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A unique solution to a fourth-order three-point boundary value problem

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Abstract: In this study, it is aimed to examine the solutions of the following nonlocal boundary value problem

$$y^{(4)} + g(x, y) = 0, x \in [c, d], y(c) = y'(c) = y''(c) = 0, y(d) = \lambda y(\xi).$$

Here, $\xi \in (c, d), \lambda \in \mathbb{R}, g \in C([c, d] \times \mathbb{R}, \mathbb{R})$ and $g(x, 0) \neq 0$. It is concentrated on applications of Green's function that corresponds to the above problem to derive existence and uniqueness results for the solutions. One example is also given to demonstrate the results.

Key words: Nonlocal boundary value problems, Green's function, existence and uniqueness of solutions

1. Introduction

We investigate the following boundary value problem

$$y^{(4)} + g(x, y) = 0, x \in [c, d], \tag{1.1}$$

$$y(c) = y'(c) = y''(c) = 0, y(d) = \lambda y(\xi). \tag{1.2}$$

Here, $\xi \in (c, d), \lambda \in \mathbb{R}, g \in C([c, d] \times \mathbb{R}, \mathbb{R})$ and $g(x, 0) \neq 0$.

Fourth-order three-point boundary value problems for ordinary differential equations can be expressed as an important phenomenon since it emerges in the studies of applied mathematics, physics, and engineering. Therefore this phenomenon can be further discussed and improved to be used in the future studies in those fields. In order to obtain more precise models for the solutions, the usage of nonlocal boundary conditions can be accepted since it considers the values inside the domain rather than considering only the boundary conditions.

This paper aims to find a result for the existence of a unique solution of (1.1)–(1.2) for a class of functions g. The corresponding Green's function is obtained to be used in the contracting mapping theorem to obtain the results for the unique solutions of these particular equations. If the value λ is taken as to be equal to 0, this means one gets a two-point boundary value problem. Therefore it can be said that the two-point boundary value problem is a special case of the problem (1.1)–(1.2).

Many researches are conducted for the nonlinear multipoint boundary value problems and existence of their solutions. Some of those researches are [1, 6, 7] including their referenced works. In [3], Schauder's fixed

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point theorem, the upper and lower solution method, and topological degree theory are used in order to conclude to the existence of unbounded solutions for a fourth-order three-point boundary value problem on a half-line.

Green's function plays an important role in the theory of boundary value problems. In the boundary value problems theory, applying the Green's function is highly functional. The background of this approach with the Green's function is well addressed in [8]. Using the Green's function method for the fourth-order three-point boundary value problems was heavily studied in [4, 5, 9], which promoted current research. The existence of nontrivial solution for a fifth-order three-point boundary value problem was proved by using the Leray–Schauder nonlinear alternative in [2].

The remaining of the paper is arranged as follows. In Section 2, we construct Green's function implementing integral equation formula and some additional assumptions. Section 3 is assigned to estimation of the Green's function. In Section 4, we prove our main theorem on the existence and uniqueness for solution of the considered problem. Also, one example is given to illustrate the result. The conclusion is set out in Section 5.

2. Computation of the Green's function

At the first instance let us construct Green's function for the following three-point boundary value problem

$$v^{(4)} + w(x) = 0, x \in [c, d], \tag{2.1}$$

$$v(c) = v'(c) = v''(c) = 0, v(d) = 0, \tag{2.2}$$

and afterwards, supposing that the solution of the following three-point boundary value problem

$$y^{(4)} + w(x) = 0, x \in [c, d], \tag{2.3}$$

$$y(c) = y'(c) = y''(c) = 0, y(d) = \lambda y(\xi), \tag{2.4}$$

can be stated as follows:

$$y(x) = v(x) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3) v(\xi)$$

, where a_0, a_1, a_2 and a_3 are constants that will be specified, we will obtain Green's function for the problem (2.3)–(2.4).

Proposition 2.1 *If $w : [c, d] \to \mathbb{R}$ is a continuous function, then boundary value problem (2.1)–(2.2) has a unique solution*

$$v(x) = \int_c^x \left[\frac{(c-x)^3(d-s)^3}{6(c-d)^3} - \frac{(x-s)^3}{6}\right] w(s) ds + \int_x^d \left[\frac{(c-x)^3(d-s)^3}{6(c-d)^3}\right] w(s) ds,$$

that we can rewrite as

$$v(x) = \int_c^d B(x, s) w(s) ds,$$

where

$$B(x, s) = \begin{cases} \frac{(c-x)^3(d-s)^3}{6(c-d)^3} - \frac{(x-s)^3}{6}, & c \le s \le x \le d, \\ \frac{(c-x)^3(d-s)^3}{6(c-d)^3}, & c \le x \le s \le d. \end{cases} \tag{2.5}$$

Proof It is well known that the problem (2.1)–(2.2) is equivalent to solving the integral equation

$$v(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 - \frac{1}{6}\int_c^x (x-s)^3 w(s)ds,$$

where b_0, b_1, b_2 and b_3 are some real constants. Using boundary conditions (2.2), we can obtain

$b_0 = \frac{c^3}{6(c-d)^3}\int_c^d (d-s)^3 w(s)\,ds, b_1 = -\frac{c^2}{2(c-d)^3}\int_c^d (d-s)^3 w(s)\,ds,$
$b_2 = \frac{c}{2(c-d)^3}\int_c^d (d-s)^3 w(s)\,ds, b_3 = -\frac{1}{6(c-d)^3}\int_c^d (d-s)^3 w(s)\,ds.$

Thus, we get

$$\begin{aligned} v(x) &= \int_c^d \frac{(c-x)^3(d-s)^3}{6(c-d)^3} w(s)ds - \frac{1}{6}\int_c^x (x-s)^3 w(s)ds \\ &= \int_c^x \frac{(c-x)^3(d-s)^3}{6(c-d)^3} w(s)ds + \int_x^d \frac{(c-x)^3(d-s)^3}{6(c-d)^3} w(s)ds - \frac{1}{6}\int_c^x (x-s)^3 w(s)ds \\ &= \int_c^x \left[\frac{(c-x)^3(d-s)^3}{6(c-d)^3} - \frac{(x-s)^3}{6}\right] w(s)ds + \int_x^d \left[\frac{(c-x)^3(d-s)^3}{6(c-d)^3}\right] w(s)ds. \end{aligned}$$

□

The uniqueness follows from the fact that the corresponding homogeneous problem to BVP (2.1)–(2.2) only admits the trivial zero solution. Hence, the proof of Proposition 2.1 is completed.

Proposition 2.2 *Assume* $w : [c,d] \to \mathbb{R}$ *is a continuous function. If* $\lambda(c-\xi)^3 \neq (c-d)^3, (c \neq \xi)$, *then boundary value problem (2.3)-(2.4) has a unique solution*

$$y(x) = v(x) + \frac{\lambda(c-x)^3}{(c-d)^3 - \lambda(c-\xi)^3} v(\xi),$$

that we can rewrite as

$$y(x) = \int_c^d G(x,s) w(s)ds,$$

where

$$G(x,s) = B(x,s) + \frac{\lambda(c-x)^3}{(c-d)^3 - \lambda(c-\xi)^3} B(\xi,s). \tag{2.6}$$

Proof Let $y(x) = v(x) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3)v(\xi)$,where a_0, a_1, a_2 and a_3 are constants that will be identified using boundary conditions (2.4) and $v(x) = \int_c^d B(x,s)w(s)ds$. So,

$$\begin{aligned} y(c) &= v(c) + (a_0 + a_1 c + a_2 c^2 + a_3 c^3)v(\xi) = (a_0 + a_1 c + a_2 c^2 + a_3 c^3)v(\xi), \\ y'(c) &= v'(c) + (a_1 + 2a_2 c + 3a_3 c^2)v(\xi) = (a_1 + 2a_2 c + 3a_3 c^2)v(\xi), \\ y''(c) &= v''(c) + (2a_2 + 6a_3 c)v(\xi) = (2a_2 + 6a_3 c)v(\xi), \\ y(d) &= v(d) + (a_0 + a_1 d + a_2 d^2 + a_3 d^3)v(\xi) = (a_0 + a_1 d + a_2 d^2 + a_3 d^3)v(\xi), \\ y(\xi) &= v(\xi) + (a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3)v(\xi) = v(\xi)(a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + 1). \end{aligned}$$

We get

$$(a_0 + a_1c + a_2c^2 + a_3c^3)v(\xi) = 0,$$
$$(a_1 + 2a_2c + 3a_3c^2)v(\xi) = 0,$$
$$(2a_2 + 6a_3c)v(\xi) = 0,$$
$$(a_0 + a_1d + a_2d^2 + a_3d^3)v(\xi) = \lambda v(\xi)(a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + 1),$$

or

$$\begin{cases} a_0 + a_1c + a_2c^2 + a_3c^3 = 0, \\ a_1 + 2a_2c + 3a_3c^2 = 0, \\ 2a_2 + 6a_3c = 0, \\ (1-\lambda)a_0 + (d-\lambda\xi)a_1 + (d^2-\lambda\xi^2)a_2 + (d^3-\lambda\xi^3)a_3 = \lambda. \end{cases}$$

Solving the system, we have

$$a_0 = \frac{c^3\lambda}{(c-d)^3 - \lambda(c-\xi)^3}, a_1 = \frac{3c^2\lambda}{(d-c)^3 - \lambda(\xi^- c)^3},$$

$$a_2 = \frac{3c\lambda}{(c-d)^3 - \lambda(c-\xi)^3}, a_3 = \frac{\lambda}{(c-d)^3 - \lambda(\xi-c)^3}.$$

Therefore

$$\begin{aligned} y(x) &= v(x) + \left(\frac{c^3\lambda}{(c-d)^3 - \lambda(c-\xi)^3} + \frac{3c^2\lambda x}{(d-c)^3 - \lambda(\xi-c)^3}\right. \\ &\quad \left. + \frac{3c\lambda x^2}{(c-d)^3 - \lambda(c-\xi)^3} + \frac{\lambda x^3}{(d-c)^3 - \lambda(\xi-c)^3}\right)v(\xi) \\ &= v(x) + \left(\frac{c^3\lambda + 3c\lambda x^2}{(c-d)^3 - \lambda(c-\xi)^3} + \frac{3c^2\lambda x + \lambda x^3}{(d-c)^3 - \lambda(\xi-c)^3}\right)v(\xi) \\ &= v(x) + \lambda\left(\frac{(c-x)^3}{(c-d)^3 - \lambda(c-\xi)^3}\right)v(\xi). \end{aligned}$$

Let us prove the uniqueness. Assume that $q(x)$ is also a solution of (2.3)-(2.4), that is

$$q^{(4)}(x) + w(x) = 0, x \in [c,d],$$
$$q(c) = q'(c) = q''(c) = 0, q(d) = \lambda q(\xi).$$

Let $p(x) = q(x) - y(x), x \in [c,d]$. Due to linearity property of derivative operator, we have

$$p^{(4)}(x) = q^{(4)}(x) - y^{(4)}(x) = -w(x) + w(x) = 0, x \in [c,d].$$

Therefore $p(x) = b_0 + b_1x + b_2x^2 + b_3x^3$, where b_0, b_1, b_2 and b_3 are constants that we will specify. We have

$$p(c) = q(c) - y(c) = 0,$$
$$p'(c) = q'(c) - y'(c) = 0,$$
$$p''(c) = q''(c) - y''(c) = 0,$$
$$p(d) = q(d) - y(d) = \lambda q(\xi) - \lambda y(\xi) = \lambda(q(\xi) - y(\xi)) = \lambda p(\xi),$$

or

$$\begin{aligned}
p(c) &= b_0 + b_1 c + b_2 c^2 + b_3 c^3 = 0,\\
p'(c) &= b_1 + 2b_2 c + 3b_3 c^2 = 0,\\
p''(c) &= 2b_2 + 6b_3 c = 0,\\
p(d) &= b_0 + b_1 d + b_2 d^2 + b_3 d^3 = \lambda(b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3) = \lambda p(\xi).
\end{aligned}$$

We get the following homogeneous system

$$\begin{cases}
b_0 + b_1 c + b_2 c^2 + b_3 c^3 = 0,\\
b_1 + 2b_2 c + 3b_3 c^2 = 0,\\
2b_2 + 6b_3 c = 0,\\
(1-\lambda)b_0 + (d-\lambda\xi)b_1 + (d^2-\lambda\xi^2)b_2 + (d^3-\lambda\xi^3)b_3 = 0.
\end{cases}$$

with determinant

$$\begin{vmatrix}
1 & c & c^2 & c^3\\
0 & 1 & 2c & 3c^2\\
0 & 0 & 2 & 6c\\
1-\lambda & d-\lambda\xi & d^2-\lambda\xi^2 & d^3-\lambda\xi^3
\end{vmatrix} = -2[(c-d)^3 - \lambda(c-\xi)^3] \neq 0.$$

So the homogeneous system has only the trivial solution and hence $p(x) \equiv 0$, $x \in [c,d]$ or $q(x) \equiv y(x), x \in [c,d]$. The proof is done. □

3. The Green's function estimation

Proposition 3.1 *Let $B(x,s)$ be the Green's function given in Proposition 2.1. Then*

$$\int_c^d \mid B(x,s) \mid ds \leq \frac{(d-c)^4}{12}$$

for $x \in [c,d]$.

Proof

$$
\begin{aligned}
\int_c^d \mid B(x,s) \mid ds &= \int_c^x \mid B(x,s) \mid ds + \int_x^d \mid B(x,s) \mid ds \\
&= \int_c^x \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} - \frac{(x-s)^3}{6} \right| ds + \int_x^d \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} \right| ds \\
&\leq \int_c^x \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} + \frac{(x-s)^3}{6} \right| ds + \int_x^d \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} \right| ds \\
&\leq \int_c^x \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} \right| ds + \int_c^x \left| \frac{(x-s)^3}{6} \right| ds + \int_x^d \left| \frac{(c-x)^3(d-s)^3}{6(c-d)^3} \right| ds \\
&= \frac{1}{6}\left| \frac{(c-x)^3}{(c-d)^3} \right| \int_c^x |(d-s)^3| ds + \frac{1}{6}\int_c^x |(x-s)^3| ds + \frac{1}{6}\left| \frac{(c-x)^3}{(c-d)^3} \right| \int_x^d |(d-s)^3| ds \\
&\leq \frac{1}{6}\left[\frac{(c-d)^4}{4} \right] + \frac{1}{6}\left[\frac{(c-d)^4}{4} \right] \\
&= \frac{(d-c)^4}{12}.
\end{aligned}
$$

□

Proposition 3.2 *The Green's function $G(x,s)$ given in (2.6) satisfies the following inequality*

$$
\int_c^d \mid G(x,s) \mid ds \leq \frac{(d-c)^4}{12} + \frac{|\lambda|(d-c)^7}{12 \mid (c-d)^3 - \lambda(c-\xi)^3 \mid}
$$

for $x \in [c,d]$.

Proof

$$
\begin{aligned}
\int_c^d \mid G(x,s) \mid ds &= \int_c^d \left| B(x,s) + \frac{\lambda(c-x)^3}{(c-d)^3 - \lambda(c-\xi)^3} B(\xi,s) \right| ds \\
&\leq \int_c^d \mid B(x,s) \mid ds + \left| \frac{\lambda(c-x)^3}{(c-d)^3 - \lambda(c-\xi)^3} \right| \int_c^d \mid B(\xi,s) \mid ds \\
&\leq \frac{(d-c)^4}{12} + \frac{\mid \lambda(c-x)^3 \mid}{\mid (c-d)^3 - \lambda(c-\xi)^3 \mid} \frac{(d-c)^4}{12} \\
&\leq \frac{(d-c)^4}{12} + \frac{\mid \lambda \mid |(d-c)^3|}{|(c-d)^3 - \lambda(c-\xi)^3|} \frac{(d-c)^4}{12} \\
&= \frac{(d-c)^4}{12} + \frac{\mid \lambda \mid (d-c)^3}{|(c-d)^3 - \lambda(c-\xi)^3|} \frac{(d-c)^4}{12} \\
&= \frac{(d-c)^4}{12} + \frac{\mid \lambda \mid (d-c)^7}{12|(c-d)^3 - \lambda(c-\xi)^3|}
\end{aligned}
$$

□

4. Existence of a unique solution

Theorem 4.1 *Suppose that $g : [c,d] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a uniform Lipschitz condition with respect to y on $[c,d] \times \mathbb{R}$, namely there is a constant L such that, for every $(x, y_1), (x, y_2) \in [c,d] \times \mathbb{R}$,*

$$| g(x, y_1) - g(x, y_2) | \leq L | y_1 - y_2 | .$$

If $(c-d)^3 \neq \lambda(c-\xi)^3, (c \neq \xi)$ and $(d-c)$ is so small that

$$\frac{(d-c)^4}{12} + \frac{| \lambda | (d-c)^7}{12|(c-d)^3 - \lambda(c-\xi)^3|} < \frac{1}{L}, \tag{4.1}$$

then there exists a unique solution of (1.1)–(1.2).

Proof Let Y be the Banach space of continuous functions on $[c,d]$ with maximum norm

$$\|y\| = max\{| y(x) | : c \leq x \leq d\}.$$

Note that $y(x)$ is a solution of (1.1)–(1.2) if and only if it is a solution of (2.3)–(2.4) with $w(x) = g(x, y(x))$. But (2.3)–(2.4) has a unique solution

$$y(x) = \int_c^d G(x,s) g(s, y(s)) ds,$$

where $G(x,s)$ is defined in Proposition 2.2. Define the operator $\Upsilon : Y \to Y$ by

$$\Upsilon y(x) = \int_c^d G(x,s) g(s, y(s)) ds,$$

for $x \in [c,d]$.

We will apply Banach fixed point theorem to show the operator Υ has a unique fixed point. Let $p, q \in Y$. Then

$$\begin{aligned}
| \Upsilon p(x) - \Upsilon q(x) | &= \left| \int_c^d G(x,s)(g(s,p(s)) - g(s,q(s))) ds \right| \\
&\leq \int_c^d | G(x,s) | . | g(s,p(s)) - g(s,q(s)) | ds \\
&\leq \int_c^d | G(x,s) | L | p(s) - q(s) | ds \leq L \int_c^d | G(x,s) | \|p - q\| ds \\
&\leq L \left[\frac{(d-c)^4}{12} + \frac{| \lambda | (d-c)^7}{12|(c-d)^3 - \lambda(c-\xi)^3|} \right] \|p - q\|, \text{for } x \in [c,d],
\end{aligned}$$

where we have used Proposition 3.2. It follows that

$$\|\Upsilon p - \Upsilon q\| \leq \beta \|p - q\|,$$

where

$$\beta = L\left[\frac{(d-c)^4}{12} + \frac{|\,\lambda\,|\,(d-c)^7}{12|(c-d)^3 - \lambda(c-\xi)^3|}\right].$$

By (4.1), $\beta < 1$ and we deduce that Υ is a contraction mapping on Y, and by the Banach contraction mapping theorem we get the asked result. □

Example 4.2 In this part we give an example to illustrate the usefulness of our main results. Let us consider the following boundary value problem

$$y^{(4)} - \sin y + 1 - y = 0, y(0) = y'(0) = y''(0) = 0, y(1) = \frac{7}{10}y\left(\frac{1}{\sqrt{3}}\right). \tag{4.2}$$

We have $g(x,y) = -\sin y + 1 - y \quad (g(x,0) = 1 \neq 0)$ and

$$\max_{0\le x\le 1}\left|\frac{\partial g}{\partial y}\right| = \max_{0\le x\le 1}|-\cos y - 1| \le \max_{0\le x\le 1}|\cos y| + 1 \le L = 2.$$

So, g is Lipschitz with respect to y on $[c,d]\times\mathbb{R}$, with Lipschitz constant $L = 2$.
Since $(c-d)^3 = -1 \neq \frac{-7}{30\sqrt{3}} = \lambda(c-\xi)^3$ and

$$\frac{(d-c)^4}{12} + \frac{|\,\lambda\,|\,(d-c)^7}{12|(c-d)^3 - \lambda(c-\xi)^3|} = \frac{1}{12} + \frac{21}{360 - 28\sqrt{3}} \cong 0.150749 < \frac{1}{L} = 0.5.$$

Now an application of Theorem 4.1 proves that the problem (4.2) has a unique solution. The graph of solution $y(x)$ is displayed in Figure.

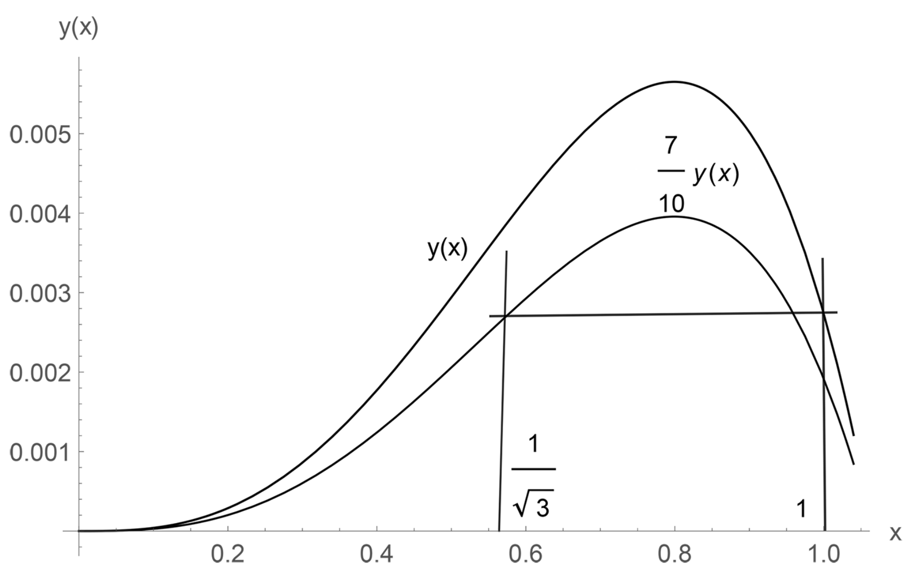

Figure. Solution of the problem (4.2).

5. Conclusion

In conclusion, we have given some sufficient conditions which show the existence and uniqueness of solutions for a nonlocal boundary value problem. An example is confirmed if the derived results can be valid.

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