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An improved Trudinger–Moser inequality and its extremal functions involving L^p-norm in $\mathbb{R}^2$

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Abstract: Let $W^{1,2}(\mathbb{R}^2)$ be the standard Sobolev space. Denote for any real number $p > 2$

$$\lambda_p = \inf_{u \in W^{1,2}(\mathbb{R}^2), u \not\equiv 0} \frac{\int_{\mathbb{R}^2}(|\nabla u|^2 + |u|^2)dx}{(\int_{\mathbb{R}^2}|u|^p dx)^{2/p}}.$$

Define a norm in $W^{1,2}(\mathbb{R}^2)$ by

$$\|u\|_{\alpha,p} = \left(\int_{\mathbb{R}^2}(|\nabla u|^2 + |u|^2)dx - \alpha(\int_{\mathbb{R}^2}|u|^p dx)^{2/p}\right)^{1/2}$$

where $0 \leq \alpha < \lambda_p$. Using the method of blow-up analysis, we prove that for $p > 2$ and $0 \leq \alpha < \lambda_p$, the supremum

$$\sup_{u \in W^{1,2}(\mathbb{R}^2),\, \|u\|_{\alpha,p} \leq 1} \int_{\mathbb{R}^2}(e^{4\pi u^2} - 1 - 4\pi u^2)dx$$

can be attained by some function $u_0 \in W^{1,2}(\mathbb{R}^2)$ with $\|u_0\|_{\alpha,p} = 1$.

Key words: Trudinger–Moser inequality, extremal function, blow-up analysis

1. Introduction and main result

Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $W_0^{1,2}(\Omega)$ be the standard Sobolev space. The classical Trudinger–Moser inequality [20, 22, 23, 27, 35] states the following:

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|\nabla u\|_2^2 \leq 1} \int_\Omega e^{\gamma u^2} dx < +\infty, \quad \forall \gamma \leq 4\pi; \tag{1.1}$$

when $\gamma > 4\pi$, the integral in (1.1) is still finite for any $u \in W_0^{1,2}(\Omega)$, but the supremum is infinity. Here and in the sequel, $\|\cdot\|_s$ denotes the usual L^s-norm for any $s > 0$ with respect to the Lebesgue measure.

An interesting question about (1.1) is whether extremal function exists or not. The first result for the attainability was proved by Carleson and Chang [7] when Ω is a unit ball in $\mathbb{R}^2$. Then this result was extended

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by Struwe [26] when Ω is close to a ball in measure, by Flucher [14] to arbitrary domains in $\mathbb{R}^2$, by Lin [19] to a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, and by Adimurthi and Druet [2] to the following modified form: Let

$$\lambda(\Omega) = \inf_{u \in W_0^{1,2}(\Omega), u \not\equiv 0} \|\nabla u\|_2^2 / \|u\|_2^2$$

be the first eigenvalue of the Laplacian operator with respect to Dirichlet boundary condition. For any α, $0 \leq \alpha < \lambda(\Omega)$

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|\nabla u\|_2^2 \leq 1} \int_\Omega e^{4\pi u^2 (1+\alpha \|u\|_2^2)} dx < +\infty; \tag{1.2}$$

the supremum is infinity when $\alpha \geq \lambda(\Omega)$. We can easily see that (1.2) gives more information than (1.1). Then this result was extended by Yang [28–30] to higher dimensional Euclidean domain and closed Riemannian surface. Later Lu and Yang [13] generalized L^2-norm in (1.2) to L^q-norm for any real number $q > 1$. Precisely, for any $q > 1$, define

$$\overline{\lambda}(\Omega) = \inf_{u \in W_0^{1,2}(\Omega), u \not\equiv 0} \|\nabla u\|_2^2 / \|u\|_q^2.$$

Then for any $0 \leq \alpha < \overline{\lambda}(\Omega)$, there holds

$$\sup_{u \in W_0^{1,2}(\Omega),\, \|\nabla u\|_2 = 1} \int_\Omega e^{4\pi u^2 (1+\alpha \|u\|_q^2)} dx < +\infty;$$

the above supremum is infinity when $\alpha > \overline{\lambda}(\Omega)$. They also derived in [13] the existence of the extremal functions for sufficiently small $\alpha > 0$.

Another meaningful extension of (1.1) is to construct Trudinger–Moser inequality for unbounded domain. Earlier works in this direction were by Cao [6], Panda [21], do Ó [10], Adachi and Tanaka [1]. Later Ruf [24] (for two dimensional case) and Li and Ruf [18] (for N-dimensional case $N > 2$) established the critical Trudinger–Moser inequality which states that

$$\sup_{u \in W^{1,N}(\mathbb{R}^N),\, \|u\|_{W^{1,N}(\mathbb{R}^N)} \leq 1} \int_{\mathbb{R}^N} \left(e^{\alpha_N |u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha_N^k |u|^{kN/(N-1)}}{k!} \right) dx < +\infty, \tag{1.3}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$, and $\|\cdot\|_{W^{1,N}(\mathbb{R}^N)}$ denotes the standard Sobolev norm on $W^{1,N}(\mathbb{R}^N)$, namely

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} (|\nabla u|^N + |u|^N) dx \right)^{1/N}.$$

In fact, they also obtained the existence of extremal functions. Based on the Young inequality and the argument of Schwarz rearrangement, Adimurthi and Yang [4] introduced a very simple proof of the critical Trudinger–Moser inequality in $\mathbb{R}^N$, as well as its singular version. Precisely, one of the conclusions in [4] is that for $N \geq 2$, $\tau > 0$, $0 \leq \beta < 1$ and $0 < \gamma \leq 1 - \beta$, there holds

$$\sup_{u \in W^{1,N}(\mathbb{R}^N),\, \|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{1}{|x|^{N\beta}} \left(e^{\alpha_N \gamma |u|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{(\alpha_N \gamma)^k |u|^{kN/(N-1)}}{k!} \right) dx < +\infty, \tag{1.4}$$

where $\|u\|_{1,\tau} = \left(\int_{\mathbb{R}^N}(|\nabla u|^N + \tau|u|^N)dx\right)^{1/N}$. Extremal functions for (1.4) was obtained by Li and Yang [16] via the method of blow-up analysis. Recently, an analog of (1.4) with the norm $\|u\|_{1,\tau}$ is replaced by a norm involving the L^p-norm was obtained by Li [15]. For more works on singular Trudinger–Moser inequalities, we refer the reader to [3, 9, 33, 34].

It was proved by do Ó and Souza [11] that for $0 \le \alpha < 1$, there holds

$$\sup_{u\in W^{1,2}(\mathbb{R}^2),\,\|u\|_{W^{1,2}(\mathbb{R}^2)}=1}\int_{\mathbb{R}^2}\left(e^{4\pi(1+\alpha\|u\|_2^2)u^2} - 1 - 4\pi(1+\alpha\|u\|_2^2)u^2\right)dx < +\infty;$$

for any $\alpha > 1$, the supremum is infinity. The existence of extremal functions was also obtained in [11] by blow-up analysis.

Motivated by [11] and [15], we shall prove the existence of extremal functions for a class of Trudinger–Moser inequality involving L^p-norms. To be specific, let $p > 2$, denote

$$\lambda_p = \inf_{u\in W^{1,2}(\mathbb{R}^2),u\not\equiv 0}\frac{\int_{\mathbb{R}^2}(|\nabla u|^2+|u|^2)dx}{(\int_{\mathbb{R}^2}|u|^p dx)^{2/p}}. \tag{1.5}$$

The fact $\lambda_p > 0$ is based on a direct method of variation. For $0 \le \alpha < \lambda_p$, we define a norm in $W^{1,2}(\mathbb{R}^2)$ by

$$\|u\|_{\alpha,p} = \left(\int_{\mathbb{R}^2}(|\nabla u|^2+|u|^2)dx - \alpha(\int_{\mathbb{R}^2}|u|^p dx)^{2/p}\right)^{1/2}. \tag{1.6}$$

Our main result can be stated as follows.

Theorem 1.1 *Let $p > 2$ be a real number, λ_p and $\|\cdot\|_{\alpha,p}$ be defined as in (1.5) and (1.6) respectively. For any fixed α, $0 \le \alpha < \lambda_p$, there exists some $u_0 \in W^{1,2}(\mathbb{R}^2)$ with $\|u_0\|_{\alpha,p} = 1$ such that*

$$\int_{\mathbb{R}^2}(e^{4\pi u_0^2} - 1 - 4\pi u_0^2)dx = \sup_{u\in W^{1,2}(\mathbb{R}^2),\,\|u\|_{\alpha,p}\le 1}\int_{\mathbb{R}^2}(e^{4\pi u^2} - 1 - 4\pi u^2)dx.$$

Following Carleson and Chang [7], Li [17], Li and Ruf [18], and Yang [32], we prove Theorem 1.1 via the method of blow-up analysis. The remaining part of this article is organized as follows. In Section 2, we prove the existence of maximizers of the subcritical functionals. In Section 3, we analyze the asymptotic behavior of the maximizers. In Section 4, using the argument of Carleson and Chang [7], we derive an upper bound estimates of the critical functional under the assumption that blow-up occurs. In Section 5, we construct test functions to conclude the existence of extremals. For notational convenience, $\mathbb{B}_R$ represents a ball centered at the origin with the radius R, and $\mathbb{B}_R^c$ means the complement of $\mathbb{B}_R$ in $\mathbb{R}^2$. The same letter C will be used to denote constants. And we do not distinguish sequence and subsequence.

2. The subcritical case

In this section, we prove the existence of maximizer for the subcritical functional

$$J_{4\pi-\epsilon}(u) = \int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u^2} - 1 - (4\pi-\epsilon)u^2)dx$$

for any $0 < \epsilon < 4\pi$. For simplicity, set

$$\Lambda_{\alpha,p} = \sup_{u\in W^{1,2}(\mathbb{R}^2),\, \|u\|_{\alpha,p}\leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1 - 4\pi u^2)dx$$

and

$$\Lambda_{\alpha,p,\epsilon} = \sup_{u\in W^{1,2}(\mathbb{R}^2),\, \|u\|_{\alpha,p}\leq 1} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi-\epsilon)u^2)dx.$$

Lemma 2.1 *Let $p > 2$ and $0 \leq \alpha < \lambda_p$ be fixed. Then for any $0 < \epsilon < 4\pi$, there exists some nonnegative decreasing radially and symmetric function $u_\epsilon \in C^1(\mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^2)$ satisfying $\|u_\epsilon\|_{\alpha,p} = 1$ and*

$$\Lambda_{\alpha,p,\epsilon} = \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx. \tag{2.1}$$

Moreover, the Euler-Lagrange equation of u_ϵ is

$$\begin{cases} -\Delta u_\epsilon + u_\epsilon = \frac{u_\epsilon}{\lambda_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2} - 1) + \alpha\|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1} \\ \lambda_\epsilon = \int_{\mathbb{R}^2} u_\epsilon^2 (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx. \end{cases} \tag{2.2}$$

Proof We first recall some results of the Schwarz rearrangement [12]. For any $u \in W^{1,2}(\mathbb{R}^2)$, suppose that $\widetilde{u}$ is the Schwarz rearrangement of $|u|$. Then $\widetilde{u}$ is a nonnegative decreasing radially symmetric function and satisfies

$$\int_{\mathbb{R}^2} |\nabla\widetilde{u}|^2 dx \leq \int_{\mathbb{R}^2} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^2} |\widetilde{u}|^q dx = \int_{\mathbb{R}^2} |u|^q dx \quad (\forall q \geq 2)$$

and

$$\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)\widetilde{u}^2} - 1 - (4\pi-\epsilon)\widetilde{u}^2)dx = \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi-\epsilon)u^2)dx.$$

Therefore, we have

$$\Lambda_{\alpha,p,\epsilon} = \sup_{u\in\mathscr{H}} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi-\epsilon)u^2)dx,$$

where $\mathscr{H}$ is a set consisting of all nonnegative decreasing radially symmetric functions $u \in W^{1,2}(\mathbb{R}^2)$ with $\|u\|_{\alpha,p} \leq 1$.

To prove (2.1), we use a direct method of variation. Choose a sequence of functions $u_k \in \mathscr{H}$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_k^2} - 1 - (4\pi-\epsilon)u_k^2)dx = \Lambda_{\alpha,p,\epsilon}. \tag{2.3}$$

Since $0 \leq \alpha < \lambda_p$, we have

$$\int_{\mathbb{R}^2} (|\nabla u_k|^2 + |u_k|^2)dx \leq \frac{\lambda_p}{\lambda_p - \alpha}.$$

Hence, u_k is bounded in $W^{1,2}(\mathbb{R}^2)$. Up to a subsequence, there exists some function $u_\epsilon \in \mathscr{H}$ such that, as $k \to \infty$,

$$\begin{aligned} &u_k \rightharpoonup u_\epsilon \quad \text{weakly} \quad \text{in} \quad W^{1,2}(\mathbb{R}^2), \\ &u_k \to u_\epsilon \quad \text{strongly} \quad \text{in} \quad L^r_{\mathrm{loc}}(\mathbb{R}^2) \quad (\forall r > 1), \\ &u_k \to u_\epsilon \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^2. \end{aligned}$$

Obviously, u_ϵ is also nonnegative decreasing radially and symmetric. The radial lemma [5] shows that for any $x \in \mathbb{R}^2 \backslash \{0\}$

$$|u(x)|^2 \le \frac{1}{\pi}\|u\|_2^2 \frac{1}{|x|^2}. \tag{2.4}$$

Given any $\eta > 0$, we can choose a sufficiently large number $R_0 > 0$ such that

$$\left|\int_{\mathbb{B}^c_{R_0}} u_\epsilon^p dx\right| \le \frac{\eta}{3}, \qquad \left|\int_{\mathbb{B}^c_{R_0}} u_k^p dx\right| \le \frac{\eta}{3}. \tag{2.5}$$

In view of $u_k \to u_\epsilon$ strongly in $L^r_{\mathrm{loc}}(\mathbb{R}^2)$ for any $r > 1$, there exists some positive integer k_0 such that

$$\left|\int_{\mathbb{B}_{R_0}} (u_k^p - u_\epsilon^p) dx\right| \le \frac{\eta}{3}$$

for any $k \ge k_0$. The above estimates imply that

$$\|u_k\|_p \to \|u_\epsilon\|_p \quad \text{as} \quad k \to \infty. \tag{2.6}$$

This together with the fact $u_k \rightharpoonup u_\epsilon$ weakly in $W^{1,2}(\mathbb{R}^2)$ leads to

$$\|u_\epsilon\|_{\alpha,p} \le \limsup_{k\to\infty} \|u_k\|_{\alpha,p} \le 1.$$

Observe that for any $u \in \mathscr{H}$

$$\int_{\mathbb{R}^2} |u|^2 dx \le \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) dx \le \frac{\lambda_p}{\lambda_p - \alpha}. \tag{2.7}$$

By (2.4) and (2.7), we get

$$\begin{aligned} \int_{\mathbb{B}^c_{r_0}} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi - \epsilon)u^2) dx &= \int_{\mathbb{B}^c_{r_0}} \left(\sum_{j=2}^{\infty} \frac{(4\pi - \epsilon)^j}{j!} u^{2j} \right) dx \\ &\le \sum_{j=2}^{\infty} \frac{(4\pi-\epsilon)^j \lambda_p^j}{(\lambda_p - \alpha)^j j!} \frac{1}{r_0^{2j-2}}. \end{aligned}$$

Given any $\nu > 0$, there exists a sufficiently large $r_0 > 0$ such that for all $u \in \mathscr{H}$,

$$\int_{\mathbb{B}^c_{r_0}} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi - \epsilon)u^2) dx \le \nu. \tag{2.8}$$

On the other hand

$$\|u_k - u_\epsilon\|^2_{W^{1,2}(\mathbb{R}^2)} \le 1 - \left(\int_{\mathbb{R}^2} (|\nabla u_\epsilon|^2 + u_\epsilon^2) dx - \alpha \big(\int_{\mathbb{R}^2} u_\epsilon^p dx \big)^{2/p} \right) + o_k(1).$$

Noting that $\|u_\epsilon\|_{\alpha,p} \le 1$, we immediately get

$$\limsup_{k\to\infty} \|u_k - u_\epsilon\|_{W^{1,2}(\mathbb{R}^2)} \le 1.$$

This together with (1.3), the mean theorem and the fact $u_k \to u_\epsilon$ strongly in $L^q_{\text{loc}}(\mathbb{R}^2)$ for any $q > 1$ implies that

$$\lim_{k\to\infty} \int_{\mathbb{B}_{r_0}} (e^{(4\pi-\epsilon)u_k^2} - 1 - (4\pi-\epsilon)u_k^2) dx = \int_{\mathbb{B}_{r_0}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2) dx. \tag{2.9}$$

Since $\nu > 0$ is arbitrary, we have by (2.8) and (2.9)

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_k^2} - 1 - (4\pi-\epsilon)u_k^2) dx = \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2) dx. \tag{2.10}$$

Combining (2.3) and (2.10), we conclude (2.1).

Clearly, $u_\epsilon \not\equiv 0$ and $\|u_\epsilon\|_{\alpha,p} \le 1$. Suppose $\|u_\epsilon\|_{\alpha,p} < 1$. It follows that

$$\begin{aligned}\Lambda_{\alpha,p,\epsilon} &= \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2) dx \\ &< \int_{\mathbb{R}^2} \left(e^{(4\pi-\epsilon)u_\epsilon^2/\|u_\epsilon\|^2_{\alpha,p}} - 1 - (4\pi-\epsilon)\frac{u_\epsilon^2}{\|u_\epsilon\|^2_{\alpha,p}} \right) dx \\ &\le \Lambda_{\alpha,p,\epsilon}.\end{aligned}$$

It is a contradiction. Hence, we get $\|u_\epsilon\|_{\alpha,p} = 1$.

A straightforward calculation shows that u_ϵ satisfies the Euler-Lagrange equation (2.2). Applying elliptic estimates to (2.2), we have $u_\epsilon \in C^1(\mathbb{R}^2)$. □

In view of (2.2), we show that the sequence λ_ϵ has a positive lower bound.

Lemma 2.2 *Let λ_ϵ be as in (2.2), we have*

$$\liminf_{\epsilon\to 0} \lambda_\epsilon > 0.$$

Proof For any fixed $u \in W^{1,2}(\mathbb{R}^2)$ with $\|u\|_{\alpha,p} \le 1$, there holds

$$\begin{aligned}\int_{\mathbb{R}^2} (e^{4\pi u^2} - 1 - 4\pi u^2) dx &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u^2} - 1 - (4\pi-\epsilon)u^2) dx \\ &\le \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2) dx.\end{aligned}$$

Then we have

$$\Lambda_{\alpha,p} \le \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2) dx.$$

On the other hand, we can easily see that

$$\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx \leq \Lambda_{\alpha,p}.$$

Combining the above two inequalities, we get

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx = \Lambda_{\alpha,p}. \tag{2.11}$$

Using an elementary inequality $t(e^t-1)\geq e^t-1-t$ for all $t\geq 0$, we obtain

$$\lambda_\epsilon \geq \frac{1}{4\pi-\epsilon}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx.$$

This together with (2.11) implies that

$$\liminf_{\epsilon\to 0}\lambda_\epsilon \geq \lim_{\epsilon\to 0}\frac{1}{4\pi-\epsilon}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx = \frac{\Lambda_{\alpha,p}}{4\pi} > 0.$$

We finish the proof of the lemma. □

3. Blow-up analysis

In this section, we perform the blow-up analysis.

Since $\|u_\epsilon\|_{\alpha,p}=1$ and $0\leq\alpha<\lambda_p$, we obtain that u_ϵ is bounded in $W^{1,2}(\mathbb{R}^2)$. Then there exists u_0 such that up to a subsequence,

$$\begin{aligned}
&u_\epsilon \rightharpoonup u_0 \quad \text{weakly} \quad \text{in} \quad W^{1,2}(\mathbb{R}^2),\\
&u_\epsilon \to u_0 \quad \text{strongly} \quad \text{in} \quad L^r_{\mathrm{loc}}(\mathbb{R}^2) \quad (\forall r>1),\\
&u_\epsilon \to u_0 \quad \text{a.e.} \quad \text{in} \quad \mathbb{R}^2.
\end{aligned}$$

Noting that u_ϵ is decreasing radially and symmetric, we denote

$$c_\epsilon = u_\epsilon(0) = \max_{\mathbb{R}^2} u_\epsilon.$$

If c_ϵ is bounded, applying the standard elliptic estimate to (2.2), we conclude that $u_\epsilon\to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2}(e^{4\pi u_0^2}-1-4\pi u_0^2)dx = \lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx = \Lambda_{\alpha,p}.$$

Hence, u_0 is a desire extremal function and Theorem 1.1 holds.

In the following, we assume $c_\epsilon\to+\infty$ as $\epsilon\to 0$. Then we have

Lemma 3.1 *$u_0\equiv 0$ and $|\nabla u_\epsilon|^2dx \rightharpoonup \delta_0$, where δ_0 is the Dirac measure centered at 0.*

Proof Suppose $u_0\not\equiv 0$. Analogously to the analysis of (2.6), we have

$$\|u_\epsilon\|_p \to \|u_0\|_p \quad \text{as} \quad \epsilon\to 0.$$

Consequently, there exists some $\epsilon_0 > 0$ such that

$$\|u_\epsilon - u_0\|^2_{W^{1,2}(\mathbb{R}^2)} \leq 1 - \|u_0\|^2_{\alpha,p}$$

for $0 < \epsilon < \epsilon_0$. Using $e^{m+n} - 1 = (e^m - 1)(e^n - 1) + (e^m - 1) + (e^n - 1)$ for any $m \geq 0$, $n \geq 0$, the Hölder inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)qu_\epsilon^2} - 1)dx &\leq \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q((1+\nu)(u_\epsilon-u_0)^2+(1+1/\nu)u_0^2)} - 1)dx \\
&= \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q(1+\nu)(u_\epsilon-u_0)^2} - 1)(e^{(4\pi-\epsilon)q(1+1/\nu)u_0^2} - 1)dx \\
&+ \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q(1+\nu)(u_\epsilon-u_0)^2} - 1)dx + \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q(1+1/\nu)u_0^2} - 1)dx \\
&\leq \left(\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)qq_1(1+\nu)(u_\epsilon-u_0)^2} - 1)dx\right)^{\frac{1}{q_1}} \left(\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)qq_2(1+1/\nu)u_0^2} - 1)dx\right)^{\frac{1}{q_2}} \\
&+ \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q(1+\nu)(u_\epsilon-u_0)^2} - 1)dx + \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)q(1+1/\nu)u_0^2} - 1)dx,
\end{aligned}$$

where $q > 1$, $\nu > 0$, $q_1 > 1$ and $1/q_1 + 1/q_2 = 1$. Here we also use an elementary inequality due to Yang [31, Lemma 2.1], that is, $(e^a - 1)^s \leq e^{as} - 1$ for $a \geq 0$ and $s \geq 1$. We can choose q and q_1 sufficiently close to 1 and ν sufficiently close to 0 such that $(4\pi - \epsilon)qq_1(1 + \nu)\|u_\epsilon - u_0\|^2_{W^{1,2}(\mathbb{R}^2)} < 4\pi$. In view of Trudinger–Moser inequality (1.3), we conclude that

$$\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)qu_\epsilon^2} - 1)dx \leq C \tag{3.1}$$

for some constant C depending on q. It follows (3.1) that $e^{(4\pi-\epsilon)u_\epsilon^2} - 1$ is bounded in $L^q(\mathbb{B}_1)$ for some $q > 1$. At the same time, $\|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{B}_1)$ and u_ϵ is bounded in $L^r(\mathbb{B}_1)$ for $r > 0$. Therefore, Δu_ϵ is bounded in $L^s(\mathbb{B}_1)$ for some $s > 1$. Applying the elliptic estimate to (2.2), we conclude that u_ϵ is bounded in $\mathbb{B}_{1/2}$, which contradicts $c_\epsilon \to +\infty$ as $\epsilon \to 0$. Therefore, $u_0 \equiv 0$.

We next prove $|\nabla u_\epsilon|^2 dx \rightharpoonup \delta_0$ in the sense of measure as $\epsilon \to 0$. Suppose not. There exists sufficiently small $\overline{r} > 0$ such that

$$\limsup_{\epsilon \to 0} \int_{\mathbb{B}_{\overline{r}}} |\nabla u_\epsilon|^2 dx \leq 1 - \gamma$$

for some $0 < \gamma < 1$. Note that u_ϵ is decreasing radially and symmetric. We set $\overline{u}_\epsilon(x) = u_\epsilon(x) - u_\epsilon(\overline{r})$ for $x \in \mathbb{B}_{\overline{r}}$. Then $\overline{u}_\epsilon(x) \in W_0^{1,2}(\mathbb{B}_{\overline{r}})$ satisfies $\int_{\mathbb{B}_{\overline{r}}} |\nabla \overline{u}_\epsilon|^2 dx \leq 1 - \gamma$. For any $q' > 1$, $q_1' > 1$ and $1/q_1' + 1/q_2' = 1$, we have by the Hölder inequality

$$\begin{aligned}
\int_{\mathbb{B}_{\overline{r}}} \left(\lambda_\epsilon^{-1} u_\epsilon (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)\right)^{q'} dx &\leq \frac{1}{\lambda_\epsilon^{q'}} \int_{\mathbb{B}_{\overline{r}}} u_\epsilon^{q'} (e^{(4\pi-\epsilon)q' u_\epsilon^2} - 1)dx \\
&\leq \frac{1}{\lambda_\epsilon^{q'}} \left(\int_{\mathbb{B}_{\overline{r}}} u_\epsilon^{q' q_1'} dx\right)^{1/q_1'} \left(\int_{\mathbb{B}_{\overline{r}}} e^{(4\pi-\epsilon)q' q_2' u_\epsilon^2} dx\right)^{1/q_2'}.
\end{aligned} \tag{3.2}$$

Since $\|u_\epsilon\|_{\alpha,p} \le 1$ and u_ϵ is nonnegative decreasing radially symmetric, we have

$$u_\epsilon(\overline{r}) \le \left(\frac{\lambda_p}{\pi(\lambda_p - \alpha)}\right)^{1/2} \frac{1}{\overline{r}}. \tag{3.3}$$

For any $\nu > 0$ and $x \in \mathbb{B}_{\overline{r}}$, there holds

$$u_\epsilon^2(x) \le (1+\nu)\overline{u}_\epsilon^2(x) + (1 + 1/\nu)u_\epsilon^2(\overline{r}). \tag{3.4}$$

Choosing $q' > 1$, $q_2' > 1$ sufficiently close to 1 and $\nu > 0$ sufficient small such that $q' q_2'(1+\nu)\|\nabla \overline{u}_\epsilon\|^2_{L^2(\mathbb{B}_{\overline{r}})} < 1$. Inserting (3.3) and (3.4) into (3.2) and noticing that u_ϵ is bounded in $L^s(\mathbb{B}_{\overline{r}})$ for any $s > 1$, we conclude that

$$\int_{\mathbb{B}_{\overline{r}}} \left(\lambda_\epsilon^{-1} u_\epsilon (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)\right)^{q'} dx \le C, \tag{3.5}$$

thanks to Lemma 2.2 and Trudinger–Moser inequality (1.1). One can see from (3.5) that $\lambda_\epsilon^{-1} u_\epsilon (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)$ is bounded in $L^{q'}(\mathbb{B}_{\overline{r}})$ for some $q' > 1$. Meanwhile $\|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{B}_{\overline{r}})$. We have by the standard elliptic estimate to (2.2) that u_ϵ is uniformly bounded in $\mathbb{B}_{\overline{r}/2}$ contradicting $c_\epsilon \to +\infty$ as $\epsilon \to 0$. Therefore, $|\nabla u_\epsilon|^2 dx \rightharpoonup \delta_0$. This completes the proof of the lemma. □

Let

$$r_\epsilon = \sqrt{\lambda_\epsilon} c_\epsilon^{-1} e^{-\frac{1}{2}(4\pi-\epsilon)c_\epsilon^2}.$$

Then we have the following:

Lemma 3.2 *For any $\gamma < 4\pi$, there holds*

$$\lim_{\epsilon \to 0} r_\epsilon^2 e^{\gamma c_\epsilon^2} = 0.$$

Proof Given $R > 0$, we have for any $\gamma < 4\pi$

$$\begin{aligned} r_\epsilon^2 e^{\gamma c_\epsilon^2} &= c_\epsilon^{-2} e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2} \int_{\mathbb{R}^2} u_\epsilon^2 (e^{(4\pi-\epsilon)u_\epsilon^2} - 1) dx \\ &= c_\epsilon^{-2} \int_{\mathbb{B}_R} u_\epsilon^2 e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1) dx \\ &+ c_\epsilon^{-2} \int_{\mathbb{B}_R^c} u_\epsilon^2 e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1) dx. \end{aligned} \tag{3.6}$$

Since

$$\int_{\mathbb{B}_R^c} u_\epsilon^2 (e^{(4\pi-\epsilon)u_\epsilon^2} - 1) dx = \sum_{j=1}^{\infty} \frac{(4\pi-\epsilon)^j}{j!} \int_{\mathbb{B}_R^c} u_\epsilon^{2j+2} dx.$$

We then have by the radial lemma [5],

$$\int_{\mathbb{B}_R^c} u_\epsilon^2 (e^{(4\pi-\epsilon)u_\epsilon^2} - 1) dx \le \frac{C}{R}.$$

Passing the limit $\epsilon \to 0$, we get

$$\lim_{\epsilon\to 0} c_\epsilon^{-2} \int_{\mathbb{B}_R^c} u_\epsilon^2 e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx = 0. \tag{3.7}$$

Noting that $c_\epsilon^2 \geq u_\epsilon^2$, whence by the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{B}_R} u_\epsilon^2 e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx &\leq \int_{\mathbb{B}_R} u_\epsilon^2 e^{\gamma u_\epsilon^2} dx \\ &\leq \left(\int_{\mathbb{B}_R} u_\epsilon^{2p_1} dx\right)^{1/p_1} \left(\int_{\mathbb{B}_R} e^{\gamma p_2 u_\epsilon^2} dx\right)^{1/p_2}, \end{aligned} \tag{3.8}$$

where $1/p_1 + 1/p_2 = 1$. Slightly modifying the proof of (3.5), one can get without any difficulty that

$$\int_{\mathbb{B}_R} e^{\gamma p_2 u_\epsilon^2} dx \leq C.$$

Since

$$\int_{\mathbb{B}_R} u_\epsilon^{2p_1} dx \leq \int_{\mathbb{R}^2} u_\epsilon^{2p_1} dx = o_\epsilon(1).$$

The above estimates together with (3.8) imply that

$$\lim_{\epsilon\to 0} c_\epsilon^{-2} \int_{\mathbb{B}_R} u_\epsilon^2 e^{-(4\pi-\epsilon-\gamma)c_\epsilon^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx = 0. \tag{3.9}$$

Combining(3.6), (3.7), and (3.9), we finish the proof of the lemma.

□

Define two blow-up functions

$$v_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(r_\epsilon x) \tag{3.10}$$

and

$$w_\epsilon(x) = c_\epsilon \left(u_\epsilon(r_\epsilon x) - c_\epsilon\right). \tag{3.11}$$

We state the result in the following form:

Lemma 3.3 *Let v_ϵ and w_ϵ be defined as in (3.10) and (3.11). Then $v_\epsilon \to 1$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, $w_\epsilon \to w$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where w is given by*

$$w(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2).$$

Moreover,

$$\int_{\mathbb{R}^2} e^{8\pi w(x)} dx = 1.$$

Proof By calculation, we obtain

$$\begin{aligned}-\Delta v_\epsilon(x) = &-r_\epsilon^2 v_\epsilon(x) - v_\epsilon(x) c_\epsilon^{-2} e^{-(4\pi-\epsilon)c_\epsilon^2} + c_\epsilon^{-2} v_\epsilon(x) e^{(4\pi-\epsilon)(1+v_\epsilon(x))w_\epsilon(x)} \\ &+ \alpha r_\epsilon^2 c_\epsilon^{p-2} \|u_\epsilon\|_p^{2-p} v_\epsilon^{p-1}(x) \end{aligned} \tag{3.12}$$

and

$$-\Delta w_\epsilon(x) = -r_\epsilon^2 c_\epsilon^2 v_\epsilon(x) - v_\epsilon(x) e^{-(4\pi-\epsilon)c_\epsilon^2} + v_\epsilon(x) e^{(4\pi-\epsilon)(1+v_\epsilon(x))w_\epsilon(x)}$$
$$+ \alpha r_\epsilon^2 c_\epsilon^p \|u_\epsilon\|_p^{2-p} v_\epsilon^{p-1}(x). \tag{3.13}$$

In view of (3.10) and Lemma 3.2, we have

$$r_\epsilon^2 c_\epsilon^p \|u_\epsilon\|_p^{2-p} v_\epsilon^{p-1}(x) \le \frac{r_\epsilon^2 c_\epsilon^p v_\epsilon^{p-1}(x)}{\left(\int_{\mathbb{B}_{Rr_\epsilon}} u_\epsilon^p(x)dx\right)^{1-\frac{2}{p}}} = \frac{c_\epsilon^2 r_\epsilon^{\frac{4}{p}} v_\epsilon^{p-1}(x)}{\left(\int_{\mathbb{B}_R} v_\epsilon^p(x)dx\right)^{1-\frac{2}{p}}} = o_\epsilon(1).$$

Applying the elliptic estimates to (3.12) and (3.13), we get

$$v_\epsilon \to 1 \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2),$$
$$w_\epsilon \to w \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2),$$

where w satisfies

$$\begin{cases} -\Delta w = e^{8\pi w} \quad \text{in} \quad \mathbb{R}^2 \\ w(0) = 0 = \sup_{\mathbb{R}^2} w. \end{cases}$$

By the uniqueness result obtained in [8], we have

$$w(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2) \quad \text{in} \quad \mathbb{R}^2.$$

It follows that

$$\int_{\mathbb{R}^2} e^{8\pi w(x)}dx = \int_0^{+\infty} \frac{2\pi r}{(1+\pi r^2)^2}dr = 1. \tag{3.14}$$

□

We next consider the convergence u_ϵ away from the concentration point 0. Following [2, 17], define $u_{\epsilon,\beta} = \min\{\beta c_\epsilon, u_\epsilon\}$ for $0<\beta<1$. Then we have the following:

Lemma 3.4 *For any $0<\beta<1$, there holds*

$$\limsup_{\epsilon\to 0} \int_{\mathbb{R}^2} |\nabla u_{\epsilon,\beta}|^2 dx = \beta.$$

Proof For any fixed $R>0$, testing (2.2) by $(u_\epsilon - \beta c_\epsilon)^+$, we have

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla (u_\epsilon - \beta c_\epsilon)^+|^2 dx &= \int_{\mathbb{R}^2} \nabla u_\epsilon \nabla (u_\epsilon - \beta c_\epsilon)^+ dx \\ &= -\int_{\mathbb{R}^2} u_\epsilon (u_\epsilon - \beta c_\epsilon)^+ dx + \lambda_\epsilon^{-1}\int_{\mathbb{R}^2} u_\epsilon (u_\epsilon - \beta c_\epsilon)^+ e^{(4\pi-\epsilon)u_\epsilon^2} dx \\ &- \lambda_\epsilon^{-1}\int_{\mathbb{R}^2} u_\epsilon (u_\epsilon - \beta c_\epsilon)^+ dx + \alpha\|u_\epsilon\|_p^{2-p}\int_{\mathbb{R}^2} (u_\epsilon - \beta c_\epsilon)^+ u_\epsilon^{p-1} dx \\ &\ge \int_{\mathbb{B}_{Rr_\epsilon}} (u_\epsilon - \beta c_\epsilon)^+ (\lambda_\epsilon^{-1} u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} + \alpha\|u_\epsilon\|_p^{2-p} u_\epsilon^{p-1} dx)dx + o_\epsilon(1) \\ &= (1-\beta)(1+o_\epsilon(1))\int_{\mathbb{B}_R} e^{8\pi w(x)}dx + o_\epsilon(1).\end{aligned}$$

Here, we used the fact $u_\epsilon > \beta c_\epsilon$ in $\mathbb{B}_{Rr_\epsilon}$ and

$$\alpha\|u_\epsilon\|_p^{2-p}\|(u_\epsilon-\beta c_\epsilon)^+u_\epsilon^{p-1}\|_{L^1(\mathbb{B}_{Rr_\epsilon})}\le\alpha\|u_\epsilon\|_p^2=o_\epsilon(1).$$

Hence,

$$\liminf_{\epsilon\to 0}\int_{\mathbb{R}^2}|\nabla(u_\epsilon-\beta c_\epsilon)^+|^2dx\ge(1-\beta)\int_{\mathbb{B}_R}e^{8\pi w(x)}dx.$$

In view of (3.14), letting $R\to+\infty$, we obtain

$$\liminf_{\epsilon\to 0}\int_{\mathbb{R}^2}|\nabla(u_\epsilon-\beta c_\epsilon)^+|^2dx\ge 1-\beta. \tag{3.15}$$

Similarly as above, testing (2.2) by $u_{\epsilon,\beta}$, we get

$$\liminf_{\epsilon\to 0}\int_{\mathbb{R}^2}|\nabla u_{\epsilon,\beta}|^2dx\ge\beta. \tag{3.16}$$

Note that $|\nabla u_\epsilon|^2=|\nabla u_{\epsilon,\beta}|^2+|\nabla(u_\epsilon-\beta c_\epsilon)^+|^2$ almost everywhere. Combining (3.15) and (3.16), we get the desired result. □

A consequence of Lemma 3.4 is the following.

Lemma 3.5

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx=\limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Proof For any β, $0<\beta<1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx &= \int_{u_\epsilon\le\beta c_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx\\ &\quad+\int_{u_\epsilon>\beta c_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx\\ &= I+II.\end{aligned}$$

By Lemma 3.1 and Lemma 3.4, we have $\limsup_{\epsilon\to 0}\|\nabla u_{\epsilon,\beta}\|^2_{W^{1,2}(\mathbb{R}^2)}=\beta<1$. Let $1<s<1/\beta$ and $1/s+1/t=1$. Again using the inequality $e^t-1-t\le t(e^t-1)$ for $t\ge 0$, the Hölder inequality and the Trudinger–Moser inequality (1.3), we can verify that

$$\begin{aligned}I &\le \int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2}-1-(4\pi-\epsilon)u_{\epsilon,\beta}^2)dx\\ &\le 4\pi\int_{\mathbb{R}^2}u_{\epsilon,\beta}^2(e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2}-1)dx\\ &\le 4\pi\left(\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)su_{\epsilon,\beta}^2}-1)dx\right)^{1/s}\left(\int_{\mathbb{R}^2}u_{\epsilon,\beta}^{2t}dx\right)^{1/t}\\ &\le C\left(\int_{\mathbb{R}^2}u_{\epsilon,\beta}^{2t}dx\right)^{1/t}\end{aligned} \tag{3.17}$$

for some constant C depending only on β and s. In view of the definition of $u_{\epsilon,\beta}$, we obtain that

$$\int_{\mathbb{R}^2} u_{\epsilon,\beta}^{2t} dx \le \int_{\mathbb{R}^2} u_\epsilon^{2t} dx = o_\epsilon(1). \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\lim_{\epsilon\to 0} I = 0. \tag{3.19}$$

Since $u_\epsilon \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for any $q > 1$, we get

$$\begin{aligned} II &= \int_{u_\epsilon > \beta c_\epsilon} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx + o_\epsilon(1) \\ &\le \frac{1}{\beta^2 c_\epsilon^2} \int_{u_\epsilon > \beta c_\epsilon} u_\epsilon^2 (e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx + o_\epsilon(1) \\ &\le \frac{1}{\beta^2} \frac{\lambda_\epsilon}{c_\epsilon^2} + o_\epsilon(1). \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20), we obtain

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx \le \frac{1}{\beta^2} \limsup_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Letting $\beta \to 1$, one has

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx \le \limsup_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \tag{3.21}$$

Note that

$$\begin{aligned} \frac{\lambda_\epsilon}{c_\epsilon^2} &= \int_{\mathbb{R}^2} \frac{u_\epsilon^2}{c_\epsilon^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx + \frac{4\pi-\epsilon}{c_\epsilon^2} \int_{\mathbb{R}^2} u_\epsilon^4 dx \\ &\le \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx + o_\epsilon(1). \end{aligned}$$

Thus,

$$\limsup_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \le \lim_{\epsilon\to 0} \int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx.$$

This estimate together with (3.21) implies that Lemma 3.5 holds. □

Obviously, one can derive a useful corollary from Lemma 3.5. Precisely,

Corollary 3.6 *For $\theta < 2$, there holds*

$$\limsup_{\epsilon\to 0} \frac{c_\epsilon^\theta}{\lambda_\epsilon} = 0.$$

Proof Suppose not. There exists some constant $M > 0$ such that $\lambda_\epsilon/c_\epsilon^\theta \leq M$ for $\theta < 2$. Then we have $\lambda_\epsilon/c_\epsilon^2 \to 0$ as $\epsilon \to 0$. Assume $v \in W^{1,2}(\mathbb{R}^2)$ and $\|v\|_{\alpha,p} = 1$, we have by Lemma 3.5 that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)v^2}-1-(4\pi-\epsilon)v^2)dx &\leq \lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2}-1-(4\pi-\epsilon)u_\epsilon^2)dx\\ &\leq \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}\\ &= 0.\end{aligned}$$

This is impossible since $v \not\equiv 0$. Therefore, we get the desired result. □

Lemma 3.7 *For any $\varphi \in C_0^\infty(\mathbb{R}^2)$, we have*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}\varphi\lambda_\epsilon^{-1}c_\epsilon u_\epsilon(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx = \varphi(0).$$

Proof For convenience in writing, we set

$$h_\epsilon(x) = \lambda_\epsilon^{-1}c_\epsilon u_\epsilon(e^{(4\pi-\epsilon)u_\epsilon^2}-1).$$

Let $0 < \beta < 1$ be fixed, we observe that

$$\int_{\mathbb{R}^2}\varphi h_\epsilon dx = \int_{u_\epsilon\leq\beta c_\epsilon}\varphi h_\epsilon dx + \int_{\{u_\epsilon>\beta c_\epsilon\}\setminus\mathbb{B}_{Rr_\epsilon}}\varphi h_\epsilon dx + \int_{\{u_\epsilon>\beta c_\epsilon\}\bigcap\mathbb{B}_{Rr_\epsilon}}\varphi h_\epsilon dx. \tag{3.22}$$

Now we estimate the integrals on the right-hand of (3.22) respectively. In view of an obvious analog of (3.19) and thanks to Lemma 3.4 and Corollary 3.6, we have

$$\begin{aligned}\int_{u_\epsilon\leq\beta c_\epsilon}\varphi h_\epsilon dx &= \frac{c_\epsilon}{\lambda_\epsilon}\int_{u_\epsilon\leq\beta c_\epsilon}u_\epsilon\varphi(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx\\ &\leq \frac{c_\epsilon}{\lambda_\epsilon}\left(\sup_{\mathbb{R}^2}|\varphi|\right)\int_{\mathbb{R}^2}u_{\epsilon,\beta}(e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2}-1)dx\\ &= o_\epsilon(1).\end{aligned} \tag{3.23}$$

It follows from Lemma 3.2 that $\mathbb{B}_{Rr_\epsilon} \subset \{u_\epsilon > \beta c_\epsilon\}$ for sufficiently small $\epsilon > 0$. We then obtain

$$\begin{aligned}\int_{\{u_\epsilon>\beta c_\epsilon\}\bigcap\mathbb{B}_{Rr_\epsilon}}\varphi h_\epsilon dx &= \varphi(0)(1+o_\epsilon(1))\int_{\mathbb{B}_{Rr_\epsilon}}\lambda_\epsilon^{-1}c_\epsilon u_\epsilon(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx\\ &= \varphi(0)(1+o_\epsilon(1))\left(\int_{\mathbb{B}_R}e^{8\pi w}dx + o_\epsilon(1)\right)\\ &= \varphi(0)(1+o_\epsilon(1)+o_R(1)),\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\int_{\{u_\epsilon>\beta c_\epsilon\}\setminus\mathbb{B}_{Rr_\epsilon}}\varphi h_\epsilon dx &\le \frac{1}{\beta}\left(\sup_{\mathbb{R}^2}|\varphi|\right)\int_{\{u_\epsilon>\beta c_\epsilon\}\setminus\mathbb{B}_{Rr_\epsilon}}\lambda_\epsilon^{-1}u_\epsilon^2(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx\\ &\le \frac{1}{\beta}\left(\sup_{\mathbb{R}^2}|\varphi|\right)\left(1-\int_{\mathbb{B}_{Rr_\epsilon}}\lambda_\epsilon^{-1}u_\epsilon^2(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx\right)\\ &= \frac{1}{\beta}\left(\sup_{\mathbb{R}^2}|\varphi|\right)\left(1-\int_{\mathbb{B}_R}e^{8\pi w}dx+o_\epsilon(1)\right)\\ &= o_\epsilon(1)+o_R(1). \end{aligned}\tag{3.25}$$

Inserting (3.23)–(3.25) into (3.22) and letting $\epsilon\to 0$ first, then $R\to+\infty$, we finish the proof of the lemma. □

Let us now investigate the convergence of the function sequence $c_\epsilon u_\epsilon$. We shall prove the following lemma.

Lemma 3.8 *For any $1<q<2$, there holds*

$$c_\epsilon u_\epsilon \rightharpoonup G \quad \text{weakly} \quad \text{in} \quad W^{1,q}_{\rm loc}(\mathbb{R}^2)$$

and

$$c_\epsilon u_\epsilon \to G \quad \text{in} \quad C^1_{\rm loc}(\mathbb{R}^2\setminus\{0\}),$$

where G is a Green's function and satisfies

$$-\Delta G+G=\delta_0+\alpha\|G\|_p^{2-p}G^{p-1}$$

in a distributional sense, where δ_0 is the usual Dirac measure centered at 0.

Proof First, we claim that $\|c_\epsilon u_\epsilon\|_p$ is bounded. To confirm this, we will use an idea similar to that in [13, Lemma 3.5]. Multiplying both sides of (2.2) by c_ϵ, we obtain

$$-\Delta(c_\epsilon u_\epsilon)+c_\epsilon u_\epsilon=\frac{1}{\lambda_\epsilon}c_\epsilon u_\epsilon(e^{(4\pi-\epsilon)u_\epsilon^2}-1)+\alpha\|c_\epsilon u_\epsilon\|_p^{2-p}(c_\epsilon u_\epsilon)^{p-1}.\tag{3.26}$$

Suppose $\|c_\epsilon u_\epsilon\|_p\to+\infty$ as $\epsilon\to 0$. Setting $\omega_\epsilon=c_\epsilon u_\epsilon/\|c_\epsilon u_\epsilon\|_p$, one can easily deduce from (3.26) that

$$-\Delta\omega_\epsilon+\omega_\epsilon=\frac{1}{\|c_\epsilon u_\epsilon\|_p}\frac{1}{\lambda_\epsilon}c_\epsilon u_\epsilon(e^{(4\pi-\epsilon)u_\epsilon^2}-1)+\alpha\omega_\epsilon^{p-1},\tag{3.27}$$

which together with Lemma 3.7 implies that $\Delta\omega_\epsilon$ is bounded in $L^1_{\rm loc}(\mathbb{R}^2)$. Applying the argument of Li and Ruf [18, Proposition 3.7], or do Ó and de Souza [11, Lemma 4.9], which is motivated by the idea of Struwe [25, Theorem 2.2], we conclude that ω_ϵ is bounded in $W^{1,q}_{\rm loc}(\mathbb{R}^2)$ for any $1<q<2$. We assume up to a subsequence $\omega_\epsilon\rightharpoonup\omega$ weakly in $W^{1,q}_{\rm loc}(\mathbb{R}^2)$. Testing (3.27) with $\varphi(x)\in C_0^\infty(\mathbb{R}^2)$ and letting $\epsilon\to 0$, we have

$$\int_{\mathbb{R}^2}\nabla\varphi\nabla\omega dx+\int_{\mathbb{R}^2}\varphi\omega dx=\alpha\int_{\mathbb{R}^2}\varphi\omega^{p-1}dx.\tag{3.28}$$

Since $0\le\alpha<\lambda_p$, it follows from (3.28) that $\omega\equiv 0$, which contradicts $\|\omega\|_p=1$. Therefore, $\|c_\epsilon u_\epsilon\|_p$ is bounded.

Again by Proposition 3.7 in [18] or Lemma 4.9 in [11], we conclude that $c_\epsilon u_\epsilon$ is bounded in $W^{1,q}_{\text{loc}}(\mathbb{R}^2)$ for $1<q<2$. Hence, there exists some $G\in W^{1,q}(\mathbb{R}^2)$ such that $c_\epsilon u_\epsilon \rightharpoonup G$ weakly in $W^{1,q}_{\text{loc}}(\mathbb{R}^2)$, and that $c_\epsilon u_\epsilon \to G$ strongly in $L^s_{\text{loc}}(\mathbb{R}^2)$ for $s>1$. Given any $\nu>0$, in view of that $c_\epsilon u_\epsilon$ is decreasing radially and symmetric, we can find a sufficiently large number $r>0$ such that

$$\left|\int_{\mathbb{B}_r^c}(c_\epsilon u_\epsilon)^p dx\right| \le \nu \tag{3.29}$$

and

$$\left|\int_{\mathbb{B}_r^c} G^p dx\right| \le \nu. \tag{3.30}$$

Moreover, we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}_r}(c_\epsilon u_\epsilon)^p dx = \int_{\mathbb{B}_r} G^p dx. \tag{3.31}$$

It follows from (3.29)–(3.31) that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}(c_\epsilon u_\epsilon)^p dx = \int_{\mathbb{R}^2} G^p dx.$$

Testing (3.26) with $\varphi(x)\in C_0^\infty(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}\nabla\varphi\nabla(c_\epsilon u_\epsilon)dx+\int_{\mathbb{R}^2}\varphi c_\epsilon u_\epsilon dx=\int_{\mathbb{R}^2}\varphi\frac{c_\epsilon u_\epsilon}{\lambda_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2}-1)dx+\alpha\|c_\epsilon u_\epsilon\|_p^{2-p}\int_{\mathbb{R}^2}\varphi(c_\epsilon u_\epsilon)^{p-1}dx. \tag{3.32}$$

Letting $\epsilon\to 0$, we obtain

$$\int_{\mathbb{R}^2}\nabla\varphi\nabla G dx+\int_{\mathbb{R}^2}\varphi G dx=\varphi(0)+\alpha\|G\|_p^{2-p}\int_{\mathbb{R}^2}\varphi G^{p-1}dx;$$

hence, G satisfies the following equation in a distributional sense

$$-\Delta G+G=\delta_0+\alpha\|G\|_p^{2-p}G^{p-1}.$$

Let r_0, R_0 be such that $R_0>4r_0>0$, we can choose a radially symmetric cut-off function $\eta(x)\in C_0^\infty(\mathbb{B}_{R_0}\backslash\mathbb{B}_{r_0})$ that equals 1 on $\mathbb{B}_{4r_0}\backslash\mathbb{B}_{2r_0}$. By Lemma 3.1, one has $\|\nabla(\eta u_\epsilon)\|_2\to 0$ as $\epsilon\to 0$, which implies that $e^{(4\pi-\epsilon)\eta^2u_\epsilon^2}-1$ is bounded in $L^s(\mathbb{B}_{R_0}\backslash\mathbb{B}_{r_0})$ for any $s>1$. Therefore, $e^{(4\pi-\epsilon)u_\epsilon^2}-1$ is bounded in $L^s(\mathbb{B}_{4r_0}\backslash\mathbb{B}_{2r_0})$. At the same time, $\|c_\epsilon u_\epsilon\|_p^{2-p}(c_\epsilon u_\epsilon)^{p-1}$ is bounded in $L^{\frac{p}{p-1}}(\mathbb{B}_{R_0}\backslash\mathbb{B}_{r_0})$. Applying elliptic estimates to (3.26) two times, we get $c_\epsilon u_\epsilon\to G$ in $C^1(\overline{\mathbb{B}}_{4r_0}\backslash\mathbb{B}_{3r_0})$. That is, $c_\epsilon u_\epsilon\to G$ in $C^1_{\text{loc}}(\mathbb{R}^2\backslash\{0\})$. This completes the proof of the lemma. □

Note that

$$-\Delta\left(G+\frac{1}{2\pi}\log|x|\right)=-G+\alpha\|G\|_p^{2-p}G^{p-1}\in L^s_{\text{loc}}(\mathbb{R}^2).$$

Applying elliptic estimates, we get $(G + \frac{1}{2\pi}\log|x|) \in C^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$. Therefore, the Green function G takes the following form

$$G(x) = -\frac{1}{2\pi}\log|x| + A_0 + g(x), \tag{3.33}$$

where A_0 is a constant depending on p, $g(x) \in C^1(\mathbb{R}^2)$ and $g(0) = 0$.

4. Upper bound estimate

In this section, under the assumption of $c_\epsilon \to +\infty$ as $\epsilon \to 0$, we will use a result by Carleson and Chang [7] to derive an upper bound of the integrals $\int_{\mathbb{R}^2}(e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx$.

Lemma 4.1 *For any δ, $0 < \delta < 1$, we have*

$$\int_{\mathbb{B}_\delta} |\nabla u_\epsilon|^2 dx = 1 - \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log\frac{1}{\delta} + A_0 + o_\epsilon(1) + o_\delta(1)\right),$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$.

Proof In view of the Euler-Lagrange equation (2.2) and $\|u_\epsilon\|_{\alpha,p} = 1$, we have by the divergence theorem

$$\begin{aligned}
\int_{\mathbb{B}_\delta} |\nabla u_\epsilon|^2 dx &= 1 - \int_{\mathbb{R}^2\setminus\mathbb{B}_\delta}(|\nabla u_\epsilon|^2 + u_\epsilon^2)dx - \int_{\mathbb{B}_\delta} u_\epsilon^2 dx + \alpha\left(\int_{\mathbb{R}^2} u_\epsilon^p dx\right)^{\frac{2}{p}} \\
&= 1 - \int_{\mathbb{R}^2\setminus\mathbb{B}_\delta} \frac{u_\epsilon^2}{\lambda_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx - \alpha\|u_\epsilon\|_p^{2-p}\int_{\mathbb{R}^2\setminus\mathbb{B}_\delta} u_\epsilon^p dx \\
&\quad + \int_{\partial\mathbb{B}_\delta} u_\epsilon \frac{\partial u_\epsilon}{\partial v} ds - \int_{\mathbb{B}_\delta} u_\epsilon^2 dx + \alpha\left(\int_{\mathbb{R}^2} u_\epsilon^p dx\right)^{\frac{2}{p}}.
\end{aligned}$$

Now we proceed to estimate the right five terms on the above equation respectively. A direct calculation gives

$$\int_{\mathbb{R}^2\setminus\mathbb{B}_\delta} \frac{u_\epsilon^2}{\lambda_\epsilon}(e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx = \frac{1}{c_\epsilon^2}\frac{c_\epsilon^2}{\lambda_\epsilon}\int_{\mathbb{R}^2\setminus\mathbb{B}_\delta} u_\epsilon^2(e^{(4\pi-\epsilon)u_\epsilon^2} - 1)dx = \frac{o_\epsilon(1)}{c_\epsilon^2}. \tag{4.1}$$

At the same time, one has

$$\alpha\|u_\epsilon\|_p^{2-p}\int_{\mathbb{R}^2\setminus\mathbb{B}_\delta} u_\epsilon^p dx = \frac{1}{c_\epsilon^2}\left(\alpha\|G\|_p^2 + o_\epsilon(1) + o_\delta(1)\right),$$

$$\int_{\partial\mathbb{B}_\delta} u_\epsilon \frac{\partial u_\epsilon}{\partial v} ds = \frac{1}{c_\epsilon^2}\left(\int_{\partial\mathbb{B}_\delta} G\frac{\partial G}{\partial v} ds + o_\epsilon(1)\right),$$

$$\int_{\mathbb{B}_\delta} u_\epsilon^2 dx = \frac{1}{c_\epsilon^2}\left(\int_{\mathbb{B}_\delta} G^2 dx + o_\epsilon(1)\right),$$

$$\alpha\left(\int_{\mathbb{R}^2} u_\epsilon^p dx\right)^{\frac{2}{p}} = \frac{1}{c_\epsilon^2}\left(\alpha\|G\|_p^2 + o_\epsilon(1)\right).$$

Combining all the above estimates and recalling that $G(x) = -\frac{1}{2\pi}\log|x| + A_0 + g(x)$, we get the desired result

$$\int_{\mathbb{B}_\delta} |\nabla u_\epsilon|^2 dx = 1 - \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log\frac{1}{\delta} + A_0 + o_\epsilon(1) + o_\delta(1)\right).$$

□

Lemma 4.2 *There holds*

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx \le \pi e^{4\pi A_0+1}.$$

Proof Set $s_\epsilon = \sup_{\partial\mathbb{B}_\delta} u_\epsilon = u_\epsilon(\delta)$ and $\widetilde{u}_\epsilon = (u_\epsilon - s_\epsilon)^+$, the positive part of $u_\epsilon - s_\epsilon$. Obviously, $\widetilde{u}_\epsilon \in W_0^{1,2}(\mathbb{B}_\delta)$. By Lemma 4.1, we have

$$\int_{\mathbb{B}_\delta} |\nabla \widetilde{u}_\epsilon|^2 dx \le \tau_{\epsilon,\delta} = 1 - \frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log\frac{1}{\delta} + A_0 + o_\epsilon(1) + o_\delta(1)\right). \tag{4.2}$$

Then we use Carleson and Chang's upper bounded estimate [7] and conclude that

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_\delta}\left(e^{(4\pi-\epsilon)\widetilde{u}_\epsilon^2/\tau_{\epsilon,\delta}} - 1\right)dx \le \pi e\delta^2. \tag{4.3}$$

Note that $u_\epsilon = c_\epsilon + o_\epsilon(1)$ on $\mathbb{B}_{Rr_\epsilon}$. This together (4.2) leads to that on $\mathbb{B}_{Rr_\epsilon} \subset \mathbb{B}_\delta$

$$\begin{aligned}
(4\pi-\epsilon)u_\epsilon^2 &\le 4\pi(\widetilde{u}_\epsilon + s_\epsilon)^2 \\
&\le 4\pi\widetilde{u}_\epsilon^2 + 8\pi s_\epsilon \widetilde{u}_\epsilon + o_\epsilon(1) \\
&\le 4\pi\widetilde{u}_\epsilon^2 - 4\log\delta + 8\pi A_0 + o_\epsilon(1) + o_\delta(1) \\
&\le 4\pi\widetilde{u}_\epsilon^2/\tau_{\epsilon,\delta} - 2\log\delta + 4\pi A_0 + o(1),
\end{aligned}$$

where $o(1)\to 0$ as $\epsilon\to 0$ first and then $\delta\to 0$. For any fixed $R>0$, we calculate

$$\begin{aligned}
\int_{\mathbb{B}_{Rr_\epsilon}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx &\le \delta^{-2}e^{4\pi A_0+o(1)}\int_{\mathbb{B}_{Rr_\epsilon}} e^{4\pi\widetilde{u}_\epsilon^2/\tau_{\epsilon,\delta}}dx + o(1) \\
&= \delta^{-2}e^{4\pi A_0+o(1)}\int_{\mathbb{B}_{Rr_\epsilon}} (e^{4\pi\widetilde{u}_\epsilon^2/\tau_{\epsilon,\delta}} - 1)dx + o(1) \\
&\le \delta^{-2}e^{4\pi A_0+o(1)}\int_{\mathbb{B}_\delta} (e^{4\pi\widetilde{u}_\epsilon^2/\tau_{\epsilon,\delta}} - 1)dx + o(1).
\end{aligned}$$

In view of (4.3), we get

$$\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx \le \pi e^{4\pi A_0+1}. \tag{4.4}$$

By a change of variable $x = r_\epsilon y$, there holds

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\epsilon}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx &= r_\epsilon^2 \int_{\mathbb{B}_R} e^{(4\pi-\epsilon)u_\epsilon^2(r_\epsilon y)}dy + o_\epsilon(1)\\ &= \frac{\lambda_\epsilon}{c_\epsilon^2}\left(\int_{\mathbb{B}_R} e^{8\pi w(y)}dy + o_\epsilon(1)\right) + o_\epsilon(1)\\ &= \frac{\lambda_\epsilon}{c_\epsilon^2}(1 + o_\epsilon(1) + o_R(1)).\end{aligned}$$

As a result, we obtain

$$\lim_{R\to+\infty}\lim_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon}} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx = \lim_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^2}. \tag{4.5}$$

In view of Lemma 3.5, (4.4) and (4.5), we conclude

$$\Lambda_{\alpha,p} = \lim_{\epsilon\to 0}\int_{\mathbb{R}^2} (e^{(4\pi-\epsilon)u_\epsilon^2} - 1 - (4\pi-\epsilon)u_\epsilon^2)dx \le \pi e^{4\pi A_0+1} \tag{4.6}$$

as desired. □

5. Test function computation

In this section, we will construct a family of the test function $\varphi_\epsilon(x) \in W^{1,2}(\mathbb{R}^2)$ such that $\|\varphi_\epsilon\|_{\alpha,p} = 1$ and

$$\int_{\mathbb{R}^2} (e^{4\pi\varphi_\epsilon^2} - 1 - 4\pi\varphi_\epsilon^2)dx > \pi e^{4\pi A_0+1} \tag{5.1}$$

for $\epsilon > 0$ sufficiently small. This result contradicts with (4.6) and consequently the blow up does not occur. Therefore we get the desired extremal function and complete the proof of Theorem 1.1.

For this purpose, we set

$$\varphi_\epsilon(x) = \begin{cases} c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi\frac{|x|^2}{\epsilon^2}) + b\right) & |x| \le R\epsilon \\ \frac{G(x)}{c} & |x| > R\epsilon \end{cases} \tag{5.2}$$

where $R = (-\log\epsilon)^2$, G is the Green function given as in (3.33), b and c are constants depending only on ϵ to be determined later.

In order to assure that $\varphi_\epsilon \in W^{1,2}(\mathbb{R}^2)$, we require

$$c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi R^2) + b\right) = \frac{1}{c}\left(-\frac{1}{2\pi}\log(R\epsilon) + A_0 + g(R\epsilon)\right)$$

which implies that

$$4\pi c^2 = -2\log\epsilon - 4\pi b + 4\pi A_0 + \log\pi + O(R\epsilon) + O\left(\frac{1}{R^2}\right). \tag{5.3}$$

By (3.33), we calculate

$$\int_{\mathbb{R}^2\setminus\mathbb{B}_{R\epsilon}}(|\nabla\varphi_\epsilon|^2+\varphi_\epsilon^2)dx=\frac{1}{c^2}\int_{\mathbb{R}^2\setminus\mathbb{B}_{R\epsilon}}(|\nabla G|^2+G^2)dx$$
$$=\frac{1}{c^2}\left(\alpha\|G\|_p^{2-p}\int_{\mathbb{R}^2\setminus\mathbb{B}_{R\epsilon}}G^p dx-\int_{\partial\mathbb{B}_{R\epsilon}}G\frac{\partial G}{\partial v}ds\right)$$
$$=\frac{1}{c^2}\left(\alpha\|G\|_p^2-\frac{1}{2\pi}\log(R\epsilon)+A_0+O(R\epsilon\log(R\epsilon))\right). \tag{5.4}$$

In view of (5.2), we have

$$\int_{\mathbb{R}^2\setminus\mathbb{B}_{R\epsilon}}\varphi_\epsilon^p dx=\frac{1}{c^p}\int_{\mathbb{R}^2\setminus\mathbb{B}_{R\epsilon}}G^p dx=\frac{1}{c^p}\left(\|G\|_p^p+O(R\epsilon\log(R\epsilon)\right). \tag{5.5}$$

Meanwhile, we obtain

$$\int_{\mathbb{B}_{R\epsilon}}|\nabla\varphi_\epsilon|^2 dx=\frac{1}{4\pi c^2}\int_0^{R\epsilon}\frac{2r^3}{(r^2+\frac{\epsilon^2}{\pi})^2}dr$$
$$=\frac{1}{4\pi c^2}\left(\log\pi-1+\log R^2+O\left(\frac{1}{R^2}\right)\right). \tag{5.6}$$

In addition, we require b to be bounded with respect to ϵ. Then we have

$$\int_{\mathbb{B}_{R\epsilon}}\varphi_\epsilon^2 dx=O((R\epsilon)^2(-\log\epsilon)), \tag{5.7}$$

and

$$\int_{\mathbb{B}_{R\epsilon}}\varphi_\epsilon^p dx=O((R\epsilon)^2(-\log\epsilon)^{\frac{p}{2}}). \tag{5.8}$$

Combining (5.4)–(5.8), we conclude

$$\|\varphi_\epsilon\|_{\alpha,p}^2=\frac{1}{c^2}\left(-\frac{1}{2\pi}\log\epsilon+A_0-\frac{1}{4\pi}+\frac{1}{4\pi}\log\pi+O(R\epsilon\log(R\epsilon))+O\left(\frac{1}{R^2}\right)\right).$$

Setting $\|\varphi_\epsilon\|_{\alpha,p}=1$, we get

$$c^2=-\frac{1}{2\pi}\log\epsilon+A_0-\frac{1}{4\pi}+\frac{1}{4\pi}\log\pi+O(R\epsilon\log(R\epsilon))+O\left(\frac{1}{R^2}\right). \tag{5.9}$$

It follows from (5.3) and (5.9) that

$$b=\frac{1}{4\pi}+O(R\epsilon\log(R\epsilon))+O\left(\frac{1}{R^2}\right). \tag{5.10}$$

In view of (5.9), (5.10) and the Taylor formula of $(1+t)^2$ near $t=0$, we conclude for any $x\in\mathbb{B}_{R\epsilon}$

$$4\pi\varphi_\epsilon^2(x)\geq 4\pi c^2+8\pi b-2\log\left(1+\pi\frac{|x|^2}{\epsilon^2}\right)$$
$$=-2\log\left(1+\pi\frac{|x|^2}{\epsilon^2}\right)-2\log\epsilon+4\pi A_0+\log\pi+1+O\left(\frac{1}{R^2}\right).$$

This leads to

$$\int_{\mathbb{B}_{R\epsilon}} (e^{4\pi\varphi_\epsilon^2} - 1 - 4\pi\varphi_\epsilon^2)dx \geq \pi\epsilon^{-2}e^{4\pi A_0+1+O(\frac{1}{R^2})} \int_{\mathbb{B}_{R\epsilon}} \frac{1}{\left(1+\pi\frac{|x|^2}{\epsilon^2}\right)^2}dx + +O\left(\frac{1}{R^2}\right)$$

$$= \pi\epsilon^{-2}e^{4\pi A_0+1+O(\frac{1}{R^2})} \int_0^{R\epsilon} \frac{2\pi r}{\left(1+\pi\frac{r^2}{\epsilon^2}\right)^2}dr + O\left(\frac{1}{R^2}\right)$$

$$= \pi e^{4\pi A_0+1} + O\left(\frac{1}{R^2}\right). \tag{5.11}$$

Also, on $\mathbb{R}^2\backslash\mathbb{B}_{R\epsilon}$, we calculate

$$\int_{\mathbb{R}^2\backslash\mathbb{B}_{R\epsilon}} (e^{4\pi\varphi_\epsilon^2} - 1 - 4\pi\varphi_\epsilon^2)dx \geq \frac{8\pi^2}{c^4}\int_{\mathbb{R}^2\backslash\mathbb{B}_{R\epsilon}} G^4 dx$$

$$= \frac{8\pi^2}{c^4}\left(\int_{\mathbb{R}^2} G^4 dx + o_\epsilon(1)\right). \tag{5.12}$$

Combining (5.11) and (5.12) and noting that $R^{-2}c^4 = o_\epsilon(1)$, we have

$$\int_{\mathbb{R}^2} (e^{4\pi\varphi_\epsilon^2} - 1 - 4\pi\varphi_\epsilon^2)dx \geq \pi e^{4\pi A_0+1} + \frac{8\pi^2}{c^4}\left(\int_{\mathbb{R}^2} G^4 dx + o_\epsilon(1)\right).$$

Therefore, we conclude (5.1) for sufficiently small $\epsilon \to 0$. □

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