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ŞUAYİP TOPRAKSEVEN

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On Lyapunov-type inequalities for boundary value problems of fractional Caputo-Fabrizio derivative

Şuayip TOPRAKSEVEN*
Department of Computer Science, Faculty of Engineering, Artvin Çoruh University, Artvin, Turkey

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Abstract: In this study, Lyapunov-type inequalities for fractional boundary value problems involving the fractional Caputo Fabrizio differential equation with mixed boundary conditions when the fractional order of $\beta \in (1,2]$ and Dirichlet-type boundary condition when the fractional order of $\sigma \in (2,3]$ have been derived. Some consequences of the results related to the fractional Sturm–Liouville eigenvalue problems have also been given. Additionally, we examine the nonexistence of the solution of the fractional boundary value problem.

Key words: Lyapunov-type inequality; fractional Caputo-Fabrizio derivative; high order fractional

1. Introduction

The famous Lyapunov-type inequality [9] asserts that if $y \in C^2[a,b]$ is a nontrivial solution of the boundary value problem and $p \in C[a,b]$,

$$y''(x) + p(x)y(x) = 0, \quad x \in [a,b], \tag{1.1}$$

$$y(a) = y(b) = 0, \tag{1.2}$$

then

$$\int_a^b |p(x)|\, dx > \frac{4}{b-a}. \tag{1.3}$$

The number 4 is sharp in the sense that it cannot be changed by a bigger number. There have been many extensions and generalizations of Lyapunov-type inequality in the last decades. An interesting and important extension of this type of inequality has been established by Hartman in [4]. The extension to the third order linear differential equation has been obtained in [12] stating that if $y(x)$ is a nontrivial solution of the following third order differential equation

$$y'''(x) + p(x)y(x) = 0, \quad x \in [a,b], \tag{1.4}$$

$$y(a) = y(b) = 0, \tag{1.5}$$

where $p \in C[a,b]$, then

$$\int_a^b |p(x)|\, dx > \frac{4}{(b-a)^2}. \tag{1.6}$$

*Correspondence: topraksp@artvin.edu.tr
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This inequality was derived under the condition that $y''(d) = 0$ for some $d \in [a, b]$. They also obtained the similar inequality in the case when $y(x)$ has consecutive 3 zeros. However, the latter condition implies the former one. Very recently, Lyapunov-type inequalities have been achieved for the fractional boundary value problems. In [2], the author investigated the following fractional boundary value problem of the Riemann-Liouville differential equation

$$D_a^\alpha u(t) + p(t)y(t) = 0, \quad t \in [a, b], \quad \alpha \in (1, 2), \tag{1.7}$$

$$u(a) = u(b) = 0, \tag{1.8}$$

where D_a^α is the Riemann-Liouville differential operator of order α (see below for the definition). If p is continuous function on $[a, b]$ and u is a nonzero solution of (1.7) and (1.8), then the following interesting inequality was proved in [2]:

$$\int_a^b |p(s)|\, ds > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{1.9}$$

It is worth noting that in the limit case when $\alpha \to 2$, the fractional differential equation (1.7) becomes the second order ordinary differential equation (1.1) and the Lyapunov-type inequality (1.9) turns into the classical Lyapunov's inequality (1.3).

In [3], the author replaced the Riemann-Liouville derivative by the Caputo differential operator ${}^C D_a^\alpha$ and investigated an interval in which some Mittag-Leffler functions have no real zeros. Some generalizations and extentions of Lyapunov-type inequalities to the fractional boundary value problems by using different boundary conditions have been studied in the literature. For example, the fractional boundary condition has been used in [13], a Robin boundary condition in [5] , a mixed boundary condition in [6] and a class of fractional boundary value problem have been studied in [11]. More recently, Caputo and Fabrizio proposed a fractional derivative [1] with regular kernel. Unlike the classical fractional derivative such as Riemann-Liouville and the Caputo derivative, the solution of this new fractional derivative contains no singular functions, thus it describes better for modelling material heterogeneities and structures with different scale. For more discussion on this new fractional derivative, we refer the reader to [10].

The following fractional boundary value problem have been studied in [6]

$${}^C D_a^\alpha u(t) + p(t)y(t) = 0, \quad t \in [a, b], \quad \alpha \in (1, 2],$$

subject to the mixed boundary condition

$$u'(a) = y(b) = 0, \tag{1.10}$$

and the authors proved the following Lyapunov-type inequality

$$\int_a^b (b-t)^{\alpha-1}|p(t)|\, dt \geq \Gamma(\alpha). \tag{1.11}$$

In this paper, a Lyapunov-type inequality will be established for the following boundary value problem involving the Caputo-Fabrizio fractional derivative of order $\beta \in (1, 2]$ with the boundary conditions (1.10),

$$\begin{aligned} {}^{CF}D_a^\beta(u)(x) + p(x)u(x) = 0, \quad a \leq x \leq b,\\ u'(a) = u(b) = 0, \end{aligned} \tag{1.12}$$

where ${}^{CF}D_a^{\beta}$ is the Caputo-Fabrizio differential operator of order $\beta \in (1,2]$.

On the other hand, very few results on Lyapunov-type inequality for the fractional boundary value problem have been studied in the literature when the fractional order $\alpha \in (2,3]$. To fill this gap, a Lyapunov-type inequality for the following boundary value problem of the fractional Caputo-Fabrizio differential equation of order $\sigma \in (2,3]$

$$\begin{aligned} {}^{CF}D^{\sigma}(u)(x) + p(x)u(x) = 0, \quad a \le x \le b, \\ u(a) = u''(a) = u(b) = 0, \end{aligned} \tag{1.13}$$

will be investigated. We follow the idea of the paper [12] and we assume that the nontrivial solution satisfies $u''(a) = 0$. We mention that the related works involving the Caputo-Fabrizio derivative of order $\alpha \in (1,2]$ with the Dirichlet boundary condition is presented in [8] and the Caputo-Fabrizio derivative of order $\alpha \in (2,3]$ with the mixed boundary condition is studied in [14].

The rest of the paper is organized as follows: In Section 2, preliminaries, definitions and related works that are needed in this study have been introduced. The associated Green functions and their properties for linear problems have been presented and the main results and their consequences have been given in Section 3. Nonexistence of some boundary value problems of the fractional Caputo-Fabrizio differential equation and the fractional Sturm–Liouville eigenvalue problems are given in Section 4.

2. Preliminaries and lemmas

In this preparatory section, we present some definitions and previous results of the new fractional Caputo–Fabrizio derivative that are needed in the analyses presented in this work.

Definition 2.1 *[7] Let $\alpha \in (0,1]$ and $f \in C([a,b]), a < b$. The Riemann-Liouvile fractional derivative of the function f of order α defined as*

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{\alpha} f(t)\, dt. \tag{2.1}$$

Definition 2.2 *[1] Let $\alpha \in (0,1]$ and $f \in H^1(a,b), a < b$. The Caputo fractional derivative of the function f of order α defined by*

$${}^{C}D_a^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{\alpha} f'(t)\, dt. \tag{2.2}$$

Definition 2.3 *[10] Given $a < b$ and $f \in H^1(a,b)$, the fractional Caputo-Fabrizio derivative of the function f of order $\alpha \in (0,1)$ is defined for $t \ge 0$*

$${}^{CF}D_a^{\alpha} f(x) = \frac{1}{1-\alpha} \int_a^x \exp\Big(-\frac{\alpha}{1-\alpha}(x-t)\Big) f'(t)\, dt. \tag{2.3}$$

Definition 2.4 *[10] The Caputo-Fabrizio fractional integral of a function $f \in L_1(a,b)$ of order $\alpha \in (0,1)$ is defined as*

$${}^{CF}\mathcal{I}_a^{\alpha}(f)(x) = (1-\alpha) f(x) + \alpha \int_a^x f(s) ds. \tag{2.4}$$

We note that unlike the most fractional integral operator such as the Riemann-Liouvile and Caputo fractional integral operator, the Caputo-Fabrizio fractional integral operator does not have any fractional power kernel.

The Caputo-Fabrizio fractional of a function $f \in H^1(a,b)$ of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ defined as (e.g., see [1])

$$^{CF}D_a^{\alpha+n} f(x) :=^{CF} D^{\alpha}(^{CF}D^n f(x)).$$

Theorem 2.5 *[1] Let the function $f(x)$ satisfy $f^{(k)}(a) = 0, \quad k = 1, 2, \ldots, n$, then the equality*

$$^{CF}D_a^{\alpha}(^{CF}D_a^n f(x)) =^{CF} D_a^n(^{CF}D_a^{\alpha} f(x)) \tag{2.5}$$

holds.

Definition 2.6 *For $\sigma = \alpha + 2$ with $\alpha \in (0,1)$, the Caupto-Fabrizio fractional derivative a function f of order σ defined as*

$$^{CF}D_a^{\sigma} f(x) = \frac{1}{1-\alpha} \int_a^x \exp\Big(- \frac{\alpha}{1-\alpha}(x-t)\Big) f'''(t)\, dt. \tag{2.6}$$

Note that the equality $^{CF}D_a^{\alpha}(^{CF}D_a^2 f(x)) = {}^{CF}D_a^2(^{CF}D_a^{\alpha} f(x))$ is defined unambiguously when $f''(0) = 0$. (see [1])

Definition 2.7 *For a function $f(x)$ defined on $[0,\infty)$, the Laplace transformation $F(s)$ of $f(x)$ is defined by*

$$F(s) = \mathcal{L}\Big\{ f(x) \Big\}(s) = \int_0^{\infty} \exp(-sx) f(x)\, dx.$$

Lemma 2.8 *[1] The Laplace transform of the Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is given by*

$$\mathcal{L}\Big\{^{CF}D_0^{\sigma} f(x)\Big\}(s) = \frac{s^{n+1}\mathcal{L}\{f(x)\}(s) - s^n f(0) - s^{n-1} f'(0) - \cdots - f^{(n)}(0)}{s + \alpha(1-s)}. \tag{2.7}$$

Lemma 2.9 *Let $\alpha \in (0,1]$ and h be a continuous function on $[0,1]$ satisfying the compatibility condition*

$$h(0) = 0. \tag{2.8}$$

Then the following boundary value problem of the fractional Caputo-Fabrizio differential equation has the unique solution

$$\begin{cases} ^{CF}D_0^{\beta} u(x) + h(x) = 0, \quad \beta = \alpha + 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{2.9}$$

Proof By applying the Laplace transform on the equation (2.9) and appealing Lemma 2.8, we have that

$$\frac{s^2 U(s) - s u(0) - u'(0)}{s + \alpha(1-s)} = -H(s),$$

where $U(s) := \mathcal{L}\Big\{u(x)\Big\}(s)$ and $H(s) := \mathcal{L}\Big\{h(x)\Big\}(s)$. We can rewrite the above equation as

$$U(s) = \frac{1}{s}u(0) + \frac{1}{s^2}u'(0) - \frac{1-\alpha}{s}H(s) - \frac{\alpha}{s^2}H(s).$$

Taking the inverse Laplace transform, we get

$$u(x) = u(0) + xu'(0) - (1-\alpha)\int_0^x h(t)\,dt - \alpha\int_0^x (x-t)h(t)\,dt. \tag{2.10}$$

Observe that the initial condition $u'(0) = 0$ is achieved if and only if the compatibility condition $h(0) = 0$ is satisfied. More precisely, differentiating the equation (2.10) with respect to x variable, we find that

$$u'(x) = u'(0) - (1-\alpha)h(x) - \alpha\int_0^x h(t)\,dt.$$

Substituting 0 for x gives that $u'(0) = u'(0) - (1-\alpha)h(0)$ which implies that $h(0) = 0$. The condition $u(1) = 0$ is satisfied if $u(0) = (1-\alpha)\int_0^1 h(t)\,dt - \alpha\int_0^1(1-t)h(t)\,dt$. □

Notice that if $\alpha \to 1$, then we recover the solution of the classical second order boundary value problem.

3. Main results

In this section, we derive Lyapunov-type inequalities for the fractional differential equations (1.12) and (1.13) involving the Caputo- Fabrizio fractional derivative of order $\beta \in (1,2]$ and $\sigma \in (2,3]$ respectively. We also state some important corollaries of the main results that are important for eigenvalue problems.

Lemma 3.1 *Assume that the compatibility condition $p(a)u(a) = 0$ holds. Then $u(x)$ solves the boundary value problem of the fractional Caputo-Fabrizio differential equation (1.12) if and only if it solves the following integral equation*

$$u(x) = \int_a^b K(t,x)p(t)u(t)\,dt, \tag{3.1}$$

where the Green function $K(t,x)$ is given by

$$K(t,x) = \begin{cases} \alpha(b-x), & a \le t \le x \le b, \\ (1-\alpha) + \alpha(b-t), & a \le x \le t \le b. \end{cases}$$

Proof Because the initial conditions are not at $x = 0$, we cannot directly apply the Laplace transform of the fractional derivatives. For this reason, we rewrite the fractional boundary value problem (1.12) so that the initial condition is at $x = 0$. To do this, we make use of a change of variable by letting $\eta := x - a$ so that $x = \eta + a$. Now, substituting this for x gives us that

$$\begin{aligned} {}^{CF}D_a^{\beta}u(\eta+a) &= \frac{1}{1-\alpha}\int_a^{\eta+a} \exp\Big(-\frac{\alpha}{1-\alpha}(\eta+a-t)\Big)u''(t)\,dt \\ &= \frac{1}{1-\alpha}\int_0^{\eta} \exp\Big(-\frac{\alpha}{1-\alpha}(\eta-s)\Big)u''(s+a)\,ds. \end{aligned}$$

Let $y(\eta) := u(\eta + a)$. Then, we infer from the above equation

$$^{CF}D_a^\beta u(x) =^{CF} D_a^\beta u(\eta + a) =^{CF} D_0^\beta y(\eta).$$

The boundary conditions for $y(\eta)$ now read

$$y'(0) = u'(a + 0) = u'(a) = 0,$$

$$y(b - a) = u(b - a + a) = u(b) = 0.$$

With this new variable, the boundary value problem (1.12) turns into

$$\begin{cases} ^{CF}D_0^\beta y(\eta) + q(\eta)y(\eta) = 0, \\ y'(0) = y(c) = 0, \end{cases} \tag{3.2}$$

where $q(\eta) := p(\eta + a)$ and $c := b - a$.

Now, taking the Laplace transform on the equation (3.2) gives that

$$\mathcal{L}\Big\{^{CF}D^\beta y(\eta)\Big\}(s) = \mathcal{L}\Big\{q(\eta)y(\eta)\Big\}(s).$$

Using Lemma 2.8, we find that

$$\frac{s^2Y(s) - sy(0) - y'(0)}{s + \alpha(1 - s)} = -Q(s),$$

where $Y(s) := \mathcal{L}\Big\{y(\eta)\Big\}(s)$ and $Q(s) := \mathcal{L}\Big\{q(\eta)y(\eta)\Big\}(s)$. The initial condition $y'(0) = 0$ and a little algebraic manipulation reveal that

$$Y(s) = \frac{1}{s}y(0) - \frac{1 - \alpha}{s}Q(s) - \frac{\alpha}{s^2}Q(s).$$

Now, appealing the inverse Laplace operator on the last equation leads to

$$y(\eta) = y(0) - (1 - \alpha)\int_0^\eta q(t)y(t)\,dt - \alpha\int_0^\eta (\eta - t)q(t)y(t)\,dt. \tag{3.3}$$

The boundary condition $y(c) = 0$ implies that

$$y(\eta) = \int_0^\eta \alpha(b - a - \eta)q(t)y(t)\,dt + \int_\eta^{b-a} \Big(1 - \alpha + \alpha(b - a - t)\Big)q(t)y(t)\,dt.$$

Notice that

$$u(x) = u(\eta + a) = y(\eta) = y(x - a).$$

Thus, we observe that

$$\begin{aligned} u(x) = y(x - a) &= \int_0^{x-a} \alpha(b - x)p(t + a)u(t + a)\,dt + \int_{x-a}^{b-a} \Big(1 - \alpha + \alpha(b - a - t)\Big)p(t + a)u(t + a)\,dt \\ &= \int_a^x \alpha(b - x)p(t)u(t)\,dt + \int_x^b \Big(1 - \alpha + \alpha(b - t)\Big)p(t)u(t)\,dt = \int_a^b K(t, x)p(t)u(t)\,dt. \end{aligned} \tag{3.4}$$

So, we have shown that if u solves the fractional boundary value problem (1.12), then it also solves the integral equation (3.1). Next, assume that u solves the integral equation (3.1). Implications in reverse direction show that the integral representation (3.1) solves the boundary value problem (1.12). It remains to show that the boundary conditions are satisfied. Notice that $K(t,b)=0$ implies that $u(b)=0$. Differentiating the equation (3.4) with respect to x yields

$$u'(x) = -\alpha \int_a^x p(t)u(t)\,dt - (1-\alpha)p(x)u(x).$$

Now, the compatibility condition $p(a)u(a)=0$ leads to have $u'(a)=0$. This completes the proof of the lemma. □

Lemma 3.2 *The Green function $K(t,x)$ satisfies the following bound*

$$\Big|K(t,x)\Big| \leq (1-\alpha)+\alpha(b-a), \quad x,t\in[a,b], \quad \alpha\in(0,1]. \tag{3.5}$$

Proof Observe that $0 \leq b-x \leq b-a$ and $0 \leq b-t \leq b-a$ when $x,t \in [a,b]$. Thus, we have $\alpha(b-x) \leq (1-\alpha)+\alpha(b-a)$ for $a \leq t \leq x \leq b$ and $(1-\alpha)+\alpha(b-t) \leq (1-\alpha)+\alpha(b-a)$ when $a \leq x \leq t \leq b$. The proof is now completed. □

We state the first main result of this paper in the next theorem.

Theorem 3.3 *Assume that $p\in C([a,b],\mathbb{R})$ (continuous real-valued function on $[a,b]$) and the compatibility condition $p(a)u(a)=0$ is satisfied. If the boundary value problem of the fractional Caputo-Fabrizio differential equation (1.12) of order $\beta\in(1,2]$ has a nonzero solution, then the function p satisfies*

$$\int_a^b |p(\xi)|\,d\xi > \frac{1}{(1-\alpha)+\alpha(b-a)}, \quad \alpha\in(0,1]. \tag{3.6}$$

Proof By Lemma 3.1, we infer that the solution for the fractional boundary value problem (1.12) is given by

$$u(x) = \int_a^b K(t,x)p(t)u(t)\,dt, \quad x,t\in[a,b].$$

Let $C[a,b]$ be the Banach space of continuous function on $[a,b]$ with maximum norm $\|u\| = \max_{x\in[a,b]}|u(x)|$. Then, taking maximum norm on the either side of the last equation gives that

$$\|u\| \leq \max_{x\in[a,b]} \int_a^b \Big|K(t,x)p(t)\Big|\,dt\|u\|$$

which leads to

$$1 \leq \max_{x\in[a,b]} \int_a^b \Big|K(t,x)p(t)\Big|\,dt,$$

and using the bound (3.5) on the Green function $K(t,x)$ we have

$$\int_a^b |p(t)|\,dt \geq \frac{1}{(1-\alpha)+\alpha(b-a)}.$$

□

Remark 3.4 *Letting $\alpha \to 1$ on the bound in Theorem 3.3, we can find the similar result to the one given in [6].*

Next, we state a Lyapunov-type inequality for boundary value problem of the high order fractional Caputo-Fabrizio differential equation. First, we derive the associated Green function and its properties. By using these properties of the Green function, we present the one of the main results of this paper.

Lemma 3.5 *The boundary value problem (1.13) has a solution $u(x)$ if and only if $u(x)$ has the integral representation*

$$u(x) = \int_a^b H(t,x)p(t)u(t)\,dt, \tag{3.7}$$

where the Green function $H(t,x)$ is given by

$$H(t,x) = \begin{cases} h_1(t,x), & a \le t \le x \le b, \\ h_2(t,x), & a \le x \le t \le b, \end{cases}$$

where

$$h_1(t,x) = \frac{2(1-\alpha)(x-a)(b-t) + \alpha(x-a)(b-t)^2 - 2(1-\alpha)(x-t)(b-a) - \alpha(x-t)^2(b-a)}{2(b-a)},$$

and

$$h_2(t,x) = \frac{2(1-\alpha)(x-a)(b-t) + \alpha(x-a)(b-t)^2}{2(b-a)}.$$

Proof Assume that u solves the boundary value problem (1.13). We show that u also solves the integral equation (3.7). As in the proof of Lemma 3.1, we make use of a change of variable to convert the boundary value problem (1.13) to a boundary value problem with boundary conditions $u(0) = u''(0) = 0$ and $u(c) = 0$ with $c = b-a$. If we let $\eta = x-a$ and $y(\eta) = u(\eta + a)$ as we did previously, we find that

$$^{CF}D_a^\sigma u(x) = ^{CF}D_a^\sigma u(\eta + a) = ^{CF}D_0^\sigma y(\eta).$$

The boundary conditions for $y(\eta)$ now become

$$y''(0) = u''(a+0) = u''(a) = 0,$$

$$y(0) = u(a+0) = u(a) = 0,$$

$$y(b-a) = u(b-a+a) = u(b) = 0.$$

As a result, the boundary value problem (1.13) turns into

$$\begin{cases} ^{CF}D_0^\sigma y(\eta) + q(\eta)y(\eta) = 0, \\ y(0) = y''(0) = y(c) = 0, \end{cases} \tag{3.8}$$

where $q(\eta) = p(\eta + a)$ and $c = b-a$.

Applying the Laplace operator to the equation (3.8), we get

$$\mathcal{L}\Big\{{}^{CF}D_0^{\sigma}y(\eta)\Big\}(s) = \mathcal{L}\Big\{q(\eta)y(\eta)\Big\}(s).$$

Appealing Lemma 2.8, we are led to

$$\frac{s^3Y(s) - s^2y(0) - sy'(0) - y''(0)}{s + \alpha(1-s)} = -Q(s),$$

where $Y(s) = \mathcal{L}\Big\{y(\eta)\Big\}(s)$ and $Q(s) = \mathcal{L}\Big\{q(\eta)y(\eta)\Big\}(s)$.

Equivalently, using the boundary conditions $y(0) = y''(0) = 0$, the last equation can be rewritten as

$$Y(s) = \frac{1}{s^2}y'(0) - \frac{1-\alpha}{s^2}Q(s) - \frac{\alpha}{s^3}Q(s).$$

The inverse Laplace operator then is applied to above equation to arrive at

$$y(\eta) = \eta y'(0) - (1-\alpha)\int_0^{\eta}(\eta - t)q(t)y(t)\,dt - \frac{\alpha}{2}\int_0^{\eta}(\eta-t)^2q(t)y(t)\,dt. \tag{3.9}$$

Taking into account the boundary condition $y(c) = 0$, we have

$$\begin{aligned} y(\eta) = {} & \frac{1-\alpha}{(b-a)}\int_0^{b-a}(b-a-t)\eta q(t)y(t)\,dt + \frac{\alpha}{2(b-a)}\int_0^{b-a}(b-a-t)^2\eta q(t)y(t)\,dt \\ & - (1-\alpha)\int_0^{\eta}(\eta-t)q(t)y(t)\,dt - \frac{\alpha}{2}\int_0^{\eta}(\eta-t)^2q(t)y(t)\,dt. \end{aligned}$$

or, equivalently

$$\begin{aligned} y(\eta) = {} & \int_0^{\eta}\Big(\frac{2(1-\alpha)\eta(b-a-t) + \alpha\eta(b-a-t)^2 - 2(1-\alpha)(\eta-t)(b-a)}{2(b-a)} \\ & - \frac{\alpha(\eta-t)^2(b-a)}{2(b-a)}\Big)q(t)y(t)\,dt + \int_{\eta}^{b-a}\frac{2(1-\alpha)\eta(b-a-t) + \alpha\eta(b-a-t)^2}{2(b-a)}q(t)y(t)\,dt. \end{aligned}$$

Using the relation $u(x) = y(x-a)$, we find that

$$\begin{aligned} u(x) = y(x-a) = {} & \int_0^{x-a}\Big(\frac{2(1-\alpha)(x-a)(b-a-t) + \alpha(x-a)(b-a-t)^2 - 2(1-\alpha)(x-a-t)(b-a)}{2(b-a)} \\ & - \frac{\alpha(x-a-t)^2(b-a)}{2(b-a)}\Big)p(t+a)u(t+a)\,dt \\ & + \int_{x-a}^{b-a}\frac{2(1-\alpha)(x-a)(b-a-t) + \alpha(x-a)(b-a-t)^2}{2(b-a)}p(t+a)u(t+a)\,dt \\ = {} & \int_a^x \frac{2(1-\alpha)(x-a)(b-t) + \alpha(x-a)(b-t)^2 - 2(1-\alpha)(x-t)(b-a)}{2(b-a)}p(t)u(t)\,dt \\ & - \int_a^x\frac{\alpha(x-t)^2(b-a)}{2(b-a)}p(t)u(t)\,dt + \int_x^b\frac{2(1-\alpha)(x-a)(b-t) + \alpha(x-a)(b-t)^2}{2(b-a)}p(t)u(t)\,dt \\ = {} & \int_a^b H(t,x)p(t)u(t)\,dt. \end{aligned} \tag{3.10}$$

Conversely, we now assume that u is represented by the integral equation (3.7). From the proof above, we see that u solves the fractional differential equation (1.13). We show u satisfies the boundary conditions. Since $H(t,a) = H(t,b) = 0$, we have $u(a) = u(b) = 0$. Differentiating the equation (3.10) twice to get

$$
\begin{aligned}
u''(x) = & -\alpha \int_a^x p(t)u(t)\,dt + \frac{1}{2(b-a)}\Big((2(1-\alpha)(b-x) + \alpha(b-x)^2)p(x)u(x) \\
& + p(x)u(x)\Big[2(1-\alpha)(b-x) - 2(1-\alpha)(x-a) + \alpha(b-x)^2 - 2\alpha(x-a)(b-x)\Big] \\
& + (p(x)u(x))'\Big[2(1-\alpha)(b-x)(x-a) + \alpha(x-a)(b-x)^2\Big] - p(x)u(x)(2(1-\alpha)(b-x) + \alpha(b-x)^2) \\
& - (2(1-\alpha)(x-a)(b-x) + \alpha(x-a)(b-x)^2)'p(x)u(x) \\
& - (p(x)u(x))'(2(1-\alpha)(x-a)(b-x) + \alpha(x-a)(b-x)^2)\Big).
\end{aligned}
$$

Since $u(a) = 0$, from the equation above we see that $u''(a) = 0$. Thus, the proof is now completed.

□

Lemma 3.6 *The function $H(t,x)$ has the following bound*

$$\Big|H(t,x)\Big| \leq \frac{1-\alpha}{4}(b-a) + \frac{\alpha}{4}(b-a)^2, \quad x,t \in [a,b], \quad \alpha \in (0,1]. \tag{3.11}$$

Proof Begin with simplifying the term

$$(x-a)(b-t) - (x-t)(b-a) = (x-t+t-a)(b-t) - (x-t)(b-t+t-a) = (b-x)(t-a).$$

For $a \leq t \leq x \leq b$, observe that $x-a \leq b-a$, $b-t \leq b-a$ and $b-x \leq b-t$. Moreover, we obtain that

$$
\begin{aligned}
(x-a)(b-t)^2 - (x-t)^2(b-a) &\leq (x-a)\Big((b-t)^2 - (x-t)^2(x-a)\Big) = (x-a)(b-x)(b-t+x-t) \\
&\leq 2(x-a)(b-x)(b-a) \leq 2(b-a)\frac{1}{4}(b-a)^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
h_1(t,x) &\leq \frac{1}{2(b-a)}\Big(2(1-\alpha)(b-x)(t-a) + 2\alpha(b-a)(x-a)(b-x)\Big) \\
&\leq \frac{1}{(b-a)}\Big((1-\alpha)\frac{1}{4}(b-a)^2 + \alpha(b-a)\frac{1}{4}(b-a)^2\Big) = \frac{1-\alpha}{4}(b-a) + \frac{\alpha}{4}(b-a)^2. \qquad (3.12)
\end{aligned}
$$

Here, we use the inequality $(u+v)^2 \geq 4uv$ for $u,v \in \mathbb{R}$. Similarly we can bound $h_2(t,x)$ by using the facts that $x-a \leq t-a$ and $b-t \leq b-a$ as follows:

$$
\begin{aligned}
h_2(t,x) &\leq \frac{1}{2(b-a)}\Big(2(1-\alpha)(x-a)(b-t) + \alpha(x-a)(b-t)^2\Big) \\
&\leq \frac{1}{2(b-a)}\Big(2(1-\alpha)(t-a)(b-t) + 2\alpha(t-a)(b-t)(b-t)\Big) = \frac{1}{b-a}\Big(\frac{1-\alpha}{4}(b-a)^2 + (b-a)\frac{\alpha}{4}(b-a)^2\Big) \\
&= \frac{1-\alpha}{4}(b-a) + \frac{\alpha}{4}(b-a)^2. \qquad (3.13)
\end{aligned}
$$

From the equation (3.12) and (3.13), we have for $x, t \in [a, b]$,

$$H(t, x) \leq \frac{1-\alpha}{4}(b-a) + \frac{\alpha}{4}(b-a)^2.$$

.

□

Theorem 3.7 *For* $\sigma = \alpha + 2, \quad \alpha \in (0, 1)$*, and* $p \in C([a, b], \mathbb{R})$*, if the boundary value problem of the fractional Caputo-Fabrizio differential equation (1.13) has a nonzero solution, then the function* p *satisfies the following condition*

$$\int_a^b |p(\eta)| d\eta > \frac{4}{(1-\alpha)(b-a) + \alpha(b-a)^2}. \tag{3.14}$$

Proof Let $C[a, b]$ be the Banach space with maximum norm, that is,

$$\|u\| = \max_{x \in [x_1, x_2]} |u(x)|, \quad u \in C[a, b].$$

By Lemma 3.5, the solution of the boundary value problem (1.13) has the form

$$u(x) = \int_a^b H(x, t) p(t) u(t)\, dt, \quad x, t \in [a, b].$$

Taking the maximum norm of the both side of the equation yield

$$\|u\| \leq \max_{x \in [a, b]} \int_a^b \Big| H(x, t) p(t) \Big| \|u\|\, dt,$$

which gives

$$\max_{x \in [a, b]} \int_a^b \Big| H(x, t) p(t) \Big|\, dt \geq 1.$$

The bound on the function H leads to have

$$1 \leq \Big(\frac{1-\alpha}{4}(b-a) + \frac{\alpha}{4}(b-a)^2 \Big) \int_a^b |p(t)|\, dt,$$

or

$$\int_a^b |p(t)|\, dt \geq \frac{4}{(1-\alpha)(b-a) + \alpha(b-a)^2}.$$

□

Remark 3.8 *In Theorem 3.7, if we let* $\alpha \to 1$*, then we recover the standard Lyapunov-type inequality for the third order linear ordinary differential equation (1.6).*

We now state some applications and consequences of Theorem 3.3 and Theorem 3.7. The first result of Theorem 3.3 as follows:

Corollary 3.9 *Assume that* $u(a) = 0$*. If* $\alpha \in (0, 1]$ *and the following boundary value problem of the fractional Caputo-Fabrizio differential equation has a nonzero solution,*

$$^{CF}D_a^{\beta} u(x) + \mu u(x) = 0, \quad \beta = \alpha + 1, \quad a \leq x \leq b, \tag{3.15}$$

$$u'(a) = 0, \quad u(b) = 0, \tag{3.16}$$

then the eigenvalues $\mu \in \mathbb{R}$ *satisfy*

$$|\mu| > \frac{1}{(1-\alpha)(b-a) + \alpha(b-a)^2}.$$

Proof The proof follows from Theorem 3.3 by replacing p by μ and noting that $\int_a^b |\mu|\, dt = (b-a)|\mu|$. □

Remark 3.10 *Letting* $\alpha \to 1$ *implies* $\beta = 2$*. We have in this case,* $|\mu| > \frac{1}{(b-a)^2}$*. This is the bound for the eigenvalues of the following second order differential equation*

$$u''(x) + \mu u(x) = 0, \quad a \leq x \leq b,$$

$$u'(a) = 0, \quad u(b) = 0.$$

Corollary 3.11 *If* $\alpha \in (0, 1]$ *and the following fractional Sturm-Lioville eigenvalue problem of the fractional Caputo-Fabrizio differential equation has a nonzero solution,*

$$^{CF}D_a^{\sigma} u(x) + \lambda u(x) = 0, \quad \sigma = \alpha + 2, \quad a \leq x \leq b, \tag{3.17}$$

$$u(a) = u"(0) = 0, \quad u(b) = 0, \tag{3.18}$$

then the eigenvalues $\lambda \in \mathbb{R}$ *satisfy*

$$|\lambda| > \frac{4}{(1-\alpha)(b-a)^2 + \alpha(b-a)^3}.$$

As a second result of Theorem 3.7, we state the next corollary for the zeros of eigenfunctions for the fractional Sturm-Lioville eigenvalue problem (3.17).

Theorem 3.12 *If the fractional Sturm-Lioville eigenvalue problem (3.17) and (3.18) has a nonzero solution and the eigenvalues* $\lambda \in \mathbb{R}$ *obey the inequality*

$$|\lambda| \leq \frac{4}{(1-\alpha)(b-a)^2 + \alpha(b-a)^3},$$

then the eigenfunctions have no real zeros.

4. Numerical examples

In this section, Lyapunov-type inequalities for some boundary value problems of the fractional differential equation in the sense of the Caputo-Fabrizio derivative have been presented.

Example 4.1 *If* $\sigma = \alpha + 2$ *with* $\alpha = \frac{1}{2}$ *and* $|\lambda| < 4$*, then the following fractional Sturm–Liouville eigenvalue problem*

$$^{CF}D^{\sigma}(u)(x) + \lambda u(x) = 0, \quad 0 < x < 1, \tag{4.1}$$

$$u(0) = u''(0) = 0, \quad u(1) = 0, \tag{4.2}$$

has no nonzero solution (eigenfunction).

Proof If there were nonzero solution, then we would have that $\int_0^1 p(x)\,dx \geq 4$ with $p(t) = \lambda$ by Theorem 3.7. However, this contradicts the hypothesis that $\int_0^1 p(x)\,dx < 4$. Therefore, there is no nonzero solution for the boundary value problem. □

Example 4.2 *For* $\beta = \alpha + 1, \quad \alpha \in (0,1]$*, if* λ *is an eigenvalue of fractional Sturm–Liouville eigenvalue problem*

$$^{CF}D^{\beta}(u)(x) + \lambda u(x) = 0, \quad 0 < x < 1, \tag{4.3}$$

$$u'(0) = 0, \quad u(1) = 0, \tag{4.4}$$

then

$$\lambda \geq 1.$$

The conclusion follows from Theorem 3.3 by replacing p with λ. As a consequence, we infer that if $\lambda < 1$ then the eigenfunctions of the fractional boundary value problem (4.3) and (4.4) have no real zeros.

References

[1] Caputo M, Fabrizio M. A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications 2015; 1 (2): 1-13.

[2] Ferreira RAC. A Lyapunov-type inequality for a fractional boundary value problem. Fractional Calculus and Applied Analysis 2013; 16 (4): 978-984.

[3] Ferreira RAC. On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. Journal of Mathematical Analysis and Applications 2014; 412 (2): 1058-1063.

[4] Hartman P. Ordinary Differential Equations. New York, USA: Wiley, 1964.

[5] Jleli M, Ragoub L, Samet B. A Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition. Journal of Function Spaces 2015; 1: 5.

[6] Jleli M, Samet B. Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Mathematical Inequalities and Applications 2015; 18 (2): 443-451.

[7] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam, Netherlands: Elsevier, 2006.

[8] Kirane M, Torebek T. A Lyaponive-type inequality for a fractional boundary vale problem with Caputo-Fabrizio derivative. Journal of Mathematical Inequalities 2018; 12 (4): 1005-1012.

[9] Liapounoff AM. Probleme general de la stabilite du mouvement. Annales de la Faculté des Sciences de Toulouse 1907; 2: 203-407 (in French).

[10] Losada J, Nieto JJ. Properties of a new fractional derivative without singular kernel. Progress in Fractional Differentiation and Applications 2015; 1 (2): 87-92.

[11] O'Regan D, Samet B. Lyapunov-type inequalities for a class of fractional differential equations. Journal of Inequalities and Applications 2015; 247: 10.

[12] Parhi N, Panigrahi S. On Liapunov-type inequality for third-order differential equations. Journal of Mathematical Analysis and Applications 1999; 233 (2): 445-460.

[13] Rong J, Bai C. Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions. Advances in Difference Equations 2015; 2015 (8): 10.

[14] Toprakseven S. The existence of positive solutions and a Lyapunov type inequality for boundary value problems of the fractional Caputo-Fabrizio differential equations. Sigma Journal of Engineering and Natural Sciences 2019; 37(4): 1129-1137.