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On the autocentralizer subgroups of finite p–groups

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Abstract: Let G be a finite group and $\mathrm{Aut}(G)$ be the group of automorphisms of G. Then, the autocentralizer of an automorphism $\alpha \in \mathrm{Aut}(G)$ in G is defined as $C_G(\alpha) = \{g \in G | \alpha(g) = g\}$. Let $Acent(G) = \{C_G(\alpha) | \alpha \in \mathrm{Aut}(G)\}$. If $|Acent(G)| = n$, then G is an n–autocentralizer group. In this paper, we classify all n–autocentralizer abelian groups for $n = 6$, 7 and 8. We also obtain a lower bound on the number of autocentralizer subgroups for p–groups, where p is a prime number. We show that if $p \neq 2$, there is no n–autocentralizer p–group for $n = 6, 7$. Moreover, if $p = 2$, then there is no 6–autocentralizer p–group.

Key words: Automorphism, centralizer, finite p–group, inner automorphism

1. Introduction

In this paper p denotes a prime number. We denote $\Phi(G)$, $G^{'}$, $Z(G)$, $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$, as a Frattini subgroup, commutator subgroup, the centre, the full automorphism group and the set of all inner automorphisms of G, respectively. Let G be a finite group. If $\alpha \in \mathrm{Aut}(G)$, then the autocentralizer of α in G is defined as follows:

$$C_G(\alpha) = \{g \in G \mid \alpha(g) = g\},$$

which is a subgroup of G.

In particular if $\alpha \in \mathrm{Inn}(G)$, then $\alpha = I_x$, for some $x \in G$, such that $I_x(y) = x^{-1}yx$, for all $y \in G$. Hence, $C_G(I_x)$ is the centralizer of x in G and denoted by $C_G(x)$. For a finite group G, let $Cent(G) = \{C_G(x) \mid x \in G\}$. In [3] Belcastra and Sherman proved that there is no n-centralizer group for $n = 2, 3$ and G is 4–centralizer group if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In addition they showed that if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $|Cent(G)| = p + 2$. Ashrafi in [1] proved that if G is a nonabelian p–group, then $|Cent(G)| \geq p + 2$, with equality if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now for a finite group G, let $Acent(G)$ be the set of autocentralizers of G, that is

$$Acent(G) = \{C_G(\alpha) \mid \alpha \in \mathrm{Aut}(G)\}.$$

The group G is called n–autocentralizer, if $|Acent(G)| = n$. It is obvious that G is 1–autocentralizer group if and only if G is a trivial group or $\mathbb{Z}_2$. Nasrabadi and Gholamian [6] showed the new results about the autocen-

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tralizers of finite groups. They showed that for any natural number n, there exists a finite $n-$autocentralizer group. In addition, they determined the structure of finite $n-$autocentralizer groups for $n \leq 5$. Furthermore, they concluded that if G is a finite nonabelian group, then $|Acent(G)| \geq 5$.
All aforementioned results motivate us to further consider finding the bounds for the number of autocentralizer subgroups of finite non-abelian $p-$groups. This paper consists of three sections. In Section 2, we characterize the abelian groups G with $|Acent(G)|$ =6, 7 and 8. In Section 3, we show that if G is a finite non-abelian $p-$group and not isomorphic to D_8, Q_8 and $\langle x, y \mid x^4 = y^4 = 1, yxy^{-1} = x^3 \rangle$, and $|Cent(G)| = p + 2$, then $|Acent(G)| \geq |Cent(G)|+3$. We conclude that there exists no finite nonabelian $p-$group G with $|Acent(G)| = 6$. Additionally, if p is an odd prime number, no finite nonabelian $p-$group G with $|Acent(G)| = 7$ exists. Finally, we investigate the relation between the order of G and the number of distinct autocentralizers of G. We seek the relationship between the number of distinct centralizers and the number of distinct autocentralizers of G. To do so, we directly compute the number of distinct autocentralizer subgroups of dihedral groups with small order.

Now in order to prove our main result, we need the following results.

Lemma 1.1 *[6, Lemma 2.1]*

1) Let H and K be two finite groups. Then

$$|Acent(H)| \times |Acent(K)| \leq |Acent(H \times K)|.$$

2) Let H and K be two finite groups such that $(|H|, |K|) = 1$. Then

$$|Acent(H)| \times |Acent(K)| = |Acent(H \times K)|.$$

Proposition 1.2 *[6, Proposition 2.2] Let p be a prime and G be a cyclic group of order p^n. Then*

$$|Acent(G)| = \begin{cases} n & p = 2 \\ n+1 & p \neq 2 \end{cases}$$

Lemma 1.3 *[6, Lemma 2.4] Let p be a prime and G be a cyclic group of order p. Then*

$$|Acent(G \times G)| = p + 3.$$

Remark 1.4 *[6, Remark 2.5] If G is a finite abelian group such that it has at least two direct summands of p, where p is a prime number, then it is obvious that*

$$|Acent(G \times G)| \geq p + 3.$$

2. Preliminary results

We utilize a result that is originally obtained by Nasrabadi and Gholamian [5] on the automorphism of G, where $G = \sum_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ with $n_1 > n_2 > ... > n_k$. Indeed an automorphism of $G = \sum_{i=1}^{k} \mathbb{Z}_{2^{n_i}}$ is completely determined by its action on this generating set of G. Here, we use this result to prove the following proposition.

Proposition 2.1 *Let $n > 1$ be a natural number, then*

$$|Acent(\mathbb{Z}_{2^n} \times \mathbb{Z}_2)| = 2n + 1.$$

Proof Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_2$, $(a, b) \in G$ and $\alpha \in \mathrm{Aut}(G)$. Using [5] we have

$$\alpha((a, b)) = (m_{11}a + 2^{n-1}m_{21}b, m_{12}a + m_{22}b),$$

where $m_{11}, m_{12}, m_{21}, m_{22} \in \mathbb{Z}$, m_{11} and m_{22} are odd numbers. We have one of the following cases:

1) $\alpha((a, b)) = (m_{11}a, b)$. If $m_{11} = 1$, then it is obvious that, $C_G(\alpha) = G$. Suppose that $m_{11} > 1$. Then $m_{11} = 2^t q + 1$ where $1 \leq t \leq n-1$ and q is an odd number. Therefore,

$$\begin{aligned}
C_G(\alpha_t) &= \{(a, b) \in G | \alpha((a, b)) = (a, b)\} \\
&= \{(a, b) \in G | (am_{11}, b) = (a, b)\} \\
&= \{(a, b) \in G | (m_{11} - 1)a \overset{2^n}{\equiv} 0\} \\
&= \{(a, b) \in G | a \overset{2^{n-t}}{\equiv} 0\} \\
&= \langle 2^{n-t} \rangle \times \mathbb{Z}_2 = \mathbb{Z}_{2^t} \times \mathbb{Z}_2.
\end{aligned}$$

2) $\alpha((a, b)) = (m_{11}a, a + b)$. If $m_{11} = 1$, then $C_G(\alpha) = \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2$. Let $m_{11} > 1$. So $m_{11} = 2^t q + 1$ where $1 \leq t \leq n-1$ and q is an odd number, thus

$$\begin{aligned}
C_G(\alpha_t) &= \{(a, b) \in G | \alpha((a, b)) = (a, b)\} \\
&= \{(a, b) \in G | (am_{11}, a + b) = (a, b)\} \\
&= \{(a, b) \in G | (m_{11} - 1)a \overset{2^n}{\equiv} 0,\ a \overset{2}{\equiv} 0\} \\
&= \{(a, b) \in G | a \overset{2^{n-t}}{\equiv} 0\} \\
&= \langle 2^{n-t} \rangle \times \mathbb{Z}_2 = \mathbb{Z}_{2^t} \times \mathbb{Z}_2.
\end{aligned}$$

3) $\alpha((a, b)) = (m_{11}a + 2^{n-1}b, b)$. If $m_{11} = 1$, Then $C_G(\alpha) = \mathbb{Z}_{2^n}$. Let $m_{11} > 1$. So $m_{11} = 2^t q + 1$ where $1 \leq t \leq n-1$ and q is an odd number, hence

$$\begin{aligned}
C_G(\alpha_t) &= \{(a, b) \in G | \alpha((a, b)) = (a, b)\} \\
&= \{(a, b) \in G | (am_{11} + 2^{n-1}b, b) = (a, b)\} \\
&= \{(a, b) \in G | (m_{11} - 1)a \overset{2^n}{\equiv} 2^{n-1}b\} \\
&= \{(a, b) \in G | a \overset{2^{n-t}}{\equiv} 2^{n-t-1}b\}.
\end{aligned}$$

4) $\alpha((a, b)) = (m_{11}a + 2^{n-1}b, a + b)$. If $m_{11} = 1$, then we have easily, $C_G(\alpha) = \mathbb{Z}_{2^{n-1}}$. Let $m_{11} > 1$. So

$m_{11} = 2^t q + 1$ where $1 \leq t \leq n-1$ and q is an odd number, therefore

$$\begin{aligned} C_G(\alpha_t) &= \{(a,b) \in G | \alpha((a,b)) = (a,b)\} \\ &= \{(a,b) \in G | (am_{11} + 2^{n-1}b, a+b) = (a,b)\} \\ &= \{(a,b) \in G | (m_{11}-1)a \overset{2^n}{\equiv} 2^{n-1}b,\ a \overset{2}{\equiv} 0\} \\ &= \{(a,b) \in G | a \overset{2^{n-t}}{\equiv} 2^{n-t-1}b,\ a \overset{2}{\equiv} 0\}. \end{aligned}$$

Now in this case if $t = n-1$, then $C_G(\alpha_{n-1}) = \mathbb{Z}_{2^{n-1}}$, and if $1 \leq t < n-1$, then we have

$$C_G(\alpha_{n-1}) = \{(a,b) \in G | a \overset{2^{n-t}}{\equiv} 2^{n-t-1}b\}.$$

Finally, by using the above results, one can see that

$$Acent(G) = \{G, \mathbb{Z}_{2^n}, \mathbb{Z}_{2^{n-1}}, \mathbb{Z}_{2^1} \times \mathbb{Z}_2, ..., \mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_2, \{(a,b) \in G | a \overset{2^{n-1}}{\equiv} 2^{n-1-1}b\},$$
$$..., \{(a,b) \in G | a \overset{2^{n-(n-1)}}{\equiv} 2^{n-(n-1)-1}b\}\},$$

this completes the proof.

□

Now we can determine finite abelian groups where $|Acent(G)| = 6, 7, 8$.

Proposition 2.2 *i) G is a 6-autoentralizer abelian group if and only if*

$$G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{2^6}, \mathbb{Z}_{p^5}, \mathbb{Z}_{2p^5}, \mathbb{Z}_{8p}, \mathbb{Z}_{4p^2}, \mathbb{Z}_{pq^2}, \mathbb{Z}_{2pq^2},$$

where p and q are distinct odd primes.

ii) G is a 7-autoentralizer abelian group if and only if

$$G \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2, \mathbb{Z}_{2^7}, \mathbb{Z}_{p^6}, \mathbb{Z}_{2p^6}, \mathbb{Z}_{p_i p_j^2},$$

where p is odd prime.

iii) G is an 8-autoentralizer abelian group if and only if

$$G \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_{2^8}, \mathbb{Z}_{p^7}, \mathbb{Z}_{2p^7}, \mathbb{Z}_{4p^3}, \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j^3},\ \mathbb{Z}_{2p_i} \times \mathbb{Z}_{p_j^3},$$
$$\mathbb{Z}_4 \times \mathbb{Z}_p \times \mathbb{Z}_{p_i}, \mathbb{Z}_p \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}, \mathbb{Z}_{2p} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j},$$

where p, p_i and p_j are distinct odd primes.

Proof

i) If $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{2^6}, \mathbb{Z}_{p^5}, \mathbb{Z}_{2p^5}, \mathbb{Z}_{8p}, \mathbb{Z}_{4p^2}, \mathbb{Z}_{pq^2}, \mathbb{Z}_{2pq^2}$, using Lemma1.1, Proposition 1.2, Lemma 1.3 and Remark 1.4, G is 6−autocentralizer group. Conversely, if G is 6−autocentralizer group, by Lemma 1.1, Proposition 1.2, Lemma 1.3 and Remark 1.4, $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{2^6}, \mathbb{Z}_{p^5}, \mathbb{Z}_{2p^5}, \mathbb{Z}_{8p}, \mathbb{Z}_{4p^2}$, $\mathbb{Z}_{pq^2}$, $\mathbb{Z}_{2pq^2}$.

ii) If $G \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2, \mathbb{Z}_{2^7}, \mathbb{Z}_{p^6}, \mathbb{Z}_{2p^6}$, using Lemma 1.1, Proposition 1.2, Lemma 1.3, Remark 1.4 and Proposition 2.1 G is 7–autocentralizer group. Conversely, if G is 7–autocentralizer group, by Lemma 1.1, Proposition 1.2, Lemma 1.3, Remark 1.4 and Proposition 2.1, $G \cong \mathbb{Z}_{2^3} \times \mathbb{Z}_2, \mathbb{Z}_{2^7}, \mathbb{Z}_{p^6}, \mathbb{Z}_{2p^6}$.

iii) If $G \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_{2^8}, \mathbb{Z}_{p^7}, \mathbb{Z}_{2p^7}, \mathbb{Z}_{4p^3}, \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j^3}, \mathbb{Z}_{2p_i} \times \mathbb{Z}_{p_j^3}, \mathbb{Z}_4 \times \mathbb{Z}_p \times \mathbb{Z}_{p_i}, \mathbb{Z}_p \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}, \mathbb{Z}_{2p} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}$, using Lemma 1.1, Proposition 1.2, Lemma 1.3, Remark 1.4 and Proposition 2.1, G is 8–autocentralizer group. Conversely, if G is 8–autocentralizer group, by Lemma 1.1, Proposition 1.2, Lemma1.3, Remark 1.4 and Proposition 2.1, $G \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_2, \mathbb{Z}_5 \times \mathbb{Z}_5, \mathbb{Z}_{2^8}, \mathbb{Z}_{p^7}, \mathbb{Z}_{2p^7}, \mathbb{Z}_{4p^3}, \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j^3}, \mathbb{Z}_{2p_i} \times \mathbb{Z}_{p_j^3}, \mathbb{Z}_4 \times \mathbb{Z}_p \times \mathbb{Z}_{p_i}, \mathbb{Z}_p \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}, \mathbb{Z}_{2p} \times \mathbb{Z}_{p_i} \times \mathbb{Z}_{p_j}$.

□

3. Main results

In this section we study the finite nonabelian p–groups, G, with $|Cent(G)| = p+2$, and find bounds of the $|Acent(G)|$. In [8] two techniques were provided to find the automorphisms of G. We use these techniques, where G is a 2–generated p–group of nilpotency class two.

Theorem 3.1 *Let $G \neq \langle x, y | x^4 = y^4 = 1,\ yxy^{-1} = x^3 \rangle$ be a finite 2–group such that $Z(G)$ is not cyclic and $|Cent(G)| = 4$. Then $|Acent(G)| \geq |Cent(G)| + 3$.*

Proof By [3, Theorem 3], if $|Cent(G)| = 4$, then $|G/Z(G)| \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, thus $G/Z(G) = \langle x, y, Z(G) \rangle$. Since $Z(G)$ is not cyclic, then it contains a Klien 4–subgroup $\langle a, b \rangle$, for some $a, b \in G$. Hence, we can define the automorphism α given by $\alpha(x) = xa$, $\alpha(y) = yb$ and $\alpha(c) = c$, for all $c \in Z(G)$. So $C_G(\alpha) \notin Cent(G)$. Now we consider the following cases:

1) If $\Phi(G) < Z(G)$.
There exists a not trivial set $R = \{r_1, ..., r_t\}$, such that $R = Z(G) - \Phi(G)$, thus we have $Z(G) = \langle r_1, r_2, ..., r_t, \Phi(G) \rangle$, so we can define automorphisms β and γ of G

$$\beta : \begin{cases} x \longmapsto x \\ y \longmapsto y \\ r_1 \longmapsto r_1 h, & (h \in \Phi(G),\ |h| = 2) \\ r_i \longmapsto r_i, & (2 \leq i \leq t) \\ m \longmapsto m, & (m \in \Phi(G)) \end{cases} \qquad \gamma : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \\ r_1 \longmapsto r_1 h, & (h \in \Phi(G),\ |h| = 2) \\ r_i \longmapsto r_i, & (2 \leq i \leq t) \\ m \longmapsto m, & (m \in \Phi(G)) \end{cases}$$

It is immediate to verify that $C_G(\alpha) \neq C_G(\beta)$, $C_G(\alpha) \neq C_G(\gamma)$, $C_G(\beta) \neq C_G(\gamma)$ and $C_G(\beta), C_G(\gamma) \notin Cent(G)$. Therefore $|Acent(G)| \geq |Cent(G)| + 3$.

2) If $\Phi(G) = Z(G)$.
In this case G is a nilpotent group of 2 class such that $G = \langle x, y \rangle$ (In [4] Magidin characterized the structure of two–generator 2–groups of class 2). If $Z(G)$ is an elementary abelian group, then G is isomorphic with G_1, G_2 or G_3, such that

$$G_1 = \langle x, y | x^4 = y^4 = 1, \quad yxy^{-1} = x^3 \rangle,$$
$$G_2 = \langle x, y | x^4 = y^2 = [x, y]^2 = [x, y, x] = [x, y, y] = 1 \rangle,$$
$$G_3 = \langle x, y, c | x^4 = y^4 = c^2 = 1, \quad [x, y] = c, \quad [x, c] = [y, c] = 1 \rangle.$$

If $G \cong G_1$, then by [6, Lemma 3.2], we have $|Acent(G)| = 5$. If $G \cong G_2$, then we define

$$\alpha : \begin{cases} x \longmapsto x^3 \\ y \longmapsto y[y, x] \end{cases} \qquad \beta : \begin{cases} x \longmapsto xy \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto xy \\ y \longmapsto x^2 y \end{cases}$$

Similar to case (1), it is easy to see that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha)$, $C_G(\beta)$, $C_G(\gamma) \notin Cent(G)$. So $|Acent(G)| \geq |Cent(G)| + 3$.
Also if $G \cong G_3$, then we define

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto y \\ y \longmapsto x \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^3 \\ y \longmapsto xy \end{cases}$$

We easily see that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha)$, $C_G(\beta)$, $C_G(\gamma) \notin Cent(G)$. Thus $|Acent(G)| \geq |Cent(G)| + 3$.
Next, if $Z(G)$ is not an elementary abelian group, then $o(x)$ or $o(y)$ is at least 8. Suppose that $o(x) = 2^n \geq 8$. According to the order of y, we consider the following automorphisms:

i) $o(y) = 2$.

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^3 \\ y \longmapsto x^{2^{n-1}} y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^3 \\ y \longmapsto y \end{cases}$$

ii) $o(y) \geq 4$.

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^3 \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^3 \\ y \longmapsto y^3 \end{cases}$$

We easily check that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha)$, $C_G(\beta)$, $C_G(\gamma) \notin Cent(G)$. Therefore $|Acent(G)| \geq |Cent(G)| + 3$. □

Theorem 3.2 *Let $G \neq Q_8, D_8$ be a finite 2–group such that $Z(G)$ be cyclic and $|Cent(G)| = 4$. Then* $|Acent(G)| \geq |Cent(G)| + 3$.

Proof We know $|Cent(G)| = 4$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. So

$$G/Z(G) = \{Z(G), xZ(G), yZ(G), xyZ(G)\}.$$

Since $Z(G)$ is cyclic, suppose that $Z(G) = \langle z \rangle$, for some $z \in G$ with $|z| = 2^n$. If $n = 1$, then we have $G = D_8$ or Q_8. Applying [6, Lemma 3.3], thus in this case $|Acent(G)| = 5$. Hence, let $n > 1$. If $|x| = |y| = 2$, then we can define the automorphisms α, β and γ such that

$$\alpha : \begin{cases} x \longmapsto y \\ y \longmapsto x \\ z \longmapsto z \end{cases} \qquad \beta : \begin{cases} x \longmapsto x \\ y \longmapsto y \\ z \longmapsto z^{-1} \end{cases} \qquad \gamma : \begin{cases} x \longmapsto y \\ y \longmapsto x \\ z \longmapsto z^{-1} \end{cases}$$

It is easy to check that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha),\ C_G(\beta),\ C_G(\gamma) \notin Cent(G)$. So $|Acent(G)| \geq |Cent(G)| + 3$. If $|x| < 2^{n+1}$, by replacing x by xz^i for suitable i, we get $|x| = 2$. Similarly, if $|y| < 2^{n+1}$, then we get $|y| = 2$. So suppose $|x| = 2^{n+1}$. Hence, G has a cyclic subgroup of order 2^{n+1}. We know 2–groups of order $2^{n+2} (n \geq 2)$ with a cyclic subgroup of index two with $|G : Z(G)| = 4$ is the modular group with presentation $G = \langle x, y | x^{2^{n+1}} = y^2 = 1,\ \ x^y = x^{2^n+1} \rangle$ ([7, Theorem 5.3.4]). For this group G, we can define the following automorphisms:

$$\alpha : \begin{cases} x \longmapsto xy \\ y \longmapsto yx^{2^n} \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{-1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^{2^{n-1}+1}y \\ y \longmapsto yx^{2^n} \end{cases}$$

It is obvious that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha),\ C_G(\beta),\ C_G(\gamma) \notin Cent(G)$. Therefore $|Acent(G)| \geq |Cent(G)| + 3$. □

Corollary 3.3 *If $G \neq Q_8, D_8, \langle x, y | x^4 = y^4 = 1,\ yxy^{-1} = x^3 \rangle$ is a finite 2–group such that $|Cent(G)| = 4$, then $|Acent(G)| \geq 7$.*

Proposition 3.4 *Let G be a finite non-2–group where $|Cent(G)| = 4$, then $|Acent(G)| \geq 10$.*

Proof Since $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and G is not a finite 2–group, there is a sylow p–group H of G, for some odd prime p, such that $H \leq Z(G)$. Hence H is abelian and normal in G. By Schur Zassenhause Theorem, there is a p'–subgroup K of G such that $G = HK$. As $H \leq Z(G)$, we also have that K is normal in G. Thus, $G \cong H \times K$. Since G is nilpotent of class 2, K is nilpotent of class 2. So, by Lemma 1.1 and Proposition 1.2

$$|Acent(G)| = |Acent(H)| \times |Acent(K)| \geq 2 \times 5 = 10.$$

□

Example 3.5 *Suppose $G = \langle x, y \mid\ x^2 = y^{12} = 1,\ xyx^{-1} = y^{-5} \rangle$. It is easy to see that G is a group of nilpotency class two and $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, therefore $|Cent(G)| = 4$. By Proposition 3.4 we have $|Acent(G)| \geq 10$, Since $G \cong C_3 \times D_8$, applying Lemma 1.1(2), we have $|Acent(G)| = 10$.*

Theorem 3.6 *Let p be an odd prime number and G is a finite p–group such that $|Cent(G)| = p + 2$. Then $|Acent(G)| \geq |Cent(G)| + 3$.*

Proof By [1] if $|Cent(G)| = p + 2$, then $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence

$$G/Z(G) = \{x^i y^j Z(G) \mid 0 \leq i, j \leq p - 1\}.$$

If $Z(G)$ is not cyclic, then it contains an abelian p-subgroup $\langle a, b \rangle$ such that is isomorphic with $\mathbb{Z}_p \times \mathbb{Z}_p$. Hence, we can define the automorphism α of G given by $\alpha(x) = xa, \alpha(y) = yb, \alpha(c) = c$, for all $c \in Z(G)$. Now it is obvious that $C_G(\alpha) \notin Cent(G)$. If $Z(G)$ is an elementary abelian group, then we consider the following cases

1) $\Phi(G) < Z(G)$.
Similar to the proof of Theorem 3.1, we define the automorphisms β and γ of G such that

$$\beta : \begin{cases} x \longmapsto x \\ y \longmapsto y \\ r_1 \longmapsto r_1 h, & (h \in \Phi(G),\ |h| = p) \\ r_i \longmapsto r_i, & (2 \leq i \leq t) \\ m \longmapsto m, & (m \in \Phi(G)) \end{cases} \qquad \gamma : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \\ r_1 \longmapsto r_1 h, & (h \in \Phi(G),\ |h| = p) \\ r_i \longmapsto r_i, & (2 \leq i \leq t) \\ m \longmapsto m, & (m \in \Phi(G)) \end{cases}$$

Clearly, $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\beta), C_G(\gamma) \notin Cent(G)$. Thus $|Acent(G)| \geq |Cent(G)| + 3$.

2) $\Phi(G) = Z(G)$.
In this case G is a nilpotent group of 2 class such that $G = \langle x, y \rangle$ (In [2] Bacon and Kappe gave a classification of two–generator p–groups of nilpotency class 2 (p odd)). Now if $Z(G)$ is an elementary abelian group, then G is isomorphic with $G_1,\ G_2$ or G_3 such that

$$G_1 = \langle x, y | x^{p^2} = y^{p^2} = 1, \quad y^{-1}xy = x^{p+1} \rangle,$$
$$G_2 = \langle x, y, c | x^{p^2} = y^{p^2} = c^p = 1, \quad [x, y] = c, [x, c] = [y, c] = 1 \rangle,$$
$$G_3 = \langle x, y | x^{p^2} = y^p = c^p = 1, \quad [x, y] = c, \quad [x, c] = [y, c] = 1 \rangle.$$

then we define the automorphisms $\alpha,\ \beta$ and γ such that

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{-1} \\ y \longmapsto y[x, y] \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^2 \\ y \longmapsto y \end{cases}$$

One can easily check that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$, and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$. Therefore $|Acent(G)| \geq |Cent(G)| + 3$.
Now suppose $Z(G) = \langle z \rangle$ for some $z \in G$ with $|z| = p^n$. If $n = 1$, then G is isomorphic with G_1, G_2 such that

$$G_1 = \langle x, y | x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1} \rangle,$$
$$G_2 = \langle x, y, c | x^p = y^p = z^p = 1,\ [x, y] = z,\ [x, z] = [y, z] = 1 \rangle.$$

If $G \cong G_1$, we can define the automorphisms $\alpha,\ \beta$ and γ such that

$$\alpha : \begin{cases} x \longmapsto xy \\ y \longmapsto x^p y \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{-1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^{p+2} \\ y \longmapsto x^p y \end{cases}$$

It is clear to see that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$, and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$. Therefore $|Acent(G)| \geq |Cent(G)| + 3$.
Similarly, if $G \cong G_2$, then we can define the automorphisms $\alpha,\ \beta$ and γ such that

$$\alpha : \begin{cases} x \longmapsto xy \\ y \longmapsto y[x, y] \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{-1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x \\ y \longmapsto y^{-1} \end{cases}$$

We can easily see that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$, and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$. So $|Acent(G)| \geq |Cent(G)| + 3$.

Let $n > 1$. If $|x| = |y| = p$, then we can define the automorphisms α, β and γ of G such that

$$\alpha : \begin{cases} x \longmapsto y \\ y \longmapsto x \\ z \longmapsto z \end{cases} \qquad \beta : \begin{cases} x \longmapsto y \\ y \longmapsto x \\ z \longmapsto z^{-1} \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x \\ y \longmapsto y \\ z \longmapsto z^{-1} \end{cases}$$

It is clear that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$, therefore $|Acent(G)| \geq |Cent(G)| + 3$. If $|x| < p^{n+1}$, then by replacing a by xz^i for suitable i, we get $|x| = p$. Similarly, if $|y| < p^{n+1}$, then we get $|y| = p$. So suppose $|x| = p^{n+1}$. Hence, G has a cyclic subgroup of order p^{n+1}. We know a nonabelian p-group with a cyclic subgroup of index p, is the modular group with presentation ([7, Theorem 5.3.4]).

$$G \cong \langle x, y | x^{p^{n+1}} = y^p = 1, \ x^y = x^{1+p^n} \rangle.$$

For this group G, we can define the automorphisms α, β and γ of G such that

$$\alpha : \begin{cases} x \longmapsto xy \\ y \longmapsto x^{p^n} y \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{-1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^{p+1} y \\ y \longmapsto x^{p^n} y \end{cases}$$

It is obvious that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$, thus $|Acent(G)| \geq |Cent(G)| + 3$.

Now if $Z(G)$ is not an elementary abelian group, then $o(x)$ or $o(y)$ is at least p^3, if $o(x) = p^n \geq p^3$ and $o(y) = p$ or $o(y) = p^2$, then there exist α, β and $\gamma \in \mathrm{Aut}(G)$ such that

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{p+1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x^{p+1} \\ y \longmapsto x^{p^{n-1}} y \end{cases}$$

It is clear that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$. Hence $|Acent(G)| \geq |Cent(G)| + 3$.

Now if $o(y) > p^2$, we define the automorphisms α, β and γ of G such that

$$\alpha : \begin{cases} x \longmapsto xa \\ y \longmapsto yb \end{cases} \qquad \beta : \begin{cases} x \longmapsto x^{p+1} \\ y \longmapsto y \end{cases} \qquad \gamma : \begin{cases} x \longmapsto x \\ y \longmapsto y^{p+1} \end{cases}$$

Easily, we see that $C_G(\alpha) \neq C_G(\beta), C_G(\alpha) \neq C_G(\gamma), C_G(\beta) \neq C_G(\gamma)$ and $C_G(\alpha), C_G(\beta), C_G(\gamma) \notin Cent(G)$, thus $|Acent(G)| \geq |Cent(G)| + 3$.

□

Immediate from Theorems 3.1, 3.2 and 3.6, we get the following corollary.

Corollary 3.7 *If G is a finite nonabelian p–group, with $|Cent(G)| = p+2$, then G is not a 6–autocentralizer p–group. Also, no finite nonabelian 7–autocentralizer p–group exists, where p is an odd prime number.*

Remark 3.8 *It might be mistaken that if G and H are finite groups while $|G| \leq |H|$, then $|Acent(G)| \leq |Acent(H)|$. But this is not necessarily true. As shown in Table, we compute the number of distinct autoentralizers of some dihedral groups. For example, $|D_{18}| < |D_{20}|$, but $|Acent(D_{18})| > |Acent(D_{20})|$. Moreover, if $|Cent(G)| \leq |Cent(H)|$ it is not necessarily true that $|Acent(G)| \leq |Acent(H)|$. For example $|Cent(D_{16})| \leq |Cent(D_{14})|$, but $|Acent(D_{14})| \leq |Acent(D_{16})|$. Also, we can find groups G and H such that $|Acent(G)| = |Acent(H)|$, but $G \ncong H$. For example $|Acent(S_3)| = |Acent(D_8)|$, but $S_3 \ncong D_8$.*

Table . Dihedral groups D_{2n} when $n \leq 12$.

$G = D_{2n}$	$D_6 \cong S_3$	D_8	D_{10}	D_{12}	D_{14}	D_{16}	D_{18}	D_{20}	D_{22}	D_{24}
$\|Cent(G)\|$	5	4	7	5	9	6	11	7	13	8
$\|Acent(G)\|$	5	5	7	6	9	10	15	8	13	16

4. Conclusion

We conclude our paper with a question about the number of distinct autocentralizer of dihedral groups. Is it true that if D_{2p} is a dihedral group, where p is an odd prime number, then $|Cent(D_{2p})| = |Acent(D_{2p})|$? This question could be of potential usefulness for the readers to carry out further research.

Proof Suppose that $G = \langle a, b \mid a^p = b^2 = 1,\ a^b = a^{-1} \rangle$, where p is an odd prime number. The elements of D_{2p} are then $1, a, ..., a^{p-1}, b, ab, ..., a^{p-1}b$. By [1, Lemma 2.2], we have $C_G(a^i) = \langle a \rangle$ and $C_G(a^i b) = \langle a^i b \rangle$, for $1 \leq i \leq p-1$. On the other hand, $C_G(b) = \langle b \rangle$. Therefore $|Cent(G)| = p+2$. We show that $|Acent(G)| = |Cent(G)| = p+2$. Any automorphism of G is a map defined by $\alpha_{k,l}(a) = a^k$ and $\alpha_{k,l}(b) = ba^l$, where $gcd(k,p) = 1,\ 1 \leq k \leq p-1$ and $0 \leq l \leq p-1$. There are four cases:

i) If $k = 1,\ l = 0$. Then $\alpha_{k,l} = id$ and $C_G(\alpha_{k,l}) = G$.

ii) If $k = 1,\ l \neq 0$. Then $C_G(\alpha_{k,l}) = \langle a \rangle$.

iii) If $k \neq 1,\ l = 0$. Since p is an odd prime number, one can check easily that $\alpha_{k,l}(a^i) \neq a^i$ and $\alpha_{k,l}(a^i b) \neq a^i b$, for every $1 \leq i \leq p-1$. Then $C_G(\alpha_{k,l}) = \langle b \rangle$.

iv) If $k \neq 1,\ l \neq 0$. Similarly case (iii), we have $\alpha_{k,l}(a^i) \neq a^i$. Clearly $\alpha_{k,l}(b) \neq b$. Then $a^i b \in C_G(\alpha_{k,l})$ if and only if $\alpha_{k,l}(a^i b) = a^{ik} b a^l = a^i b$. It implies that $a^{ik-i} = a^{-l}$. This is true if and only if $i(k-1) \overset{p}{\equiv} -l$. Since $k \neq 1$ and $l \neq 0$, we have $i \overset{p}{\equiv} -l(k-1)^{-1}$. Thus, for fixed k and l, there is a unique i such that $C_G(\alpha_{k,l}) = \langle a^i b \rangle$. One can check that when fixing $k \neq 1$, we can obtain every $1 \leq i \leq p-1$, by changing l. Hence $\langle a^i b \rangle \in Acent(G)$, for every $1 \leq i \leq p-1$.

Therefore $Acent(G) = \{G, \langle a \rangle, \langle b \rangle, \langle a^i b \rangle\}$, for every $1 \leq i \leq p-1$, this completes the proof. □

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