we obtain:

**Proposition 3.2** *The heat kernel* $h_s$ *satisfies the following properties for all* $(x,t)$ *in* $\mathbb{K}$

(1) $\left( L + \dfrac{\partial}{\partial s} \right)(h_s(x,t)) = 0.$

(2) *For* $s,\ t > 0$, $h_s * h_t = h_{s+t}$.

(3) $h_s(x,t) \geq 0$, $h_s(x,t) = h_s(x,-t)$ *and* $\displaystyle\int_{\mathbb{K}} h_s(x,t)\, d\eta_\alpha(x,t) = 1$.

(4) *For all* $\varrho > 0$, *we have* $h_{\varrho^2 s}(\delta_\varrho(x,t)) = \varrho^{-2(|\alpha|+n+1)} h_s(x,t)$.
*Furthermore,*

$$\mathfrak{F}(h_s)(m,\lambda) = e^{-s\ |(m,\lambda)|_{\widehat{\mathbb{K}}}}.$$

A reasoning similar to [6, Lemma 3.5] allows us to prove the following inequality:

**Lemma 3.3**

$$h_s(x,t) \leq C s^{-(|\alpha|+n+1)} e^{-\frac{A}{s}|(x,t)|^2_{\mathbb{K}}},$$

*where* $A$ *and* $C$ *are two positive constants.*

For the proof of the main result of this section namely the inequality of Heisenberg, we will need the following two lemmas:

**Lemma 3.4** *The function* $h_s$ *belongs to* $L^2_\alpha(\mathbb{K})$ *and we have*

$$\|h_s\|_{2,\eta_\alpha} = \mathcal{D}_{\alpha,n} s^{-\frac{|\alpha|+n+1}{2}} \ , \tag{3.3}$$

*where*

$$\mathcal{D}_{\alpha,n} = \left( \frac{\pi^{n-1}}{2^{|\alpha|}} \int_0^{+\infty} \left( \frac{u}{\sinh(4u)} \right)^{|\alpha|+n} du \right)^{\frac{1}{2}} .$$

**Proof** By the Plancherel formula (2.2), we have

$$\begin{aligned} \|h_s\|^2_{2,\eta_\alpha} &= \|\mathfrak{F}(h_s)\|^2_{2,\gamma_\alpha} \\ &= (2\pi)^{n-1} \int_{\mathbb{R}} e^{-4|\lambda|(|\alpha|+n)s} \prod_{k=1}^{n} \left( \sum_{m_k \in \mathbb{N}} L^{\alpha_k}_{m_k}(0)\, e^{-8|\lambda| m_k s} \right) |\lambda|^{|\alpha|+n} d\lambda \ . \end{aligned}$$

Using the generating function identity (2.1) for each Laguerre polynomial $L^{\alpha_k}_{m_k}$ we get

$$\begin{aligned} \|h_s\|^2_{2,\eta_\alpha} &= (2\pi)^{n-1} \int_{\mathbb{R}} \left( \frac{e^{-4|\lambda|s}}{1-e^{-8|\lambda|s}} \right)^{|\alpha|+n} |\lambda|^{|\alpha|+n} d\lambda \\ &= (2\pi)^{n-1} \int_{\mathbb{R}} \left( \frac{|\lambda|}{2\sinh(4|\lambda|s)} \right)^{|\alpha|+n} d\lambda \\ &= \frac{\pi^{n-1}}{2^{|\alpha|}} s^{-(|\alpha|+n+1)} \int_0^{+\infty} \left( \frac{u}{2\sinh(4u)} \right)^{|\alpha|+n} du \ . \end{aligned}$$

□

We denote by $\{H^s \mid s>0\}$ the heat semigroup associated to the operator $L$, where $H^s(x,t) = f * h_s(x,t)$.

**Lemma 3.5** *Let* $0 < a < |\alpha|+n+1$, *then for all* $f$ *in* $L^2_\alpha(\mathbb{K})$ *there exists a positive constant* $C$ *such that*

$$\|H^s(f)\|_{2,\eta_\alpha} \le C s^{-\frac{a}{2}} \||(x,t)|^a_{\mathbb{K}} f\|_{2,\eta_\alpha} \ . \tag{3.4}$$

**Proof** For $r>0$, we put $f_r = f\chi_{\mathcal{B}_{\mathbb{K},r}}$ and $f^r = f - f_r$. Then

$$|f^r(x,t)| \le r^{-a} |f(x,t)|\, |(x,t)|^a_{\mathbb{K}} \ .$$

The above relation together with relation (2.3) and the fact that $\|h_s\|_{1,\eta_\alpha} = 1$ give us the following inequalities

$$\|H^s(f^r)\|_{2,\eta_\alpha} \le \|f^r\|_{2,\eta_\alpha} \|h_s\|_{1,\eta_\alpha} \le \|f^r\|_{2,\eta_\alpha} \le r^{-a} \||(x,t)|^a_{\mathbb{K}} f\|_{2,\eta_\alpha} \ . \tag{3.5}$$

On the other hand, we use relations (2.3) and (3.3) to find

$$\begin{aligned}
\left\|H^s(f_r)\right\|_{2,\eta_\alpha} &= \left\|f_r * h_s\right\|_{2,\eta_\alpha} \\
&\le \left\|f_r\right\|_{1,\eta_\alpha}\left\|h_s\right\|_{2,\eta_\alpha} \\
&\le \left\|h_s\right\|_{2,\eta_\alpha}\left\||(x,t)|_{\mathbb{K}}^{-a}\chi_{\mathcal{B}_{\mathbb{K}},r}\right\|_{2,\eta_\alpha}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha} \\
&\le \left(\frac{\omega_{\alpha,n}}{2(-a+|\alpha|+n+1)}\right)^{\frac{1}{2}} r^{(-a+|\alpha|+n+1)}\mathcal{D}_{\alpha,n} \\
&\times\ s^{-\frac{|\alpha|+n+1}{2}}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha} \\
&\le C_{\alpha,n} r^{(-a+|\alpha|+n+1)} s^{-\frac{|\alpha|+n+1}{2}}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha}\ ,
\end{aligned} \tag{3.6}$$

where

$$C_{\alpha,n} = \mathcal{D}_{\alpha,n}\left(\frac{\omega_{\alpha,n}}{2(-a+|\alpha|+n+1)}\right)^{\frac{1}{2}}.$$

Using relations (3.5) and (3.6), we deduce that, for all $r>0$,

$$\begin{aligned}
\left\|H^s(f)\right\|_{2,\eta_\alpha}\left\|f * h_s\right\|_{2,\eta_\alpha} &\le \left\|f_r * h_s\right\|_{2,\eta_\alpha} + \left\|f^r * h_s\right\|_{2,\eta_\alpha} \\
&\le r^{-a}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha}\left(1 + C_{\alpha,n} r^{(|\alpha|+n+1)} s^{-\frac{|\alpha|+n+1}{2}}\right).
\end{aligned}$$

In particular for $r=\sqrt{s}$, we get

$$\left\|f * h_s\right\|_{2,\eta_\alpha} \le (1 + C_{\alpha,n})\, s^{-\frac{a}{2}}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha}.$$

□

**Theorem 3.6** *Let $a,\ b>0$. Then, there exists $C := C(\alpha,n,a,b) > 0$ such that for all $f\in L^2_\alpha(\mathbb{K})$ we have*

$$\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha}^{\frac{2b}{a+2b}}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{a}{a+2b}} \ge C\left\|f\right\|_{2,\eta_\alpha}. \tag{3.7}$$

**Proof**

1. Assume that $0<a<|\alpha|+n+1$ and $b\le 1$. By using relation (3.4), we get

$$\begin{aligned}
\left\|f\right\|_{2,\eta_\alpha} &\le \left\|H_s(f)\right\|_{2,\eta_\alpha} + \left\|(1-H_s)(f)\right\|_{2,\eta_\alpha} \\
&\le C_1 s^{-\frac{a}{2}}\left\||(x,t)|_{\mathbb{K}}^{a} f\right\|_{2,\eta_\alpha} + \left\|(1-H_s)(sL)^{-b}(sL)^{b} f\right\|_{2,\eta_\alpha}.
\end{aligned} \tag{3.8}$$

   Let $g=(sL)^b f$. Then, for $t\ge 0$, since the function: $t\longmapsto(1-e^{-t})\,t^{-b}$ is bounded when $b\le 1$, we deduce by the Plancherel formula (2.2) and relation (3.2) that

$$\begin{aligned}
\left\|(1-H_s)(sL)^{-b} g\right\|_{2,\eta_\alpha} &= \left\|\left(1-e^{-s|(m,\lambda)|_{\widehat{\mathbb{K}}}}\right)\left(s\ |(m,\lambda)|_{\widehat{\mathbb{K}}}\right)^{-b}\mathfrak{F}(g)\right\|_{2,\gamma_\alpha} \\
&\le C_2 s^{b}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}.
\end{aligned} \tag{3.9}$$

Substituting (3.9) in (3.8), we get

$$\|f\|_{2,\eta_\alpha} \leq C\left(s^{-\frac{a}{2}}\,\||(x,t)|_{\mathbb{K}}^{a}\,f\|_{2,\eta_\alpha} + s^{b}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}\right).$$

Optimizing in $s$, we obtain for all $0<a<|\alpha|+n+1$ and $b\leq 1$

$$\|f\|_{2,\eta_\alpha} \leq C\,\||(x,t)|_{\mathbb{K}}^{a}\,f\|_{2,\eta_\alpha}^{\frac{2b}{a+2b}}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{a}{a+2b}}. \tag{3.10}$$

2. Let $0<a<|\alpha|+n+1$ and $b>1$. Then, we have $\dfrac{|(m,\lambda)|_{\widehat{\mathbb{K}}}}{\varepsilon} \leq 1+\left(\dfrac{|(m,\lambda)|_{\widehat{\mathbb{K}}}}{\varepsilon}\right)^{b}$ for all $\varepsilon>0$. It follows that

$$\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha} \leq \varepsilon\,\|f\|_{2,\eta_\alpha} + \varepsilon^{1-b}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}.$$

Optimizing in $\varepsilon$, we get

$$\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha} \leq b\,(b-1)^{\frac{1}{b}-1}\,\|f\|_{2,\eta_\alpha}^{1-\frac{1}{b}}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{1}{b}}.$$

We get the desired inequality from the above relation together with the following inequality

$$\|f\|_{2,\eta_\alpha} \leq C\,\||(x,t)|_{\mathbb{K}}^{a}\,f\|_{2,\eta_\alpha}^{\frac{2}{a+2}}\,\||(m,\lambda)|_{\widehat{\mathbb{K}}}\,\mathfrak{F}(f)\|_{2,\gamma_\alpha}^{\frac{a}{a+2}}$$

which is deduced from relation (3.10).

3. If $a\geq|\alpha|+n+1$, then for all $\varepsilon>0$ we have $\dfrac{|(x,t)|_{\mathbb{K}}}{\varepsilon} \leq 1+\dfrac{|(x,t)|_{\mathbb{K}}^{a}}{\varepsilon^{a}}$. It follows

$$\||(x,t)|_{\mathbb{K}}\,f\|_{2,\eta_\alpha} \leq \varepsilon\,\|f\|_{2,\eta_\alpha} + \varepsilon^{1-a}\,\||(x,t)|_{\mathbb{K}}^{a}\,f\|_{2,\eta_\alpha}.$$

Optimizing in $\varepsilon$, we get

$$\||(x,t)|_{\mathbb{K}}\,f\|_{2,\eta_\alpha} \leq a\,(a-1)^{\frac{1}{a}-1}\,\|f\|_{2,\eta_\alpha}^{1-\frac{1}{a}}\,\||(x,t)|_{\mathbb{K}}^{a}\,f\|_{2,\eta_\alpha}^{\frac{1}{a}}. \tag{3.11}$$

From relation (3.10), we have

$$\|f\|_{2,\eta_\alpha} \leq C\,\||(x,t)|_{\mathbb{K}}\,f\|_{2,\eta_\alpha}^{\frac{2b}{1+2b}}\left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b}\,\mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{1}{1+2b}}. \tag{3.12}$$

Combining relations (3.11) and (3.12), we get the result.

□

## 4. Local uncertainty principle

In this section, we establish three inequalities in their sharpest form that constitute local uncertainty principle associated with the multivariate Laguerre function.

**Theorem 4.1** *Let $\beta$ be a real number such that $0<\beta<|\alpha|+n+1$, there is a constant $K:=K(\alpha,\beta,n)$ such that for all $f\in L^2_\alpha(\mathbb{K})$ and every measurable set $E\subset\widehat{\mathbb{K}}$ with $0<\gamma_\alpha(E)<\infty$, we have*

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha}\leq K(\gamma_\alpha(E))^{\frac{\beta}{2(|\alpha|+n+1)}}\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha},$$

*where*

$$K=\left(\frac{\omega_{\alpha,n}(-\beta+|\alpha|+n+1)}{2\beta^2}\right)^{\frac{\beta}{2(|\alpha|+n+1)}}\frac{|\alpha|+n+1}{-\beta+|\alpha|+n+1}.$$

**Proof** Let $0<\beta<|\alpha|+n+1$ and $f\in L^2_\alpha(\mathbb{K})$. For all $r>0$, we have

$$\begin{aligned}\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} &\leq \|\mathfrak{F}(f\chi_{B_r})\chi_E\|_{2,\gamma_\alpha}+\left\|\mathfrak{F}\left(f\chi_{B_r^c}\right)\chi_E\right\|_{2,\gamma_\alpha}\\ &\leq (\gamma_\alpha(E))^{\frac12}\|\mathfrak{F}(f\chi_{B_r})\|_{\infty,\gamma_\alpha}+\left\|\mathfrak{F}\left(f\chi_{B_r^c}\right)\right\|_{2,\gamma_\alpha}\\ &\leq (\gamma_\alpha(E))^{\frac12}\|f\chi_{B_r}\|_{1,\eta_\alpha}+\left\|\mathfrak{F}\left(f\chi_{B_r^c}\right)\right\|_{2,\gamma_\alpha}. \qquad (4.1)\end{aligned}$$

On the one hand, by Hölder's inequality, we get

$$\begin{aligned}\|f\chi_{B_r}\|_{1,\eta_\alpha} &\leq \left\||(x,t)|^{-\beta}_{\mathbb{K}}\chi_{B_r}\right\|_{2,\eta_\alpha}\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha}\\ &\leq A_{\alpha,\beta,n}r^{-\beta+|\alpha|+n+1}\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha}, \qquad (4.2)\end{aligned}$$

where

$$A_{\alpha,\beta,n}=\left(\frac{\omega_{\alpha,n}}{2(-\beta+|\alpha|+n+1)}\right)^{\frac12}.$$

On the other hand, we apply Plancherel formula (2.2), we get

$$\begin{aligned}\left\|\mathfrak{F}\left(f\chi_{B_r^c}\right)\right\|_{2,\gamma_\alpha} &= \left\|f\chi_{B_r^c}\right\|_{2,\eta_\alpha}\\ &\leq \left\||(x,t)|^{-\beta}_{\mathbb{K}}\chi_{B_r^c}\right\|_{\infty,\eta_\alpha}\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha}\\ &\leq r^{-\beta}\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha}. \qquad (4.3)\end{aligned}$$

Now, combining the relations (4.1), (4.2), and (4.3), the following relation holds for all $r>0$,

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha}\leq\left(r^{-\beta}+(\gamma_\alpha(E))^{\frac12}A_{\alpha,\beta,n}r^{-\beta+|\alpha|+n+1}\right)\left\||(x,t)|^\beta_{\mathbb{K}}f\right\|_{2,\eta_\alpha}.$$

We minimize the above relation to find the desired inequality. □

**Theorem 4.2** *Let* $\beta$ *be a real number such that* $\beta > |\alpha| + n + 1$, *there is a constant* $M := M(\alpha, \beta, n)$ *such that for every* $f \in L^2_\alpha(\mathbb{K})$ *and every measurable set* $E \subset \widehat{\mathbb{K}}$ *with* $0 < \gamma_\alpha(E) < \infty$, *we have*

$$\left\|\mathfrak{F}(f)\chi_E\right\|_{2,\gamma_\alpha} \leq M\left(\gamma_\alpha(E)\right)^{\frac{1}{2}} \|f\|_{2,\eta_\alpha}^{1-\frac{|\alpha|+n+1}{\beta}} \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|_{2,\eta_\alpha}^{\frac{|\alpha|+n+1}{\beta}},$$

*where*

$$M = \left(\frac{\pi\omega_{\alpha,n}}{2\left(\beta-(|\alpha|+n+1)\right)\sin\left(\pi\frac{|\alpha|+n+1}{\beta}\right)}\right)^{\frac{1}{2}} \left(\frac{|\alpha|+n+1}{\beta-(|\alpha|+n+1)}\right)^{-\frac{|\alpha|+n+1}{2\beta}}.$$

**Proof** Since $\beta > |\alpha| + n + 1$, the function

$$(x,t) \longmapsto \left(1+|(x,t)|^{2\beta}_{\mathbb{K}}\right)^{-1}$$

belongs to $L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$. By using Hölder's inequality, a straightforward calculus gives us the following inequalities, for every function $f$ in $L^2_\alpha(\mathbb{K})$,

$$\begin{aligned}
\|f\|^2_{1,\eta_\alpha} &\leq \left\|\left(1+|(x,t)|^{2\beta}_{\mathbb{K}}\right)^{-\frac{1}{2}}\right\|^2_{2,\eta_\alpha} \left\|\left(1+|(x,t)|^{2\beta}_{\mathbb{K}}\right)^{\frac{1}{2}} f\right\|^2_{2,\eta_\alpha} \\
&\leq \left\|\left(1+|(x,t)|^{2\beta}_{\mathbb{K}}\right)^{-\frac{1}{2}}\right\|^2_{2,\eta_\alpha} \left(\|f\|^2_{2,\eta_\alpha} + \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|^2_{2,\eta_\alpha}\right) \\
&\leq \frac{\pi\omega_{\alpha,n}}{2\beta\sin\left(\pi\frac{|\alpha|+n+1}{\beta}\right)} \left(\|f\|^2_{2,\eta_\alpha} + \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|^2_{2,\eta_\alpha}\right). \qquad (4.4)
\end{aligned}$$

Let $\varrho > 0$, we replace $f$ by $f_\varrho$ in relation (4.4). Since $\|f_\varrho\|^2_{2\eta_\alpha} = \varrho^{-2(|\alpha|+n+1)}\|f\|^2_{2,\eta_\alpha}$ and $\left\||(x,t)|^\beta_{\mathbb{K}} f_\varrho\right\|^2_{2,\eta_\alpha} = \varrho^{2\beta-2(|\alpha|+n+1)}\left\||(x,t)|^\beta_{\mathbb{K}} f\right\|^2_{2,\eta_\alpha}$, we get for all $\varrho > 0$

$$\|f\|^2_{1,\eta_\alpha} \leq \frac{\pi\omega_{\alpha,n}}{2\beta\sin\left(\pi\frac{|\alpha|+n+1}{\beta}\right)} \left(\varrho^{-2(|\alpha|+n+1)}\|f\|^2_{2,\eta_\alpha} + \varrho^{2\beta-2(|\alpha|+n+1)}\left\||(x,t)|^\beta_{\mathbb{K}} f\right\|^2_{2,\eta_\alpha}\right).$$

Minimizing over $\varrho > 0$, we get

$$\|f\|^2_{1,\eta_\alpha} \leq M^2 \|f\|_{2,\eta_\alpha}^{2-\frac{2(|\alpha|+n+1)}{\beta}} \left\||(x,t)|^\beta_{\mathbb{K}} f\right\|_{2,\eta_\alpha}^{2\frac{|\alpha|+n+1}{\beta}}.$$

The inequality above gives us the result since we have

$$\left\|\mathfrak{F}(f)\chi_E\right\|_{2,\gamma_\alpha} \leq \left\|\mathfrak{F}(f)\right\|_{\infty,\gamma_\alpha}\left(\gamma_\alpha(E)\right)^{\frac{1}{2}} \leq \|f\|_{1,\eta_\alpha}\left(\gamma_\alpha(E)\right)^{\frac{1}{2}}.$$

□

**Theorem 4.3** *Let $\beta = |\alpha| + n + 1$, then there is a constant $\Omega := \Omega(\alpha, n)$ such that for all nonzero function $f \in L^2_\alpha(\mathbb{K})$ and measurable set $E \subset \widehat{\mathbb{K}}$ such that $0 < \gamma_\alpha(E) < \infty$, we have*

$$\|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} \leq \Omega(\gamma_\alpha(E))^{\frac{1}{2(|\alpha|+n+1)}} \|f\|_{2,\eta_\alpha}^{1-\frac{1}{\beta}} \left\| |(x,t)|_{\mathbb{K}}^{\beta} f \right\|_{2,\eta_\alpha}^{\frac{1}{\beta}},$$

*where*

$$\Omega = \left(\frac{\omega_{\alpha,n}(|\alpha|+n)}{2}\right)^{\frac{1}{2(|\alpha|+n+1)}} \left(1 + \frac{1}{|\alpha|+n}\right) \beta(\beta-1)^{\frac{1}{\beta}-1}.$$

**Proof** Since $\beta = |\alpha| + n + 1 > 1$, we have for all $\varepsilon > 0$

$$\frac{|(x,t)|_{\mathbb{K}}}{\varepsilon} \leq 1 + \frac{|(x,t)|_{\mathbb{K}}^{\beta}}{\varepsilon^{\beta}}; \ (x,t) \in \mathbb{K}.$$

Thus, we get

$$\||(x,t)|_{\mathbb{K}} f\|_{2,\eta_\alpha} \leq \varepsilon \|f\|_{2,\eta_\alpha} + \varepsilon^{1-\beta} \left\| |(x,t)|_{\mathbb{K}}^{\beta} f \right\|_{2,\eta_\alpha}.$$

Minimizing the right hand side on $\varepsilon$, we get

$$\||(x,t)|_{\mathbb{K}} f\|_{2,\eta_\alpha} \leq \beta(\beta-1)^{\frac{1}{\beta}-1} \|f\|_{2,\eta_\alpha}^{1-\frac{1}{\beta}} \left\| |(x,t)|_{\mathbb{K}}^{\beta} f \right\|_{2,\eta_\alpha}^{\frac{1}{\beta}}.$$

The result follows from the relation above and the following inequality which is deduced from Theorem 4.1

$$\begin{aligned} \|\mathfrak{F}(f)\chi_E\|_{2,\gamma_\alpha} &\leq \left(\frac{\omega_{\alpha,n}(|\alpha|+n)}{2}\right)^{\frac{1}{2(|\alpha|+n+1)}} \left(1 + \frac{1}{|\alpha|+n}\right) \\ &\times (\gamma_\alpha(E))^{\frac{1}{(|\alpha|+n+1)}} \||(x,t)|_{\mathbb{K}} f\|_{2,\eta_\alpha}. \end{aligned}$$

□

As an application of the local uncertainty principle, based on the work of Ghobber and Jaming [3], we are able to regain the Heisenberg's inequality as it is detailed below

1. **First case :** Assume that $0 < a < |\alpha| + n + 1$ and $b > 0$. Using Plancherel formula (2.2) and Theorem 4.1, we get for all $r > 0$

$$\begin{aligned} \|f\|_{2,\eta_\alpha}^2 &= \|\mathfrak{F}(f)\|_{2,\gamma_\alpha}^2 \\ &= \left\|\chi_{\mathcal{B}_{\widehat{\mathbb{K}},r}} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2 + \left\|\chi_{\mathcal{B}^c_{\widehat{\mathbb{K}},r}} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2 \\ &\leq K^2 \left(\gamma_\alpha\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right)\right)^{\frac{a}{|\alpha|+n+1}} \||(x,t)|_{\mathbb{K}}^a f\|_{2,\eta_\alpha}^2 + r^{-2b} \left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^b \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2. \end{aligned}$$

By relation (2.4), we deduce that

$$\|f\|_{2,\eta_\alpha}^2 \leq K^2 \Xi_{\alpha,n}^{\frac{a}{|\alpha|+n+1}} r^a \||(x,t)|_{\mathbb{K}}^a f\|_{2,\eta_\alpha}^2 + r^{-2b} \left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^b \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2,$$

where $K$ is given in Theorem 4.1.

By minimizing the right hand side of the above inequality over $r>0$, we get the following inequality:

$$\||(x,t)|_{\mathbb{K}}^{a} f\|_{2,\eta_\alpha}^{\frac{2b}{a+2b}} \left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{a}{a+2b}} \geq \Sigma_{\alpha,n,a,b} \|f\|_{2,\eta_\alpha},$$

where

$$\Sigma_{\alpha,n,a,b} = \left(K^2 \Xi_{\alpha,n}^{\frac{a}{|\alpha|+n+1}}\right)^{-\frac{b}{a+2b}} \left(\frac{2b}{a}\right)^{-\frac{a}{2(a+2b)}} \left(1+\frac{a}{2b}\right)^{-\frac{1}{2}}.$$

2. **Second case :** Assume that $a>|\alpha|+n+1$ and $b>0$. We use Plancherel formula (2.2) and Theorem 4.2, we find

$$\begin{aligned} \|f\|_{2,\eta_\alpha}^2 &\leq M^2 \gamma_\alpha\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right) \|f\|_{2,\eta_\alpha}^{2-2\frac{|\alpha|+n+1}{a}} \||(x,t)|_{\mathbb{K}}^{a} f\|_{2,\eta_\alpha}^{2\frac{|\alpha|+n+1}{a}} \\ &+ \left\|\chi_{\mathcal{B}_{\widehat{\mathbb{K}},r}^c} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2, \end{aligned} \tag{4.5}$$

where $M$ is given in Theorem 4.2.

Using Plancherel formula (2.2) again, we get

$$\left\|\chi_{\mathcal{B}_{\widehat{\mathbb{K}},r}^c} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2 \leq \|f\|_{2,\eta_\alpha}^{2-2\frac{|\alpha|+n+1}{a}} \left\|\chi_{\mathcal{B}_{\widehat{\mathbb{K}},r}^c} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{2\frac{|\alpha|+n+1}{a}}. \tag{4.6}$$

Substituting (4.6) in (4.5), we obtain

$$\begin{aligned} \|f\|_{2,\eta_\alpha}^{2\frac{|\alpha|+n+1}{a}} &\leq M^2 r^{|\alpha|+n+1} \Xi_{\alpha,n} \||(x,t)|_{\mathbb{K}}^{a} f\|_{2,\eta_\alpha}^{2\frac{|\alpha|+n+1}{a}} \\ &+ r^{-\frac{2b}{a}(|\alpha|+n+1)} \left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{2\frac{|\alpha|+n+1}{a}}. \end{aligned}$$

We minimize the right hand side of the above inequality, we get

$$\|f\|_{2,\eta_\alpha} \leq H_{\alpha,n,a,b} \||(x,t)|_{\mathbb{K}}^{a} f\|_{2,\eta_\alpha}^{\frac{2b}{a+2b}} \left\||(m,\lambda)|_{\widehat{\mathbb{K}}}^{b} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^{\frac{a}{a+2b}},$$

where

$$H_{\alpha,n,a,b} = \left(\left(M^2 \Xi_{\alpha,n} \frac{2b}{a}\right)^{-\frac{2b}{a+2b}} \left(1+\frac{2b}{a}\right)\right)^{\frac{a}{2(|\alpha|+n+1)}}.$$

3. **Third case :** Assume that $a=|\alpha|+n+1$ and $b>0$. Using Plancherel formula (2.2) and Theorem 4.3, we get for all $r>0$

$$\|f\|_{2,\eta_\alpha}^2 \leq \Omega^2 \left(\gamma_\alpha\left(\mathcal{B}_{\widehat{\mathbb{K}},r}\right)\right)^{\frac{1}{|\alpha|+n+1}} \|f\|_{2,\eta_\alpha}^{2-\frac{2}{a}} \||(x,t)|_{\mathbb{K}}^{a} f\|_{2,\eta_\alpha}^{\frac{2}{a}} + \left\|\chi_{\mathcal{B}_{\widehat{\mathbb{K}},r}^c} \mathfrak{F}(f)\right\|_{2,\gamma_\alpha}^2, \tag{4.7}$$

where $\Omega$ is given in Theorem 4.3.

Again the Plancherel formula (2.2) allows us to say that

$$\left\|\chi_{\mathcal{B}^c_{\widehat{\mathbb{K}},r}}\mathfrak{F}(f)\right\|^2_{2,\gamma_\alpha} \leq \|f\|^{2-\frac{2}{a}}_{2,\eta_\alpha}\left\|\chi_{\mathcal{B}^c_{\widehat{\mathbb{K}},r}}\mathfrak{F}(f)\right\|^{\frac{2}{a}}_{2,\gamma_\alpha}. \tag{4.8}$$

Substituting (4.8) in (4.7), we get

$$\|f\|^{\frac{2}{a}}_{2,\eta_\alpha} \leq r\Omega^2\Xi_{\alpha,n}^{\frac{1}{|\alpha|+n+1}}\,\||(x,t)|^a_{\mathbb{K}}f\|^{\frac{2}{a}}_{2,\eta_\alpha} + r^{-\frac{2b}{a}}\left\||(m,\lambda)|^b_{\widehat{\mathbb{K}}}\mathfrak{F}(f)\right\|^{\frac{2}{a}}_{2,\gamma_\alpha}.$$

Minimizing over $r>0$, we get

$$\|f\|_{2,\eta_\alpha} \leq E^{\frac{a}{2}}_{\alpha,n,a,b}\,\||(x,t)|^a_{\mathbb{K}}f\|^{\frac{2b}{a+2b}}_{2,\eta_\alpha}\left\||(m,\lambda)|^b_{\widehat{\mathbb{K}}}\mathfrak{F}(f)\right\|^{\frac{a}{a+2b}}_{2,\gamma_\alpha},$$

where

$$E_{\alpha,n,a,b} = \left(\Omega^2\Xi_{\alpha,n}^{\frac{1}{|\alpha|+n+1}}\right)^{\frac{2b}{a+2b}}\left(\frac{2b}{a}\right)^{\frac{a}{a+2b}}\left(1+\frac{a}{2b}\right).$$

□

## References


[1] De Bruijn NG. Uncertainty principle in Fourier analysis. In: Inequalities, Shisha O (editor). Proceedings of Symposium held at Wright-Patterson air force base; Ohio, USA 1967. pp. 57-71.

[2] Faris WG. Inequalities and uncertainty principles. Journal of Mathematical Physics 1978; 19: 461-466. doi: 10.1063/1.523667

[3] Ghobber S, Jaming P. Uncertainty principles for integral operators. Studia Mathematica 2014; 220 (3): 197-220. doi: 10.4064/sm-220-3-1

[4] Guliyev VS. Polar coordinates in Laguerre hypergroup. Khazar Journal of Mathematics 2006; 2 (3): 11-20.

[5] Heisenberg W. Uber den anschaulichen inhalt der quantentheoretiscchen kinematic und mechanik. Zeitschrift für Physik 1927; 43: 172-198 (in German).

[6] Huang J, Liu H. The weak type (1,1) estimates of maximal functions on the Laguerre hypergroup. Canadian Mathematical Bulletin 2010; 53 (3): 491-502. doi: 10.4153/cmb-2010-058-6

[7] Kamoun L, Selmi R. Harmonic analysis associated to the multivariate Laguerre function. Mediterranean Journal of Mathematics 2018; 15 (2): Article 39. doi: 10.1007/s00009-018-1085-0

[8] Price JF. Inequalities and local uncertainty principles. Journal of Mathematical Physics 1983; 24: 1711-1714. doi: 10.1063/1.525916

[9] Price JF. Sharp local uncertainty inequalities. Studia Mathematica 1987; 85: 37-45. doi: 10.4064/sm-85-1-37-45

[10] Rahmouni A. Uncertainty inequalities on Laguerre hpergroup. Mediterranean Journal of Mathematics 2013; 10 (1): 333-351. doi: 10.1007/s00009-012-0198-0

[11] Rassias JM. On the Heisenberg-Pauli-Weyl inequality. JIPAM. Journal of Inequalities in Pure and Applied Mathematics 2004; 5 (1): Article 4.

[12] Rassias JM. On the Heisenberg-Weyl inequality. JIPAM. Journal of Inequalities in Pure and Applied Mathematics 2005; 6 (1): Article 11.

[13] Rassias JM. On the sharpened Heisenberg-Weyl inequality. JIPAM. Journal of Inequalities in Pure and Applied Mathematics 2006; 7 (3): Article 80.

[14] Rösler M. An uncertainty principle for the Dunkl transform. Bulletin of the Australian Mathematical Society 1999; 59 (3): 353-360. doi: 10.1017/s0004972700033025

[15] Rösler M, Voit M. An uncertainty principle for Hankel transform. Proceeding of the American Mathematical Society 1999; 127 (1): 183-194. doi: 10.1090/s0002-9939-99-04553-0

[16] Schwartz L. Topologie générale et analyse fonctionnelle. Paris, France : Hermann, 1970 (in French).

[17] Shimeno N. A note on the uncertainty principle for the Dunkl transform. Journal of Mathematical Sciences (University of Tokyo) 2001; 8 (1): 33-42.

[18] Stein EM. Singular integrals and differentiability properties of functions. New Jersey, USA : Princeton University Press, 1970.