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Extension of Montgomery identity via Taylor polynomial on time scales

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Abstract: An extension in Montgomery identity with the help of Taylor's formula on time scales is provided in the paper, which is used to establish Ostrowski type inequality, midpoint inequality and trapezoid type inequality on time scales in generalized forms. The weighted version of obtained Montgomery identity and respective Ostrowski inequality are also addressed at the end of the paper.

Key words: Mathematical inequalities, time scales calculus

1. Introduction

An identity due to Montgomery is utilized to obtain a number of novel inequalities such as Ostrowski type inequality, trapezoid inequality, Mohajani inequality, Čebyšev and Grüss inequalities:
The Montgomery identity given by Pečarić in [17] is stated as:
Let $\psi : [\alpha, \beta] \to \mathbb{R}$ and $\psi' : [\alpha, \beta] \to \mathbb{R}$ be integrable on $[\alpha, \beta]$, then

$$\psi(y) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \psi(v)dv + \int_{\alpha}^{\beta} \hat{R}(y, v)\psi'(v)dv, \tag{1.1}$$

where

$$\hat{R}(y, v) = \left\{ \begin{array}{ll} \frac{v-\alpha}{\beta-\alpha}, & \alpha \leq v < y, \\ \frac{v-\beta}{\beta-\alpha}, & y \leq v \leq \beta. \end{array} \right\}.$$

The interpolation (1.1) is used by many authors to generalized inequalities for higher order convex function, namely Jensen's inequaity [12], Jensen-Stefensen's inequality [13], Sherman's inequality [14, 15], cyclic refinement of Jensen's inequality [8], Popoviciu's inequality [9, 16], combinatorial improvements of Jensen's inequality [18], Levinson's inequality [1] via time scales theory [2].

The Montgomery weighted identity (obtained in [19]) states, for $y \in [\alpha, \beta]$,

$$\psi(y) = \int_{\alpha}^{\beta} z(v)\psi(v)dv + \int_{\alpha}^{\beta} \hat{R}_z(y, v)\psi'(v)dv, \tag{1.2}$$

where $\psi : [\alpha, \beta] \to \mathbb{R}$ is differentiable on $[\alpha, \beta]$, while $\psi' : [\alpha, \beta] \to \mathbb{R}$ integrable on the same interval $[\alpha, \beta]$ and $z : [\alpha, \beta] \to [0, \infty\rangle$ is normalized weight function, i.e. $\int_{\alpha}^{\beta} z(v)dv = 1$, where as $Z(v) = \int_{\alpha}^{v} z(y)dy$ for $v \in [\alpha, \beta]$

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and the weighted Peano kernel is

$$\hat{R}_z(y,v) = \left\{ \begin{array}{cc} Z(v), & \alpha \le v < y, \\ Z(v)-1, & y \le v \le \beta. \end{array} \right\}.$$

The development in the field of time scales was started by German mathematician Hilger in 1988 as a theory to have ability to contain difference and differential calculus both together in a consistent way. From that point forward, numerous creators have thought about many integral inequalities on this topic. Bohner and Matthews [6] obtained the following Montgomery identity and respective Ostrowski inequality in the context of time scales. Some of the important integral inequalities can also be seen in [21–24].

Theorem 1.1 *[6, Lemma 3.1] Let* $\alpha, \beta, w, u \in \mathbb{T},\ \alpha < \beta$ *and* $\psi : [\alpha,\beta]_{\mathbb{T}} = [\alpha,\beta] \bigcap \mathbb{T} \to \mathbb{R}$ *be differentiable, then*

$$\psi(u) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \psi^{\sigma}(w)\Delta w + \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \hat{R}(u,w)\psi^{\Delta}(w)\Delta w, \tag{1.3}$$

where

$$\hat{R}(u,w) = \left\{ \begin{array}{cc} w-\alpha, & \alpha \le w < u, \\ w-\beta, & u \le w \le \beta. \end{array} \right\}.$$

The following is the weighted identity on the same topic:

Theorem 1.2 *[25] Let* $\alpha, \beta, w, u \in \mathbb{T},\ \alpha < \beta$ *and* $\psi : [\alpha,\beta]_{\mathbb{T}} = [\alpha,\beta] \bigcap \mathbb{T} \to \mathbb{R}$ *be differentiable, then*

$$\psi(u) = \int_{\alpha}^{\beta} z(w)\psi^{\sigma}(w)\Delta w + \int_{\alpha}^{\beta} \hat{R}_z(u,w)\psi^{\Delta}(w)\Delta w, \tag{1.4}$$

where

$$\hat{R}_z(u,w) = \left\{ \begin{array}{cc} Z(w), & \alpha \le w < u, \\ Z(w)-1, & u \le w \le \beta. \end{array} \right\}.$$

Where $z : [\alpha,\beta] \to [0,\infty), \int_{\alpha}^{\beta} z(u)du = 1$ and $Z(u) = \int_{\alpha}^{u} z(y)dy$ for $u \in [\alpha,\beta]$ and $Z(u) = 0$ for $u < \alpha$, $Z(u) = 1$ for $u > \beta$.
In the paper, we extend Montgomery identities (1.3), (1.4) by using Taylor series in time scales settings and we prove the respective trapezoid and Ostrowski type inequalities for this scenario. The sharpness of the constants on the RHS of Ostrowski inequality is also addressed. Moreover special cases of obtained Ostrowski inequality include generalized midpoint inequality.

2. Preliminary results

Generalized polynomials on time scales are the functions $g, h : \mathbb{T}^2 \to \mathbb{R}, \hat{l} \in \mathbb{N}_0$ defined recursively as follows: $g_0(u,w) = h_0(u,w) = 1,\ \forall u,w \in \mathbb{T}$ and for given $g_{\hat{l}}, h_{\hat{l}}$ with $\hat{l} \in \mathbb{N}$,

$$h_{\hat{l}+1}(u,w) = \int_{w}^{u} h_{\hat{l}}(\tau,w)\Delta\tau, \quad g_{\hat{l}+1}(u,w) = \int_{w}^{u} g_{\hat{l}}(\sigma(\tau),w)\Delta\tau.$$

If $h_{\hat{l}}^{\Delta}(u,w)$ denotes for each fixed w, the derivative of $h_{\hat{l}+1}(u,w)$ according to the variable u, then

$$h_{\hat{l}+1}^{\Delta}(u,w) = h_{\hat{l}}(u,w), \quad g_{\hat{l}+1}^{\Delta}(u,w) = g_{\hat{l}}(\sigma(u),w) \text{ for } \hat{l} \in \mathbb{N}_0, u \in \mathbb{T}^k.$$

Also $\mathbb{T}^k = \left\{ \begin{array}{cc} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \sup \mathbb{T} < \infty \\ \mathbb{T} & if \sup \mathbb{T} = \infty. \end{array} \right\}.$

Further

$$h_{\hat{l}}(u,w) = (-1)^{\hat{l}} g_{\hat{l}}(w,u).$$

Taylor formula for arbitrary time scale $\mathbb{T}$ is as what follows:

Theorem 2.1 *[7, Theorem 1.113] Let $m \in \mathbb{N}$, ψ is m times differentiable on $\mathbb{T}^{\hat{l}^m}$. Then for $u \in \mathbb{T}$, we have*

$$\psi(w) = \sum_{\hat{l}=0}^{m-1} h_{\hat{l}}(w,u)\psi^{\Delta^{\hat{l}}}(u) + \int_{u}^{\rho^{m-1}(w)} h_{m-1}(w,\sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta, \tag{2.1}$$

where $h_{\hat{l}} : \mathbb{T}^2 \to \mathbb{R}$, $\hat{l} \in \mathbb{N}_0$ represents generalized polynomial on time scales with the following values $h_0(w,u) = 1$, $h_2(w,u) = w - u$, $\forall u, w \in \mathbb{T}$ and $h_{\hat{l}}(u,w) = \int_w^u h_{\hat{l}-1}(\eta,w)\Delta\eta$, $h_{\hat{l}}^{\Delta}(u,w) = h_{\hat{l}-1}(u,w)$ for $\hat{l} \in \mathbb{N}, u \in \mathbb{T}^k$.

Elvan Akın proved the following result to exchange the integrals on time scales.

Theorem 2.2 *[3, Theorem 10] Assume $\alpha < \beta \in \mathbb{T}$ and $\Psi(\eta,w)$ is a real-valued function on $\mathbb{T} \times \mathbb{T}$, then*

$$\int_{\alpha}^{\beta} \int_{\alpha}^{\eta} \Psi(\eta,w)\Delta w \Delta\eta = \int_{\alpha}^{\beta} \int_{\sigma(w)}^{\beta} \Psi(\eta,w)\Delta\eta\Delta w.$$

See also [11, Lemma 1] for exchange of integrals on time scales.

Remark 2.3 *From [7, Theorem 1.109], it is easy to obtain*

$$g_m(u,\rho^l(u)) = 0 \ \forall m \in \mathbb{N}, \ 0 \le l \le m-1. \tag{2.2}$$

The following lemma is used to proof the results.

Lemma 2.4 *The function $h_{\hat{l}}$ for $u \in \mathbb{T}$ satisfies*

$$h_m(\rho^{\hat{l}}(u), \sigma(u)) = 0, \ \forall m \in \mathbb{N}, \ 0 \le \hat{l} \le m-2. \tag{2.3}$$

Proof By using Remark 2.3, we can write $g_m(u,\rho^{\hat{l}}(u)) = 0 \ \forall m \in \mathbb{N}, \ 0 \le \hat{l} \le m-1$. Since $h_m(u,w) = (-1)^m g_m(w,u)$, $\forall m \in \mathbb{N}$. Therefore

$$h_m(\rho^{\hat{l}}(u), u) = 0, \ \forall m \in \mathbb{N}, \ 0 \le \hat{l} \le m-1.$$

By using [7, Theorem 1.16 (iw)],

$$h_m(\rho^{\hat{l}}(u), \sigma(u)) = h_m(\rho^{\hat{l}}(u), u) + \mu(u) h_{m-1}(\rho^{\hat{l}}(u), u).$$

$$\Rightarrow h_m(\rho^{\hat{l}}(u), \sigma(u)) = 0, \;\; \forall m \in \mathbb{N}, \;\; 0 \leq \hat{l} \leq m-2.$$

□

3. Generalized Montgomery identity on time scales

Theorem 3.1 *Let* $m \in \mathbb{N}$, ψ *is* m *times differentiable on* $\mathbb{T}^{\hat{l}^m}$. *Let* $u \in \mathbb{T}$, *then we have*

$$\begin{aligned} \psi(u) &= \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \psi^{\sigma}(v)\Delta v + \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2} \psi^{\Delta^{\hat{l}+1}}(u)\Big\{h_{\hat{l}+2}(\sigma(\alpha), u) - h_{\hat{l}+2}(\sigma(\beta), u)\Big\} \\ &+ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} Q_m(u,\eta)\psi^{\Delta^m}(\eta)\Delta\eta, \end{aligned} \tag{3.1}$$

where

$$Q_m(u,\eta) = \left\{ \begin{array}{ll} -h_m(\sigma(\alpha), \sigma(\eta)), & \eta \in [\alpha, \rho^{m-2}(u)), \\ -h_m(\sigma(\beta), \sigma(\eta)), & \eta \in [\rho^{m-2}(u), \beta]. \end{array} \right\}.$$

Proof Suppose ψ^{Δ} is $m-1$ times differentiable, then by replacing m with $m-1$, ψ with ψ^{Δ} in (2.1), we have

$$\psi^{\Delta}(v) = \sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(v,u)\psi^{\Delta^{\hat{l}+1}}(u) + \int_{u}^{\rho^{m-2}(s)} h_{m-2}(v, \sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta. \tag{3.2}$$

We can rewrite (1.3) as

$$\begin{aligned} \psi(u) &= \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} \psi^{\sigma}(v)\Delta v + \frac{1}{\beta-\alpha}\int_{\alpha}^{u} (v-\alpha)\psi^{\Delta}(v)\Delta s \\ &+ \frac{1}{\beta-\alpha}\int_{u}^{\beta} (v-\beta)\psi^{\Delta}(v)\Delta v. \end{aligned} \tag{3.3}$$

By using (3.2) in (3.3), we have

$$
\begin{aligned}
\psi(u) &= \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v \\
&+ \frac{1}{\beta-\alpha}\int_{\alpha}^{u}(v-\alpha)\sum_{\hat{l}=0}^{m-2}h_{\hat{l}}(v,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta v \quad (3.4)\\
&+ \frac{1}{\beta-\alpha}\int_{u}^{\beta}(v-\beta)\sum_{\hat{l}=0}^{m-2}h_{\hat{l}}(v,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta v \quad (3.5)\\
&+ \frac{1}{\beta-\alpha}\int_{\alpha}^{u}(v-\alpha)\int_{u}^{\rho^{m-2}(v)}h_{m-2}(v,\sigma(\eta))\psi^{\Delta^{m}}(\eta)\Delta\eta\Delta v \quad (3.6)\\
&- \frac{1}{\beta-\alpha}\int_{u}^{\beta}(v-\beta)\int_{\rho^{m-2}(v)}^{u}h_{m-2}(v,\sigma(\eta))\psi^{\Delta^{m}}(\eta)\Delta\eta\Delta v. \quad (3.7)
\end{aligned}
$$

By making calculations for integral in (3.4), we obtain

$$
\begin{aligned}
\int_{\alpha}^{u}(v-\alpha)\sum_{\hat{l}=0}^{m-2}h_{\hat{l}}(v,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta v &= \sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\int_{\alpha}^{u}h_{1}(v,\alpha)h^{\Delta}_{\hat{l}+1}(v,u)\Delta v,\\
&= \sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\{h_{\hat{l}+2}(\sigma(\alpha),u)-h_{\hat{l}+2}(\sigma(u),u)\}.
\end{aligned}
\tag{3.8}
$$

Similarly (3.5) gives

$$
\int_{u}^{\beta}(v-\beta)\sum_{\hat{l}=0}^{m-2}h_{\hat{l}}(v,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta v = \sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\{h_{\hat{l}+2}(\sigma(u),u)-h_{\hat{l}+2}(\sigma(\beta),u)\}.
\tag{3.9}
$$

By making calculations for (3.6), we have

$$\begin{aligned}\int_{\alpha}^{u}(v-\alpha)\int_{u}^{\rho^{m-2}(v)} h_{m-2}(v,\sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta v &= \int_{\alpha}^{\rho^{m-2}(u)} \psi^{\Delta^m}(\eta)\int_{\sigma(\eta)}^{\alpha}(v-\alpha)h_{m-2}(v,\sigma(\eta))\Delta v\Delta\eta,\\ &= \int_{\alpha}^{\rho^{m-2}(u)} \psi^{\Delta^m}(\eta)\int_{\sigma(\eta)}^{\alpha}(v-\alpha)h^{\Delta}_{m-1}(v,\sigma(\eta))\Delta v\Delta\eta,\\ &= -\int_{\alpha}^{\rho^{m-2}(u)} \psi^{\Delta^m}(\eta)\int_{\sigma(\eta)}^{\alpha} h_{m-1}(\sigma(v),\sigma(\eta))\Delta v\Delta\eta,\\ &= -\int_{\alpha}^{\rho^{m-2}(u)} \psi^{\Delta^m}(\eta)h_m(\sigma(\alpha),\sigma(\eta))\Delta\eta. \end{aligned} \tag{3.10}$$

Similarly (3.7) gives

$$\int_{\alpha}^{u}(v-\alpha)\int_{u}^{\rho^{m-2}(v)} h_{m-2}(v,\sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta v = -\int_{\rho^{m-2}(u)}^{\beta} \psi^{\Delta^m}(\eta)h_m(\sigma(\beta),\sigma(\eta))\Delta\eta. \tag{3.11}$$

Use (3.8)–(3.11) in (3.4)–(3.7) respectively to get the desired result. □

Remark 3.2
- *If we use* $\mathbb{T}=\mathbb{R}$ *in Theorem 3.1,* (3.1) *becomes (2.2) in [4].*
- *If we use* $\mathbb{T}=\mathbb{Z}$ *in Theorem 3.1,* (3.1) *takes the form*

$$\begin{aligned}\psi(u) &= \frac{1}{\beta-\alpha}\sum_{v=\alpha}^{\beta-1}\psi(v+1)+\frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\Delta^{\hat{l}+1}\psi(u)\left\{\frac{(\alpha+1-u)^{(\hat{l}+2)}}{(\hat{l}+2)!}-\frac{(\beta+1-u)^{(\hat{l}+2)}}{(\hat{l}+2)!}\right\}\\ &+ \frac{1}{\beta-\alpha}\sum_{\eta=\alpha}^{\beta-1}\Delta^m\psi(\eta)Q_m(u,\eta),\end{aligned}$$

where

$$Q_m(u,\eta)=\begin{bmatrix} -\frac{(\alpha-\eta)^{(m)}}{(m)!}, & \eta\in[\alpha,u-m+2)\,, \\ -\frac{(\beta-\eta)^{(m)}}{m!}, & \eta\in[u-m+2,\beta)\,. \end{bmatrix}.$$

- *If we use* $\mathbb{T} = q^{\mathbb{Z}}$, $q > 1$ *in Theorem 3.1, (3.1) attains the form*

$$\psi(u) = \frac{1}{\beta-\alpha}\sum_{v=\alpha}^{\beta-1}\psi(qv) + \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\Delta^{\hat{l}+1}\psi(u)\Bigg\{\prod_{\mu_2=0}^{\hat{l}+1}\frac{(q\alpha - q^{\mu_2}u)}{\sum\limits_{\mu_1=0}^{\mu_2}q^{\mu_2}} - \prod_{\mu_2=0}^{\hat{l}+1}\frac{(q\beta - q^{\mu_2}u)}{\sum\limits_{\mu_1=0}^{\mu_2}q^{\mu_2}}\Bigg\}$$

$$+ \frac{1}{\beta-\alpha}\sum_{\eta=\alpha}^{\beta-1}\Delta^m\psi(\eta)Q_m(u,\eta),$$

where

$$Q_m(u,\eta) = \begin{cases} -\prod\limits_{\mu_2=0}^{m-1}\frac{(q\alpha - q^{\mu_2+1}\eta)}{\sum\limits_{\mu_1=0}^{\mu_2}q^{\mu_2}}, & \eta \in \left[\alpha, uq^{2-m}\right), \\ -\prod\limits_{\mu_2=0}^{m-1}\frac{(q\beta - q^{\mu_2+1}\eta)}{\sum\limits_{\mu_1=0}^{\mu_2}q^{\mu_2}}, & \eta \in \left[uq^{2-m}, \beta\right). \end{cases}$$

3.1. Trapezoid type inequalities

Theorem 3.3 *Under the conditions of Theorem 3.1, we obtain following generalized trapezoid inequality*

$$\begin{aligned} \Bigg|\frac{\psi(\alpha)+\psi(\beta)}{2} &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v \\ &- \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\alpha)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\alpha) - h_{\hat{l}+2}(\sigma(\beta),\alpha)\Big\} \\ &- \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\beta)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\beta) - h_{\hat{l}+2}(\sigma(\beta),\beta)\Big\}\Bigg| \\ &\leq \frac{1}{2(\beta-\alpha)}\|\psi^{\Delta^m}\|_p\left(\int_{\alpha}^{\beta}|Q_m(\alpha,\eta)+Q_m(\beta,\eta)|^q\Delta\eta\right)^{\frac{1}{q}}. \end{aligned} \tag{3.12}$$

Proof Rewrite (3.1) by replacing $u = \alpha$ and $u = \beta$ in the following forms,

$$\begin{aligned} \psi(\alpha) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v &+ \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\alpha)\left\{h_{\hat{l}+2}(\sigma(\alpha),\alpha) - h_{\hat{l}+2}(\sigma(\beta),\alpha)\right\} \\ &+ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}Q_m(\alpha,\eta)\psi^{\Delta^m}(\eta)\Delta\eta, \end{aligned} \tag{3.13}$$

and

$$\psi(\beta) = \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v + \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\beta)\left\{h_{\hat{l}+2}(\sigma(\alpha),\beta) - h_{\hat{l}+2}(\sigma(\beta),\beta)\right\} + \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}Q_m(\beta,\eta)\psi^{\Delta^m}(\eta)\Delta\eta. \tag{3.14}$$

Add (3.13) and (3.14) and divide the resultant by 2 to get

$$\begin{aligned}\frac{\psi(\alpha)+\psi(\beta)}{2} &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\alpha)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\alpha) - h_{\hat{l}+2}(\sigma(\beta),\alpha)\Big\} \\ &- \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\beta)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\beta) - h_{\hat{l}+2}(\sigma(\beta),\beta)\Big\}, \\ &= \frac{1}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\{Q_m(\alpha,\eta)+Q_m(\beta,\eta)\}\psi^{\Delta^m}(\eta)\Delta\eta. \end{aligned} \tag{3.15}$$

Use Hölder's inequality on R.H.S of (3.15) to obtain

$$\begin{aligned}\Bigg|\frac{\psi(\alpha)+\psi(\beta)}{2} &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\alpha)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\alpha) - h_{\hat{l}+2}(\sigma(\beta),\alpha)\Big\} \\ &- \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\beta)\Big\{h_{\hat{l}+2}(\sigma(\alpha),\beta) - h_{\hat{l}+2}(\sigma(\beta),\beta)\Big\}\Bigg|, \\ &\leq \frac{1}{2(\beta-\alpha)}\|\psi^{\Delta^m}\|_p\left(\int_{\alpha}^{\beta}\Big|Q_m(\alpha,\eta)+Q_m(\beta,\eta)\Big|^q\Delta\eta\right)^{\frac{1}{q}}, \end{aligned}$$

which is required trapezoid inequality. Furthermore, we can write

$$Q_m(\alpha,\eta)+Q_m(\beta,\eta) = -2\Big[h_m(\sigma(\alpha),\sigma(\eta)) + h_m(\sigma(\beta),\sigma(\eta))\Big].$$

□

Remark 3.4 *For* $\mathbb{T} = \mathbb{R}$, *(3.12) becomes as*

$$\begin{aligned}\Bigg|\frac{\psi(\alpha)+\psi(\beta)}{2} &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi(v)dv + \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{(\hat{l}+1)}(\alpha)\left\{\frac{(\beta-\alpha)^{(\hat{l}+2)}}{(\hat{l}+2)!}\right\}\\ &- \frac{1}{2(\beta-\alpha)}\sum_{\hat{l}=0}^{m-2}\psi^{(\hat{l}+1)}(\beta)\left\{\frac{(\alpha-\beta)^{(\hat{l}+2)}}{(\hat{l}+2)!}\right\}\Bigg|\\ &\leq \frac{1}{2(\beta-\alpha)}\|\psi^{(m)}\|_p\left(\int_{\alpha}^{\beta}|Q_m(\alpha,u)+Q_m(\beta,u)|^q du\right)^{\frac{1}{q}},\end{aligned}$$

where

$$Q_m(\alpha,u)+Q_m(\beta,u) = -2\left[\frac{(\alpha-u)^m}{m!}+\frac{(\beta-u)^m}{m!}\right].$$

Remark 3.5 *In the Theorem 3.3 take* $m = 2$ *and* $q = 1$. *In this case (3.12) takes the form*

$$\begin{aligned}\Bigg|\frac{\psi(\alpha)+\psi(\beta)}{2} &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{2(\beta-\alpha)}\psi^{\Delta}(\alpha)\Big\{h_2(\sigma(\alpha),\alpha)-h_2(\sigma(\beta),\alpha)\Big\}\\ &- \frac{1}{2(\beta-\alpha)}\psi^{\Delta}(\beta)\Big\{h_2(\sigma(\alpha),\beta)-h_2(\sigma(\beta),\beta)\Big\}\Bigg|\\ &\leq \frac{1}{2(\beta-\alpha)}\|\psi^{\Delta^2}\|_\infty\int_{\alpha}^{\beta}|Q_2(\alpha,\eta)+Q_2(\beta,\eta)|\Delta\eta,\end{aligned}$$

where

$$Q_2(\alpha,\eta)+Q_2(\beta,\eta) = -2\Big[h_2(\sigma(\alpha),\sigma(\eta))+h_2(\sigma(\beta),\sigma(\eta))\Big].$$

3.2. Ostrowski type inequalities

Theorem 3.6 *With all assumptions of Theorem 3.1,assume additionally (p,q) as a pair of conjugate exponents, that is* $1 \leq p, q < \infty, \frac{1}{p}+\frac{1}{q} = 1$. *Then*

$$\begin{aligned}\Bigg|\psi(u) &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\Big\{h_{\hat{l}+2}(\sigma(\alpha),u)-h_{\hat{l}+2}(\sigma(\beta),u)\Big\}\Bigg|\\ &\leq \frac{1}{\beta-\alpha}\|\psi^{\Delta^m}\|_p\left(\int_{\alpha}^{\beta}|Q_m(u,\eta)|^q\Delta\eta\right)^{\frac{1}{q}}.\end{aligned} \tag{3.16}$$

The constant $(\int_{\alpha}^{\beta}|Q_m(u,\eta)|^q\Delta\eta)^{\frac{1}{q}}$ *is sharp for* $1 < p \leq \infty$. *Also it is best possible for* $p = 1$.

Proof We use the identity (3.1) along with Hölder inequality, consequences

$$\begin{aligned}\Bigg|\psi(u) &- \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\Big\{h_{\hat{l}+2}(\sigma(\alpha),u) - h_{\hat{l}+2}(\sigma(\beta),u)\Big\}\Bigg| \\ &\leq \frac{1}{\beta-\alpha}\|\psi^{\Delta^m}\|_p\left(\int_{\alpha}^{\beta}|Q_m(u,\eta)|^q\Delta\eta\right)^{\frac{1}{q}}.\end{aligned}$$

Let us denote $D_1(\eta) = Q_m(t,\eta)$. To see the sharpness of the constant $(\int_{\alpha}^{\beta}|D_1(\eta)|^q\Delta\eta)^{\frac{1}{q}}$ we need a function u so that the equality in (3.16) is resulted.

For $1 < p < \infty$, take ψ so that

$$\psi^{\Delta^m}(\eta) = sgnD_1(\eta).|D_1(\eta)|^{\frac{1}{p-1}}.$$

For $p = \infty$, take

$$\psi^{\Delta^m}(\eta) = sgnD_1(\eta).$$

For $p = 1$, we shall prove

$$\left|\int_{\alpha}^{\beta}D_1(\eta)\psi^{\Delta^m}(\eta)\Delta\eta\right| \leq \max_{\eta\in[\alpha,\beta]_{\mathbb{T}}}|D_1(\eta)|\left(\int_{\alpha}^{\beta}|\psi^{\Delta^m}(\eta)|\Delta\eta\right). \tag{3.17}$$

Suppose $|D_1(\eta)|$ attains its maximum value at $\eta_0 \in [\alpha,\beta]_{\mathbb{T}}$. Firstly $D_1(\eta_0) > 0$ and $\epsilon > 0$ is such that $\epsilon < \beta - \eta_0$. Define $\psi_\epsilon(\eta)$ by

$$\psi_\epsilon(\eta) = \left\{\begin{array}{cc} 0, & \alpha \leq \eta < \eta_0, \\ \frac{1}{\epsilon}h_m(\eta,\eta_0), & \eta_0 \leq \eta < \eta_0+\epsilon, \\ \frac{1}{m}h_{m-1}(\eta,\eta_0), & \eta_0+\epsilon \leq \eta \leq \beta. \end{array}\right\}.$$

For $\eta_0 \leq \eta < \eta_0 + \epsilon$, the expression for derivatives are

$$\begin{aligned}\psi'_\epsilon(\eta) &= \frac{1}{\epsilon}h_m^{\Delta}(\eta,\eta_0) = \frac{1}{\epsilon}h_{m-1}(\eta,\eta_0). \\ \psi''_\epsilon(\eta) &= \frac{1}{\epsilon}h_{m-1}^{\Delta}(\eta,\eta_0) = \frac{1}{\epsilon}h_{m-2}(\eta,\eta_0).\end{aligned}$$

Continue in the same way, to get

$$\begin{aligned}\psi_\epsilon^{\Delta^m}(\eta) &= \frac{1}{\epsilon}h_{m-m}(\eta,\eta_0) = \frac{1}{\epsilon}h_0(\eta,\eta_0) \\ &= \frac{1}{\epsilon}. \qquad \because h_0 = 1.\end{aligned}$$

For $\eta_0 + \epsilon \leq \eta \leq \beta$,

$$\begin{aligned}\psi'_\epsilon(\eta) &= \frac{1}{m}h_{m-1}^{\Delta}(\eta,\eta_0) = \frac{1}{m}h_{m-2}(\eta,\eta_0). \\ \psi''_\epsilon(\eta) &= \frac{1}{m}h_{m-2}^{\Delta}(\eta,\eta_0) = \frac{1}{m}h_{m-3}(\eta,\eta_0).\end{aligned}$$

Continue in the same way, to get

$$\psi_{\epsilon}^{\Delta^m}(\eta) = \frac{1}{m} h_{m-m}^{\Delta}(\eta,\eta_0) = \frac{1}{m} h_0^{\Delta} = 0.$$

Then, for small enough ϵ

$$\left|\int_{\alpha}^{\beta} D_1(\eta)\psi^{\Delta^m}(\eta)\Delta\eta\right| = \left|\int_{\eta_0}^{\eta_0+\epsilon} D_1(\eta)\frac{1}{\epsilon}\Delta\eta\right| = \frac{1}{\epsilon}\int_{\eta_0}^{\eta_0+\epsilon} D_1(\eta)\Delta\eta.$$

Now from (3.17), we have

$$\frac{1}{\epsilon}\int_{\eta_0}^{\eta_0+\epsilon} D_1(\eta)\Delta\eta \leq D_1(\eta_0)\int_{\eta_0}^{\eta_0+\epsilon}\frac{1}{\epsilon}\Delta\eta = D_1(\eta_0).$$

Since

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\eta_0}^{\eta_0+\epsilon} D_1(\eta)\Delta\eta = D_1(\eta_0).$$

Therefore the statement yields. In case $D_1(\eta_0) < 0$, take

$$\psi_{\epsilon}(\eta) = \left\{\begin{array}{cc} \frac{1}{m}h_{m-1}(\eta,\eta_0+\epsilon), & \alpha \leq \eta < \eta_0, \\ \frac{-1}{\epsilon}h_m(\eta,\eta_0+\epsilon), & \eta_0 \leq \eta < \eta_0+\epsilon, \\ 0, & \eta_0+\epsilon \leq \eta \leq \beta. \end{array}\right\},$$

and working on the same steps as above the result yields. □

Corollary 3.7 *Under the same assumptions of Theorem 3.6 and for $p = 1$, we have*

$$\begin{aligned} \left|\psi(u) - \frac{1}{\beta-\alpha}\int\limits_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v \right. &- \left.\frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(u)\Big\{h_{\hat{l}+2}(\sigma(\alpha),u) - h_{\hat{l}+2}(\sigma(\beta),u)\Big\}\right| \\ &\leq \frac{1}{\beta-\alpha}\|\psi^{\Delta^m}\|\max\Big\{|-h_m(\sigma(\alpha),\rho^{m-3}(u))|, |-h_m(\sigma(\beta),\rho^{m-3}(u))|\Big\}, \end{aligned} \tag{3.18}$$

and the constant located on the right hand side of the above inequality is best possible.

Proof By using (3.1)

$$\begin{aligned} \int_{\alpha}^{\beta}|Q_m(u,\eta)|^q\Delta\eta &= \int_{\alpha}^{\rho^{m-2}(u)}|Q_m(u,\eta)|^q\Delta\eta + \int_{\rho^{m-2}(u)}^{\beta}|Q_m(u,\eta)|^q\Delta\eta, \\ &= \int_{\alpha}^{\rho^{m-2}(u)}|-h_m(\sigma(\alpha),\sigma(\eta))|^q\Delta\eta + \int_{\rho^{m-2}(u)}^{\beta}|-h_m(\sigma(\beta),\sigma(\eta))|^q\Delta\eta. \end{aligned}$$

For $p = 1 \Rightarrow q = \infty$, we have

$$\begin{aligned}\sup_{\eta\in[\alpha,\beta]}|Q_m(u,\eta)| &= \max\left\{\sup_{\eta\in[\alpha,\rho^{m-2}(u)]}|-h_m(\sigma(\alpha),\sigma(\eta))|, \sup_{\eta\in[\rho^{m-2}(t),\beta]}|-h_m(\sigma(\beta),\sigma(\eta))|\right\},\\ &= \max\left\{|-h_m(\sigma(\alpha),\sigma(\rho^{m-2}(u)))|,|-h_m(\sigma(\beta),\sigma(\rho^{m-2}(u)))|\right\},\\ &= \max\left\{|-h_m(\sigma(\alpha),\rho^{m-3}(u))|,|-h_m(\sigma(\beta),\rho^{m-3}(u))|\right\}.\end{aligned}$$

By using above expression in (3.16) we get (3.18). □

Remark 3.8 *Choose $m = 2$ in Corollary 3.7, In this case (3.18) takes the form*

$$\begin{aligned}\left|\psi(u) - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v\right. &- \left.\frac{1}{\beta-\alpha}\psi^{\Delta^2}(u)\left\{h_2(\sigma(\alpha),u) - h_2(\sigma(\beta),u)\right\}\right|\\ &\leq \frac{1}{\beta-\alpha}\|\psi^{\Delta^2}\|\max\left\{|-h_2(\sigma(\alpha),\rho^{-1}(u))|,|-h_2(\sigma(\beta),\rho^{-1}(u))|\right\}.\end{aligned}$$

Remark 3.9 *By using $\mathbb{T} = \mathbb{R}$ in Remark 3.8, we have [4, Corollary 1].*

3.3. Generalized midpoint inequality

Corollary 3.10 *Under the conditions of Theorem 3.6, we find the following inequality of generalized midpoint*

$$\begin{aligned}\left|\psi(\frac{\alpha+\beta}{2})\right. &- \left.\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\psi^{\sigma}(v)\Delta v - \frac{1}{\beta-\alpha}\sum_{\hat{l}=0}^{m-2}\psi^{\Delta^{\hat{l}+1}}(\frac{\alpha+\beta}{2})\left\{h_{\hat{l}+2}(\sigma(\alpha),\frac{\alpha+\beta}{2}) - h_{\hat{l}+2}(\sigma(\beta),\frac{\alpha+\beta}{2})\right\}\right|\\ &\leq \frac{1}{\beta-\alpha}\|\psi^{\Delta^m}\|_p\left(\int_{\alpha}^{\beta}\left|Q_m\left(\frac{\alpha+\beta}{2},\eta\right)\right|^q\Delta\eta\right)^{\frac{1}{q}},\end{aligned}$$

where

$$Q_m(\frac{\alpha+\beta}{2},\eta) = \left\{\begin{array}{ll} -h_m(\sigma(\alpha),\sigma(\eta)), & \eta\in[\alpha,\rho^{m-2}(\frac{\alpha+\beta}{2})),\\ -h_m(\sigma(\beta),\sigma(\eta)), & \eta\in[\rho^{m-2}(\frac{\alpha+\beta}{2}),\beta].\end{array}\right\}.$$

Proof Use $u = \frac{\alpha+\beta}{2}$ in Theorem 3.6, to fulfill the requirement. □

4. Generalized weighted Montogomery identity on time scales

Theorem 4.1 *Let $m \in \mathbb{N}$, If ψ be m times differentiable on $\mathbb{T}^{l^m}$ and $u \in \mathbb{T}, z : [\alpha, \beta]_{\mathbb{T}} = [\alpha, \beta] \bigcap \mathbb{T} \to [0, \infty)$ is a density probability function, then*

$$\begin{aligned}\psi(u) &= \int_{\alpha}^{\beta} z(s)\psi^{\sigma}(s)\Delta s - \sum_{\hat{l}=0}^{m-2} \psi^{\Delta^{\hat{l}+1}}(u)\left\{\int_{\alpha}^{\beta} z(y)h_{\hat{l}+1}(\sigma(y), u)\Delta y\right\} \\ &+ \int_{\alpha}^{\beta} Q_{z,m}(u, \eta)\psi^{\Delta^m}(\eta)\Delta\eta, \qquad (4.1)\end{aligned}$$

where

$$Q_{z,m}(u, \eta) = \left\{\begin{array}{ll} Z(\alpha)h_{m-1}(\alpha, \sigma(\eta)) - \int_{\sigma(\eta)}^{\alpha} Z^{\Delta}(s)h_{m-1}(\sigma(s), \sigma(\eta))\Delta s, & \eta \in [\alpha, \rho^{m-2}(u)), \\ (Z(\beta) - 1)h_{m-1}(\beta, \sigma(\eta)) - \int_{\sigma(\eta)}^{\beta} (Z(s) - 1)^{\Delta}h_{m-1}(\sigma(s), \sigma(\eta))\Delta s, & \eta \in [\rho^{m-2}(u), \beta]. \end{array}\right\}.$$

Proof Since ψ^{Δ} is $m-1$ times differentiable, therefore by replacing m with $m-1$, and ψ with ψ^{Δ} in (2.1), we have

$$\psi^{\Delta}(s) = \sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(s, u)\psi^{\Delta^{\hat{l}+1}}(u) + \int_{u}^{\rho^{m-2}(s)} h_{m-2}(s, \sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta. \qquad (4.2)$$

(1.4) takes the shape

$$\psi(u) = \int_{\alpha}^{\beta} z(s)\psi^{\sigma}(s)\Delta s + \int_{\alpha}^{u} Z(s)\psi^{\Delta}(s)\Delta s + \int_{u}^{\beta} (Z(s) - 1)\psi^{\Delta}(s)\Delta s. \qquad (4.3)$$

Now by using (4.2) in (4.3), we have

$$\begin{aligned}\psi(u) &= \int_{\alpha}^{\beta} z(s)\psi^{\sigma}(s)\Delta s + \int_{\alpha}^{u} Z(s)\sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(s, u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta s \qquad (4.4) \\ &+ \int_{u}^{\beta} (Z(s) - 1)\sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(s, u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta s \\ &+ \int_{\alpha}^{u} Z(s)\int_{u}^{\rho^{m-2}(s)} h_{m-2}(s, \sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta s \\ &- \int_{u}^{\beta} (Z(s) - 1)\int_{\rho^{m-2}(s)}^{u} h_{m-2}(s, \sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta s.\end{aligned}$$

By using Theorem 2.2,

$$\int_{\alpha}^{u} Z(s) \sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(s,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta s = -\sum_{\hat{l}=0}^{m-2} \psi^{\Delta^{\hat{l}+1}}(u) \int_{\alpha}^{u} z(y)h_{\hat{l}+1}(\sigma(y),u)\Delta y. \tag{4.5}$$

Similarly

$$\int_{u}^{\beta} (Z(s)-1) \sum_{\hat{l}=0}^{m-2} h_{\hat{l}}(s,u)\psi^{\Delta^{\hat{l}+1}}(u)\Delta s = -\sum_{\hat{l}=0}^{m-2} \psi^{\Delta^{\hat{l}+1}}(u)\Big\{ \int_{u}^{\beta} z(y)h_{\hat{l}+1}(\sigma(y),u)\Delta y\Big\}. \tag{4.6}$$

By using Theorem 2.2 and (2.3), we have

$$\int_{\alpha}^{u} Z(s) \int_{u}^{\rho^{m-2}(s)} h_{m-2}(s,\sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta s =$$

$$\int_{\alpha}^{\rho^{m-2}(u)} \psi^{\Delta^m}(\eta)\Big\{Z(\alpha)h_{m-1}(\alpha,\sigma(\eta)) - \int_{\sigma(\eta)}^{\alpha} Z^{\Delta}(s)h_{m-1}(\sigma(s),\sigma(\eta))\Delta s\Big\}\Delta\eta. \tag{4.7}$$

Similarly,

$$\int_{u}^{\beta} (Z(s)-1) \int_{u}^{\rho^{m-2}(s)} h_{m-2}(s,\sigma(\eta))\psi^{\Delta^m}(\eta)\Delta\eta\Delta s =$$

$$-\int_{\rho^{m-2}(u)}^{\beta} \psi^{\Delta^m}(\eta)\Big\{(Z(\beta)-1)h_{m-1}(\beta,\sigma(\eta)) - \int_{\sigma(\eta)}^{\beta} (Z(s)-1)^{\Delta}h_{m-1}(\sigma(s),\sigma(\eta))\Delta s\Big\}\Delta\eta. \tag{4.8}$$

Use (4.5)–(4.8) in (4.4), to get the required result. □

4.1. Weighted Ostrowski inequality

Corollary 4.2 *With assumptions of the Theorem 4.1, additionally by taking (p,q) as a pair of conjugate exponents, that is $1 \le p,q \le \infty, \frac{1}{p}+\frac{1}{q}=1$. we find*

$$\Big|\psi(t) - \int_{\alpha}^{\beta} z(s)\psi^{\sigma}(s)\Delta s + \sum_{\hat{l}=0}^{m-2} \psi^{\Delta^{\hat{l}+1}}(u)\Big\{\int_{\alpha}^{\beta} z(y)h_{\hat{l}+1}(\sigma(y),u)\Delta y\Big\}\Big|$$

$$\le \|\psi^{\Delta^m}\|_p \left(\int_{\alpha}^{\beta} \Big|Q_{z,m}(u,\eta)\Big|^q \Delta\eta\right)^{\frac{1}{q}}. \tag{4.9}$$

The constant $(\int_{\alpha}^{\beta} |Q_{z,m}(u,\eta)|^{q} \Delta\eta)^{\frac{1}{q}}$ *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof Proof is similar to proof of Theorem 3.6. □

Remark 4.3 *By using* $\mathbb{T} = \mathbb{R}$ *in Section 4, we have [4, Theorems 1, 2].*

Remark 4.4 *Remaining results appeared in Section 3 can be proved for the weighted Montogomery identity and Ostrowski's inequality.*

5. Conclusion

In this paper, we obtained extension in Montgomery identity with the help of Taylor formula on time scales. Further, it is applied to find extended form of Ostrowski type inequality, midpoint inequality and trapezoid type inequality on time scales in generalized forms along with the weighted version of obtained Montgomery identity and respective Ostrowski's inequality. As special cases, our inequalities contain the results proved in [4] when $\mathbb{T} = \mathbb{R}$. Moreover, our results can be used to prove certain integral inequalities including Čebyšev-Grüss type inequality, Steffensen's inequality and Popoviciu type inequality in more generalized settings, e.g., it is possible to extend the results given in [5, 10, 20] with the help of new inequalities presented here.

Conflict of interest

The authors declare that they have no conflicts of interest.

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