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Benedicks and Donoho-Stark type theorems

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Abstract: In this paper, we prove a Benedicks type theorem and a Donoho-Stark type theorem, for the generalized Fourier transform \mathcal{F}_α associated to some differential operators that we call Flensted-Jensen operators, in various spaces such $L_\alpha^1(\mathbb{K})$, $L_\alpha^2(\mathbb{K})$ and $L_\alpha^1(\mathbb{K}) \cap L_\alpha^2(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}_+ \times \mathbb{R}$.

Key words: Generalized Fourier transform, Benedicks theorem, Donoho-Stark theorem, uncertainty principle

1. Introduction

The uncertainty principle is a characterization of a quantum mechanical system. This principle says that one cannot measure, simultaneously and as accurately as one wants, the position and momentum of a quantum particle. In harmonic analysis, the uncertainty principle can be summarized by the following sentence:

a nonzero function and its Fourier transform cannot be localized as precisely as one wishes.

We can distinguish two formulations of this principle, quantitative and qualitative. In 1927, W. Heisenberg [13] gave a physical interpretation of the quantitative uncertainty principle that he wrote in the form of the following formula called Heisenberg inequality:

$$\forall f \in L^2(\mathbb{R}), \quad \int_{\mathbb{R}} x^2 |f(x)|^2 dx \cdot \int_{\mathbb{R}} y^2 |\widehat{f}(y)|^2 dy \geq \frac{1}{4} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^2,$$

where \widehat{f} is the Fourier transform of f , defined for all $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, by

$$\widehat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixy} dx.$$

Equality cases are realized only by Gaussians of the form

$$f(x) = C e^{-ax^2}, \quad x \in \mathbb{R},$$

where C and a are constants with $a > 0$.

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By a qualitative uncertainty principle one means a result that, without giving quantitative estimates for a function f and its Fourier transform \widehat{f} , says that f and \widehat{f} cannot both be sharply localized unless $f = 0$. Several authors have published works in the context of the qualitative uncertainty principle. We can cite for example, [1–7, 12, 14–16]. For further references about uncertainty principle, we refer the reader to the book [11] and the survey [10].

In [7], Donoho and Stark studied a new version of qualitative uncertainty principle. This uncertainty principle relies on the notion of ε -concentrated, where a function f belongs to $L^2(\mathbb{R})$ called ε -concentrated on a measurable set E if

$$\|f - f|_E\|_2 \leq \varepsilon \|f\|_2.$$

Both others in [7] established that if f is ε_1 -concentrated on E and \widehat{f} is ε_2 -concentrated on F , then

$$m(E)m(F) \geq (1 - \varepsilon_1 - \varepsilon_2)^2$$

The aim of this paper is to establish Benedicks and Donoho-Stark type theorems associated to the following operators, that we call Flensted-Jensen operators,

$$\begin{cases} D &= \frac{\partial}{\partial \theta}, \\ D_\alpha &= \frac{\partial^2}{\partial y^2} + [(2\alpha + 1) \coth y + \tanh y] \frac{\partial}{\partial y} - \frac{1}{\cosh^2 y} \frac{\partial^2}{\partial \theta^2} + (\alpha + 1)^2, \end{cases}$$

where $\alpha > 0$ and $(y, \theta) \in \mathbb{K} = [0, +\infty[\times \mathbb{R}$.

This system was first considered by Flensted-Jensen in [9] for $\alpha = n - 1$, where n is a positive integer, in the frame work of simply connected semisimple Lie group. The operators D and $[D_{n-1} - n^2]$ with the identity generate the algebra $\mathbf{D}(\widetilde{G}/K)$ of left invariant differential operators on \widetilde{G}/K , where \widetilde{G} is the universal covering group of $G = \mathbf{U}(n, 1)$ and K is the subgroup $\mathbf{U}(n)$. A several works on the theory of uncertainty principle, related to the operators D and D_α were studied in [15–17].

The outline of this paper is given as follows:

Section 2 is devoted to recall some results concerning the harmonic analysis associated to the operators D and D_α . In section 3, we prove a Benedicks type theorem. In the last section we obtain a various versions of Donoho-Stark theorem.

2. Preliminaries

For $(y, \theta) \in \mathbb{K}$, the following system

$$\begin{cases} Du(y, \theta) &= i\lambda u(y, \theta), \\ D_\alpha u(y, \theta) &= -\mu^2 u(y, \theta), \quad \lambda, \mu \in \mathbb{R} \\ u(0, 0) &= 1, \quad \frac{\partial u}{\partial y}(0, \theta) = 0, \quad \theta \in \mathbb{R} \end{cases}$$

has a unique solution given by

$$\varphi_{\lambda, \mu}(y, \theta) = e^{i\lambda\theta} (\cosh y)^\lambda \varphi_\mu^{\alpha, \lambda}(y),$$

where $\varphi_\mu^{\alpha,\lambda}$ is the Jacobi function defined by

$$\varphi_\mu^{\alpha,\lambda}(y) = {}_2F_1\left(\frac{\alpha + \lambda + 1 + i\mu}{2}, \frac{\alpha + \lambda + 1 - i\mu}{2}; \alpha + 1; -\sinh^2 y\right).$$

Recall that ${}_2F_1$ is the Gaussian hypergeometric function (see [8]).

From [18], we have

$$\sup_{(y,\theta) \in \mathbb{K}} |\varphi_{\lambda,\mu}(y, \theta)| = 1, \quad (\lambda, \mu) \in \widehat{\mathbb{K}} \tag{2.1}$$

Let $\mathbb{L} = \mathbb{R} \times [0, +\infty[$ and $\widehat{\mathbb{K}} = \mathbb{L} \cup \Omega$, where

$$\Omega = \bigcup_{m \in \mathbb{N}} D_m^+ \cup D_m^-.$$

with

$$D_m^+ = \{(\alpha + 2m + 1 + \eta, i\eta) \mid \eta > 0\} \quad \text{and} \quad D_m^- = \{(-\alpha - 2m - 1 - \eta, i\eta) \mid \eta > 0\}.$$

Let $1 \leq p < +\infty$. Consider $L_\alpha^p(\mathbb{K})$, the space of measurable functions f on \mathbb{K} verifying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(y, \theta)|^p dm_\alpha(y, \theta) \right)^{\frac{1}{p}} < +\infty,$$

where

$$dm_\alpha(y, \theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1} \cosh y dy d\theta.$$

For $p = \infty$, we put

$$\|f\|_{\infty,m_\alpha} = \text{ess sup}_{(y,\theta) \in \mathbb{K}} |f(y, \theta)|.$$

The generalized Fourier transform of f associated to Flensted-Jensen operators is given by

$$\forall (\lambda, \mu) \in \widehat{\mathbb{K}}, \quad \mathcal{F}_\alpha f(\lambda, \mu) = \int_{\mathbb{K}} f(y, \theta) \varphi_{-\lambda,\mu}(y, \theta) dm_\alpha(y, \theta).$$

where $f \in L_\alpha^1(\mathbb{K})$.

Denote γ_α the positive measure, defined on $\widehat{\mathbb{K}}$ by

$$\begin{aligned} \int_{\widehat{\mathbb{K}}} g d\gamma_\alpha(\lambda, \mu) &= \frac{1}{(2\pi)^2} \int_{\mathbb{R} \times [0, +\infty[} g(\lambda, \mu) \frac{d\lambda d\mu}{|C_1(\lambda, \mu)|^2} \\ &+ \frac{1}{(2\pi)^2} \sum_{m=0}^{+\infty} \left\{ \int_0^{+\infty} g(\alpha + 2m + 1 + \eta, i\eta) C_2(\alpha + 2m + 1 + \eta, i\eta) d\eta \right. \\ &\quad \left. + \int_0^{+\infty} g(-\alpha - 2m - 1 - \eta, i\eta) C_2(-\alpha - 2m - 1 - \eta, i\eta) d\eta \right\}, \end{aligned}$$

where

$$C_1(\lambda, \mu) = \frac{2^{\alpha+1-i\mu} \Gamma(\alpha + 1) \Gamma(i\mu)}{\Gamma\left(\frac{\alpha + \lambda + 1 + i\mu}{2}\right) \Gamma\left(\frac{\alpha - \lambda + 1 + i\mu}{2}\right)}, \quad (\lambda, \mu) \in \mathbb{R} \times]0, +\infty[$$

and

$$C_2(\lambda, \mu) = -i \operatorname{Res}_{z=\mu} \left[C_1(\lambda, z) C_1(\lambda, -z) \right]^{-1}, \quad (\lambda, \mu) \in \Omega.$$

We have from [18] the following inversion formula

$$\mathcal{F}_\alpha^{-1} g(y, \theta) = \int_{\widehat{\mathbb{K}}} g(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu), \tag{2.2}$$

For $1 \leq p < +\infty$, denote $L_\alpha^p(\widehat{\mathbb{K}})$ the space of measurable functions $g : \widehat{\mathbb{K}} \mapsto \mathbb{C}$ verifying

$$\|g\|_{p, \gamma_\alpha} = \left(\int_{\widehat{\mathbb{K}}} |g(\lambda, \mu)|^p d\gamma_\alpha(\lambda, \mu) \right)^{\frac{1}{p}} < +\infty.$$

For $p = \infty$, we denote

$$\|g\|_{\infty, \gamma_\alpha} = \operatorname{ess\,sup}_{(\lambda, \mu) \in \widehat{\mathbb{K}}} |g(\lambda, \mu)|$$

The generalized Fourier transform \mathcal{F}_α extended to an isometry between $L_\alpha^2(\mathbb{K})$ and $L_\alpha^2(\widehat{\mathbb{K}})$. In particular, for $f \in L_\alpha^2(\widehat{\mathbb{K}})$, we have the Plancherel formula

$$\|\mathcal{F}_\alpha f\|_{2, \gamma_\alpha} = \|f\|_{2, m_\alpha}. \tag{2.3}$$

For $f \in L_\alpha^1(\mathbb{K})$, we have

$$\|\mathcal{F}_\alpha f\|_{\infty, \gamma_\alpha} \leq \|f\|_{1, m_\alpha}. \tag{2.4}$$

In the following sections, we consider $E \subset \mathbb{K}$ and $F \subset \widehat{\mathbb{K}}$ tow measurable subsets. For a function $f \in L_\alpha^2(\mathbb{K})$, we denote by

T_E the time-limiting operator

$$T_E f = \chi_E f,$$

P_F the frequency-limiting operator

$$\mathcal{F}_\alpha(P_F f) = \chi_F \mathcal{F}_\alpha(f),$$

where χ_A is the characteristic function of the set A .

If $0 < \gamma_\alpha(F) < \infty$, then for $f \in L_\alpha^2(\mathbb{K})$ we have

$$P_F f(y, \theta) = \int_F \mathcal{F}_\alpha f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu). \tag{2.5}$$

The operators P_F is bounded from $L_\alpha^2(\mathbb{K})$ into itself and

$$\|P_F f\|_{2, m_\alpha} \leq \|f\|_{2, m_\alpha}. \tag{2.6}$$

3. Benedicks type theorem

In order to prove the main theorem of this section, we start by proving the following lemmas.

Lemma 3.1 *If $0 < m_\alpha(E) < \infty$ and $\gamma_\alpha(F) < \infty$, then the Hilbert-Schmidt norm of $P_F T_E$ is finite and we have*

$$\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}.$$

Proof Let $f \in L^2_\alpha(\mathbb{K})$, from relation (2.5)

$$\begin{aligned} P_F T_E f(y, \theta) &= \int_F \mathcal{F}_\alpha T_E f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu) \\ &= \int_{\widehat{\mathbb{K}}} \chi_F(\lambda, \mu) \left\{ \int_{\mathbb{K}} \chi_E(s, t) f(s, t) \varphi_{-\lambda, \mu}(s, t) dm_\alpha(s, t) \right\} \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu). \end{aligned}$$

Denote

$$g_{s,t}(\lambda, \mu) = \chi_F(\lambda, \mu) \varphi_{-\lambda, \mu}(s, t)$$

and

$$\mathcal{N}(s, t, y, \theta) = \chi_E(s, t) \mathcal{F}_\alpha^{-1}(g_{s,t})(y, \theta).$$

Using Fubini's theorem, we obtain

$$P_F T_E f(y, \theta) = \int_{\mathbb{K}} f(s, t) \mathcal{N}(s, t, y, \theta) dm_\alpha(s, t).$$

\mathcal{N} is called the kernel of integral operator $P_F T_E$ and the Hilbert-Schmidt norm of this operator is given by

$$\|P_F T_E\|_{HS} = \|\mathcal{N}\|_{L^2_\alpha(\mathbb{K}) \otimes L^2_\alpha(\mathbb{K})}.$$

Therefore,

$$\|\mathcal{N}\|_{L^2_\alpha(\mathbb{K}) \otimes L^2_\alpha(\mathbb{K})} = \left(\int_{\mathbb{K}} |\chi_E(s, t)|^2 \left(\int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(g_{s,t})(y, \theta)|^2 dm_\alpha(y, \theta) \right) dm_\alpha(s, t) \right)^{\frac{1}{2}}.$$

By applying Plancherel formula (2.3), we get

$$\|\mathcal{N}\|_{L^2_\alpha(\mathbb{K}) \otimes L^2_\alpha(\mathbb{K})} = \left(\int_{\mathbb{K}} \chi_E(s, t) \left(\int_{\mathbb{K}} \chi_F(\lambda, \mu) |\varphi_{-\lambda, \mu}(s, t)|^2 d\gamma_\alpha(\lambda, \mu) \right) dm_\alpha(s, t) \right)^{\frac{1}{2}}.$$

We deduce from relation (2.1) that

$$\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}.$$

□

Lemma 3.2 *Let $f \in L^2_\alpha(\mathbb{K})$. Then*

$$(1 - \|P_F T_E\|) \|f\|_{2, m_\alpha} \leq (\|T_{E^c} f\|_{2, m_\alpha}^2 + \|P_{F^c} f\|_{2, m_\alpha}^2)^{\frac{1}{2}}$$

Proof Let I be the identity operator, we have

$$I = P_F T_E + P_F T_{E^c} + P_{F^c}.$$

For $f \in L_\alpha^2(\mathbb{K})$, we get

$$\begin{aligned} \|f - P_F T_E f\|_{2,m_\alpha}^2 &= \|P_F T_{E^c} f + P_{F^c} f\|_{2,m_\alpha}^2 \\ &= \|P_F T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2 \end{aligned}$$

It follows by using (2.6) that

$$\|f - P_F T_E f\|_{2,m_\alpha}^2 \leq \|T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2. \tag{3.1}$$

On the other hand, we have

$$\|f - P_F T_E f\|_{2,m_\alpha} \geq \|f\|_{2,m_\alpha} - \|P_F T_E f\|_{2,m_\alpha}.$$

Since

$$\|P_F T_E f\| \leq \|P_F T_E\| \|f\|_{2,m_\alpha},$$

therefore

$$\|f - P_F T_E f\|_{2,m_\alpha} \geq (1 - \|P_F T_E\|) \|f\|_{2,m_\alpha}. \tag{3.2}$$

Combining relations (3.1) and (3.2) we obtain the wanted result. □

Theorem 3.3 *Let $f \in L_\alpha^2(\mathbb{K})$. If $\text{supp}(f) \subset E$, $\text{supp}(\mathcal{F}_\alpha f) \subset F$ and $0 < m_\alpha(E)\gamma_\alpha(F) < 1$ then $f = 0$.*

Proof Let $f \in L_\alpha^2(\mathbb{K})$. from lemma 3.1, we obtain

$$\|P_F T_E\| \leq \|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)} < 1.$$

Applying lemma 3.2, we get

$$\begin{aligned} \|f\|_{2,m_\alpha}^2 &\leq (1 - \|P_F T_E\|)^{-2} (\|T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2) \\ &\leq \left(1 - \sqrt{m_\alpha(E)\gamma_\alpha(F)}\right)^{-2} (\|T_{E^c} f\|_{2,m_\alpha}^2 + \|P_{F^c} f\|_{2,m_\alpha}^2). \end{aligned}$$

Hence $\text{supp} f \subset E$ and $\text{supp} \mathcal{F}_\alpha f \subset F$ then

$$T_{E^c} f = 0 \quad \text{and} \quad P_{F^c} f = 0.$$

Therefore $f = 0$. □

4. Donoho-Stark uncertainty principle

4.1. L^2 version of Donoho-Stark theorem

We start by giving the definition of ε -concentrated functions.

Definition 4.1 Let $f \in L^2_\alpha(\mathbb{K})$, E and F be measurable subsets, respectively, of \mathbb{K} and $\widehat{\mathbb{K}}$. We call

1. f is an ε_E -concentrated on E if there exists a vanishing function g on $\mathbb{K} \setminus E$, such that

$$\|f - g\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}.$$

2. $\mathcal{F}_\alpha(f)$ is an ε_F -concentrated on F if there exists a vanishing function h on $\widehat{\mathbb{K}} \setminus F$, such that

$$\|\mathcal{F}_\alpha(f) - h\|_{2,\gamma_\alpha} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.$$

Lemma 4.2 Let $f \in L^2_\alpha(\mathbb{K})$, E and F be measurable subsets, respectively, of \mathbb{K} and $\widehat{\mathbb{K}}$. We have

1. f is ε_E -concentrated on E if and only if

$$\|f - T_E f\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}. \tag{4.1}$$

2. $\mathcal{F}_\alpha f$ is ε_F -concentrated on F if and only if

$$\|f - P_F f\|_{2,m_\alpha} \leq \varepsilon_F \|f\|_{2,m_\alpha}. \tag{4.2}$$

Proof

1. Let f be a ε_E -concentrated on E . There exists a vanishing function g on E^c , such that

$$\|f - g\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}. \tag{4.3}$$

On the other hand, we have

$$f(y, \theta) - T_E f = \chi_{E^c} f.$$

Then

$$\begin{aligned} \|f - T_E f\|_{2,m_\alpha}^2 &= \int_{\mathbb{K}} |f(y, \theta) - T_E f(y, \theta)|^2 dm_\alpha(y, \theta) \\ &= \int_{E^c} |f(y, \theta) - g(y, \theta)|^2 dm_\alpha(y, \theta) \\ &\leq \|f - g\|_{2,m_\alpha}^2. \end{aligned}$$

Then from relation (4.3), we get

$$\|f - T_E f\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}.$$

2. Let $\mathcal{F}_\alpha f$ be a ε_F -concentrated to F , then there exists a vanishing function h on F^c , such that

$$\|\mathcal{F}_\alpha(f) - h\|_{2,\gamma_\alpha} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}. \tag{4.4}$$

Moreover

$$\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_F f) = \mathcal{F}_\alpha f - \chi_F \mathcal{F}_\alpha f = \chi_{F^c} \mathcal{F}_\alpha f.$$

Then

$$\begin{aligned} \|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_F f)\|_{2,\gamma_\alpha}^2 &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_\alpha f(\lambda, \mu) - \mathcal{F}_\alpha(P_F f)(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu) \\ &= \int_{F^c} |\mathcal{F}_\alpha f(\lambda, \mu) - h(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu) \\ &\leq \|\mathcal{F}_\alpha f - h\|_{2,\gamma_\alpha}^2. \end{aligned}$$

By relation (4.4), we obtain the following result

$$\|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_F f)\|_{2,\gamma_\alpha} \leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.$$

Applying Plancherel's formula (2.3) on both terms of the above inequality we get

$$\|f - P_F f\|_{2,m_\alpha} \leq \varepsilon_F \|f\|_{2,m_\alpha}.$$

□

Lemma 4.3 For $f \in L_\alpha^2(\mathbb{K})$ we have

$$\|P_F T_E f\|_{2,m_\alpha} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)} \|f\|_{2,m_\alpha}.$$

Proof Assume that $m_\alpha(E)$ and $\gamma_\alpha(F)$ are finite. Applying Lemma 3.1 we get

$$\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}$$

considering

$$\|P_F T_E\| = \sup_{f \in L_\alpha^2(\mathbb{K}) \setminus \{0\}} \frac{\|P_F T_E f\|_{2,m_\alpha}}{\|f\|_{2,m_\alpha}} \leq \|P_F T_E\|_{HS}$$

then for $f \in L_\alpha^2(\mathbb{K}) \setminus \{0\}$ we have

$$\frac{\|P_F T_E f\|_{2,m_\alpha}}{\|f\|_{2,m_\alpha}} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}$$

which allows us to deduce the wanted result.

□

Theorem 4.4 Consider a nonzero function $f \in L_\alpha^2(\mathbb{K})$. If f is an ε_E -concentrated on E , $\mathcal{F}_\alpha f$ is an ε_F -concentrated on F and $\varepsilon_E + \varepsilon_F < 1$, then

$$\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq 1 - \varepsilon_E - \varepsilon_F.$$

Proof Let $f \in L^2_\alpha(\mathbb{K}) \setminus \{0\}$, we have

$$\|f - P_F T_E f\|_{2,m_\alpha} \leq \|f - P_F f\|_{2,m_\alpha} + \|P_F f - P_F T_E f\|_{2,m_\alpha}.$$

From relations(4.2), (2.6) and (4.1), we obtain

$$\begin{aligned} \|f - P_F T_E f\|_{2,m_\alpha} &\leq \varepsilon_F \|f\|_{2,m_\alpha} + \|f - T_E f\|_{2,m_\alpha} \\ &\leq (\varepsilon_E + \varepsilon_F) \|f\|_{2,m_\alpha}. \end{aligned}$$

which allows us to get the following inequality

$$\begin{aligned} \|P_F T_E f\|_{2,m_\alpha} &\geq \|f\|_{2,m_\alpha} - \|f - P_F T_E f\|_{2,m_\alpha} \\ &\geq (1 - \varepsilon_E - \varepsilon_F) \|f\|_{2,m_\alpha}. \end{aligned}$$

Applying lemma 4.3 we conclude that

$$\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq (1 - \varepsilon_E - \varepsilon_F).$$

□

4.2. L^1 version of Donoho-Stark theorem

In this section, we study the case of a function $f \in L^1_\alpha(\mathbb{K})$.

The operator T_E verifies the following inequality on $L^1_\alpha(\mathbb{K})$.

$$\|T_E f\|_{1,m_\alpha} \leq \|f\|_{1,m_\alpha} \tag{4.5}$$

We say that f is an ε_E -concentrated on E in $L^1_\alpha(\mathbb{K})$ if

$$\|f - T_E f\|_{1,m_\alpha} \leq \varepsilon_E \|f\|_{1,m_\alpha}.$$

We denote by $B^1_\alpha(F)$ the following subset

$$B^1_\alpha(F) = \{g \in L^1_\alpha(\mathbb{K}) \mid P_F g = g\}.$$

We say that f is an ε_F -bandlimited on F if there is a function $g \in B^1_\alpha(F)$ such that

$$\|f - g\|_{1,m_\alpha} \leq \varepsilon_F \|f\|_{1,m_\alpha}.$$

We begin with the following lemma in order to prove the Donoho-Stark type theorem on $L^1_\alpha(\mathbb{K})$.

Lemma 4.5 Consider a nonzero function $f \in B^1_\alpha(F)$, we have

$$\frac{\|T_E f\|_{1,m_\alpha}}{\|f\|_{1,m_\alpha}} \leq m_\alpha(E)\gamma_\alpha(F).$$

Proof Let $f \in B_1^\alpha(F) \setminus \{0\}$, according to relation (2.5) we get

$$f(y, \theta) = \int_{\mathbb{K}} \chi_F(\lambda, \mu) \mathcal{F}_\alpha f(\lambda, \mu) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu).$$

Therefore by Fubini's theorem, we obtain

$$f(y, \theta) = \int_{\mathbb{K}} f(s, t) \left(\int_F \varphi_{-\lambda, \mu}(s, t) \varphi_{\lambda, \mu}(y, \theta) d\gamma_\alpha(\lambda, \mu) \right) dm_\alpha(s, t).$$

From relation(2.1), we get

$$\|f\|_{\infty, m_\alpha} \leq \gamma_\alpha(F) \|f\|_{1, m_\alpha}. \tag{4.6}$$

Furthermore,

$$\|T_E f\|_{1, m_\alpha} = \int_{\mathbb{K}} \chi_E(y, \theta) |f(y, \theta)| dm_\alpha(y, \theta) \leq m_\alpha(E) \|f\|_{\infty, m_\alpha}$$

by using the relation(4.6), we get

$$\|T_E f\|_{1, m_\alpha} \leq m_\alpha(E) \gamma_\alpha(F) \|f\|_{1, m_\alpha}.$$

Then, we gain the wanted result. □

Theorem 4.6 Consider a nonzero function $f \in L_\alpha^1(\mathbb{K})$ and $\varepsilon_E, \varepsilon_F$ two real numbers such that $\varepsilon_E + \varepsilon_F < 1$. If f is ε_E -concentrated on E and ε_F -bandlimited on F in $L_\alpha^1(\mathbb{K})$ then

$$m_\alpha(E) \gamma_\alpha(F) \geq \frac{1 - \varepsilon_E - \varepsilon_F}{1 + \varepsilon_F}.$$

Proof We consider $f \in L_\alpha^1(\mathbb{K}) \setminus \{0\}$, we have

$$\|T_E f\|_{1, m_\alpha} = \|f + T_E f - f\|_{1, m_\alpha}.$$

By applying the triangular inequality, we obtain

$$\|T_E f\|_{1, m_\alpha} \geq \|f\|_{1, m_\alpha} - \|f - T_E f\|_{1, m_\alpha}.$$

Since f is ε_E -concentrated on E , then

$$\|T_E f\|_{1, m_\alpha} \geq (1 - \varepsilon_E) \|f\|_{1, m_\alpha}. \tag{4.7}$$

On the other hand, f is ε_F -bandlimited so there exists a function $g \in B_\alpha^1(F)$ such that

$$\|f - g\|_{1, m_\alpha} \leq \varepsilon_F \|f\|_{1, m_\alpha}. \tag{4.8}$$

Furthermore, from relation (4.5) we get

$$\|T_E g\|_{1, m_\alpha} \geq \|T_E f\|_{1, m_\alpha} - \|T_E f - T_E g\|_{1, m_\alpha} \geq \|T_E f\|_{1, m_\alpha} - \|f - g\|_{1, m_\alpha}.$$

Using both relations (4.7) and (4.8), we get

$$\|T_E g\|_{1,m_\alpha} \geq (1 - \varepsilon_E - \varepsilon_F) \|f\|_{1,m_\alpha}.$$

On the other hand, we have

$$\|g\|_{1,m_\alpha} \leq (1 + \varepsilon_F) \|f\|_{1,m_\alpha}.$$

Therefore,

$$\frac{\|T_E g\|_{1,m_\alpha}}{\|g\|_{1,m_\alpha}} \geq \frac{1 - \varepsilon_E - \varepsilon_F}{1 + \varepsilon_F}.$$

Then, by lemma 4.5 we obtain the wanted result. □

In the sequel, we give an $L^1_\alpha \cap L^2_\alpha$ version of Donoho-Stark theorem for the generalized Fourier transform \mathcal{F}_α .

Theorem 4.7 Consider a nonzero function $f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$. If f is ε_E -concentrated on E in $L^1_\alpha(\mathbb{K})$ and $\mathcal{F}_\alpha f$ is ε_F -concentrated on F in $L^2_\alpha(\mathbb{K})$ then

$$m_\alpha(E)\gamma_\alpha(F) \geq (1 - \varepsilon_E)^2(1 - \varepsilon_F)^2.$$

Proof Assume that $m_\alpha(E)$ and $\gamma_\alpha(F)$ are finite. For a nonzero function $f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$, we have

$$\|f\|_{2,m_\alpha} \leq \|f - P_F f\|_{2,m_\alpha} + \|P_F f\|_{2,m_\alpha}.$$

Plancherel's formula (2.3) gives us the following inequality

$$\|f\|_{2,m_\alpha} \leq \|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_F f)\|_{2,\gamma_\alpha} + \|\chi_F \mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}.$$

Since $\mathcal{F}_\alpha f$ is ε_F -concentrated on F in $L^2_\alpha(\mathbb{K})$, we obtain by using relation(4.2)

$$\begin{aligned} \|f\|_{2,m_\alpha} &\leq \varepsilon_F \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha} + \left(\int_F |\mathcal{F}_\alpha f(\lambda, \mu)|^2 d\gamma_\alpha(\lambda, \mu) \right)^{\frac{1}{2}} \\ &\leq \varepsilon_F \|f\|_{2,m_\alpha} + \sqrt{\gamma_\alpha(F)} \|\mathcal{F}_\alpha f\|_{\infty,\gamma_\alpha}. \end{aligned}$$

Furthermore from relation (2.4), we obtain

$$(1 - \varepsilon_F) \|f\|_{2,m_\alpha} \leq \sqrt{\gamma_\alpha(F)} \|f\|_{1,m_\alpha}. \tag{4.9}$$

On the other hand, we have

$$\|f\|_{1,m_\alpha} \leq \|f - T_E f\|_{1,m_\alpha} + \|T_E f\|_{1,m_\alpha}.$$

Seeing that f is ε_E -concentrated on E in $L^1_\alpha(\mathbb{K})$, we conclude from relation (4.1) that

$$\begin{aligned} \|f\|_{1,m_\alpha} &\leq \varepsilon_E \|f\|_{1,m_\alpha} + \int_E |f(y, \theta)| dm_\alpha(y, \theta) \\ &\leq \varepsilon_E \|f\|_{1,m_\alpha} + \sqrt{m_\alpha(E)} \|f\|_{2,m_\alpha}. \end{aligned}$$

Therefore,

$$(1 - \varepsilon_E) \|f\|_{1,m_\alpha} \leq \sqrt{m_\alpha(E)} \|f\|_{2,m_\alpha}. \tag{4.10}$$

Combining (4.9) and (4.10) we reach the needed result. □

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